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# Thermoelastic Dynamic Solution Using Helmholtz Displacement Potentials

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**Abstract:**
The Helmholtz's decomposition for the displacement field in terms of vector potentials is used to uncouple the system of equations governing the motion of linear thermoelastic isotropic bodies. The solution proposed by Chandrasekharaiah and Cowin (1989) is recovered in a rather different fashion.

**Keywords:** Linear Thermoelasticity, Helmholtz displacement potentials, Papkovitch-Neuber solution.

## 1. Introduction

When it comes to clarity and straightforwardness, the theoretical framework of continuum mechanics can be summarized by the universally accepted works of Truesdell and Toupin (1960) for the linear theory, Naghdi-Hsu (1961) for the treatment of elastostatics, Truesdell and Noll (1965) for the non-linear theory, Gurtin (1972) for the treatment of elastodynamics and Carlson (1972) expanding on Gurtin's work for the solution of the thermoelastic dynamic equations. Two decades later, Chandrasekharaiah and Cowin (1989) (see also Chandrasekharaiah and Cowin (1993)) propose a unified solution for the theories of thermoelasticity and poroelasticity where the governing equations are written in terms of field variables whose significance varies depending on the theories considered. In particular, the solution for the thermoelastic equations and the connection with the Papkovitch's solution (1932) is shown. The Papkovitch-Neuber approach provides a solution of the three dimensional equations of linear elasticity (assuming a homogeneous isotropic body) although it was later employed for finding analytical solutions of (uncompressible) Navier-Stokes fluid equations. The same solution was also given by Neuber (1934) at a later time. The solution is based on Helmholtz's decomposition of the displacement field into the gradient of a scalar potential plus the curl of a vector potential. It yields a simple expression for the displacement vector in terms of a harmonic vector potential and a harmonic scalar potential. Because harmonic functions are easy to construct and have distinct properties, the Papkovitch-Neuber's solution turns out to be superior to other stress functions. In this paper, we recover the dynamic solution for thermoelastic homogeneous linear isotropic bodies given by Chandrasekhariah and Cowin (1989) in a rather different fashion. Helmholtz's decomposition is used to uncouple the volumetric part of the displacement field from the temperature field. The solution of the thermoelastic dynamic problem reduces to solving a sixth order vector (wave) equation for the displacement field and a fourth order scalar (heat) equation for the temperature field. The solutions for the displacement and temperature fields for the static as well as the isothermal cases are also shown to exhibit striking similarities.

## 2. Preliminaries

Consider a deformable continuum $\mathcal{B}$ comprising infinitely many particles occupying a region $\mathcal{R}$ with a closed boundary $\partial\mathcal{R}$ and moving in a three-dimensional Euclidean space $\mathcal{E}^3$. We denote any subset of $\mathcal{B}$ by $\mathcal{S}$ and let $\mathcal{S}$ occupy a region $\mathcal{P}$ with a closed boundary $\partial\mathcal{P}$. A typical particle $X$ whose position is $X$ in a fixed reference configuration $\kappa_0$ occupies the place $\boldsymbol{x}$ in the current configuration $\kappa$ of $\mathcal{B}$ at time $t$. The motion of a continuum is a smooth invertible mapping defined by

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t), \tag{2.1}$$

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i.e. to every material point of $\kappa_0$ corresponds a material point in $\kappa$. The particle velocity $\boldsymbol{v}$, the particle acceleration $\boldsymbol{a}$, the deformation gradient $\boldsymbol{F}$ relative to $\boldsymbol{X}$, its determinant $J$ and the (symmetric) non-linear Lagrangian strain tensor $\boldsymbol{E}$ are respectively

$$\boldsymbol{v}(\boldsymbol{X},t)=\dot{\boldsymbol{x}}(\boldsymbol{X},t)=\frac{\partial\chi(\boldsymbol{X},t)}{\partial t}, \quad \boldsymbol{a}(\boldsymbol{X},t)=\ddot{\boldsymbol{x}}(\boldsymbol{X},t)=\frac{\partial^2\chi(\boldsymbol{X},t)}{\partial t^2}$$
$$\boldsymbol{F}=\frac{\partial\chi(\boldsymbol{X},t)}{\partial \boldsymbol{X}}, \quad J=\det\boldsymbol{F}, \quad \boldsymbol{E}=\frac{1}{2}(\boldsymbol{C}-\boldsymbol{I}), \tag{2.2}$$

where a superposed dot denotes the material time derivative with respect to $t$ holding $X$ fixed, $\boldsymbol{I}$ is the second order identity tensor, $\boldsymbol{C}$ is the symmetric right Cauchy-Green tensor defined by $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$ where the superscript $T$ denotes the transpose of the tensor.

The Lagrangian strain tensor $\boldsymbol{E}$ written in terms of the displacement gradient is

$$\boldsymbol{E}=\frac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T+\nabla\boldsymbol{u}^T\nabla\boldsymbol{u}) , \tag{2.3}$$

where $\nabla$ is the gradient operator and $\boldsymbol{u}$ is the relative displacement defined by

$$\boldsymbol{x}(\boldsymbol{X},\mathrm{t})=\chi(\boldsymbol{X},\mathrm{t})+\boldsymbol{u}(\boldsymbol{X},\mathrm{t}) \tag{2.4}$$

from which the six independent components of $\boldsymbol{E}$ can be determined.

In the context of a purely mechanical theory, the three fundamental postulates governing the motion of a continuum are the balance of mass, the balance of linear momentum and the balance of angular momentum.

In the (current) Eulerian configuration, the balance of mass is the continuity equation

$$\dot{\rho}+\rho\,\nabla.\dot{\boldsymbol{x}}=0, \tag{2.5}$$

where $\rho$ stands for the mass density function and is the divergence operator with respect to $\boldsymbol{x}$.

If the continuum is subjected to a force field per unit mass $\boldsymbol{b}$ acting on all parts of the body $\mathcal{B}$ and a traction (or stress) vector $\boldsymbol{t}$ acts on its boundary $\partial\mathcal{P}$, the equation governing the motion of all particles is

$$\nabla.\boldsymbol{T}+\rho\boldsymbol{b}=\rho\ddot{\boldsymbol{x}}, \tag{2.6}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor related to $\boldsymbol{t}$ by the relation $\boldsymbol{T}=\boldsymbol{tn}$ in which $\boldsymbol{n}$ denotes the outward unit normal to $\partial\mathcal{P}$.

A consequence of the balance of angular momentum is that the Cauchy stress tensor is symmetric.

The counterpart of (2.6) written in the (referential) Lagragian form is

$$\nabla.\boldsymbol{P}+\rho_0\boldsymbol{b}=\rho_0\ddot{\boldsymbol{x}}, \tag{2.7}$$

where $\nabla.$ is the divergence operator with respect to $\boldsymbol{X}$, $\boldsymbol{P}$ is the (non-symmetric) first Piola-Kirchhoff stress tensor and $\rho_0$ is related to $\rho$ by the relation $\rho_0=\rho J$. In the reference configuration, the stress vector is denoted by $\boldsymbol{p}$ and its relation to $\boldsymbol{P}$ is $\boldsymbol{p}=\boldsymbol{PN}$, where $\boldsymbol{N}$ denotes the outward unit normal to a body with boundary $\partial P_0$.

Another referential stress tensor can be introduced by means of a (Piola) transformation $\pi$:

$$\boldsymbol{T}=\pi\{\boldsymbol{S}\}=\frac{1}{J}\boldsymbol{FSF}^T, \tag{2.8}$$

where $\boldsymbol{S}$ is the second (symmetric) Piola-Kirchhoff stress tensor written as $\boldsymbol{P}=\boldsymbol{FS}$.

## 3. **Thermoelastic Dynamic Equations**

Here we follow the treatment of linear homogeneous isotropic thermoelasticity by Carlson (1972). For a body whose temperature varies, the equation of motion (2.7) is supplemented by the energy equation

$$\rho_0\dot{\varepsilon}=\rho_0 r-\nabla.\boldsymbol{q_0}+\boldsymbol{S}.\dot{\boldsymbol{E}}, \tag{3.1}$$

where $\dot{\varepsilon}$, $\gamma$ and $\boldsymbol{q_0}$ represent the time rate of change of the internal energy per unit mass, the heat supply per unit mass and the heat flux per unit mass respectively and where $\boldsymbol{S}.\dot{\boldsymbol{E}}$ is the stress power.

We recall that in the linear theory, there is no distinction between the various measures of stress. Appealing to a Taylor expansion about the point $(\boldsymbol{0},\,\theta_0)$ in the strain-temperature space, the stress tensor in a homogeneous material can be written as

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E},\theta) = \hat{\boldsymbol{S}}(\boldsymbol{0},\theta_o) + \frac{\partial \hat{\boldsymbol{S}}(\boldsymbol{0},\theta_o)}{\partial \boldsymbol{E}}.\boldsymbol{E} + \frac{\partial \hat{\boldsymbol{S}}(\boldsymbol{0},\theta_o)}{\partial(\theta-\theta_o)}(\theta-\theta_o)+..., \tag{3.2}$$

where $\theta$ stands for the absolute temperature, $\theta_0$ is the absolute temperature in the reference configuration, $\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{E}} = \boldsymbol{K}$ is the fourth order elasticity tensor and $\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{\theta}} = \boldsymbol{M}$ is the stress-temperature tensor.

Assuming a stress-free reference configuration, another form of (3.2) is

$$\boldsymbol{S}\text{=}\boldsymbol{K}\left[\boldsymbol{E}\right]\text{+}\boldsymbol{M}(\theta\text{-}\theta_o) \tag{3.3}$$

One can appeal to the Second Law of Thermodynamics to construct an entropy function $\eta = \hat{\eta}(\boldsymbol{E},\theta)$ and to write the energy equation as

$$\rho_0\dot{\eta}\theta = \rho_0 r - \nabla.\, \boldsymbol{q_0}. \tag{3.4}$$

The specific heat at constant strain $c$ can be written in terms of the internal energy or in terms of the entropy function as

$$c = \frac{\partial\hat{\varepsilon}}{\partial\theta}(\boldsymbol{E},\theta) = \theta\frac{\partial\hat{\eta}}{\partial\theta}(\boldsymbol{E},\theta). \tag{3.5}$$

By expanding the functions $\hat{\eta}$ and $\hat{\varepsilon}$ around the point $(0,\theta_0)$, (3.4) takes the from

$$-\theta_0\boldsymbol{M}.\dot{\boldsymbol{e}} + \rho_0 c_0\dot{\theta} = \rho_0 r - \nabla.\, \boldsymbol{q_0}, \quad \boldsymbol{E} \sim = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \tag{3.6}$$

where $\boldsymbol{e}$ has been obtained by neglecting the last term of (2.3) as is possible in the linearized theory and where $c_0$ refers to the value of $c$ at zero strain.

An assumption for the heat flux is that it depends on the strain field, the temperature field and the temperature gradient

$$\boldsymbol{q_O} = \hat{\boldsymbol{q}}_{\boldsymbol{O}}(\boldsymbol{E},\theta,\boldsymbol{g_O}), \quad \boldsymbol{g_O} = \nabla\theta. \tag{3.7}$$

Introducing the conductivity $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = -\frac{\partial\hat{\boldsymbol{q}}_{\boldsymbol{0}}(\boldsymbol{0},\theta,0)}{\partial\boldsymbol{g_0}}, \tag{3.8}$$

another form for the heat flux is

$$\boldsymbol{q_0} = -\kappa\boldsymbol{g_0}. \tag{3.9}$$

Finally, if isotropy is assumed, the elasticity tensor for a homogeneous body subjected to small deformations can be written as

$$\boldsymbol{K[E]} = \lambda(tr\boldsymbol{e})\boldsymbol{I} + 2\mu\boldsymbol{e}, \tag{3.10}$$

where $\lambda$ and $\mu$ are the Lamé constants and the symbol $tr$ refers to the trace. In addition, the stress-temperature tensor is spherical and the conductivity is a constant

$$\boldsymbol{M} = m\boldsymbol{I}, \quad \boldsymbol{\kappa} = k\boldsymbol{I}. \tag{3.11}$$

We can substitute (3.10) and $(3.11)_1$ into (3.3) to obtain an expression for the stress tensor

$$\boldsymbol{S} = \lambda(tr\boldsymbol{e})\boldsymbol{I} + 2\mu\boldsymbol{e} + m\boldsymbol{I}(\theta-\theta_0) \tag{3.12}$$

Similarly, substituting $(3.1)_1$ into $(3.6)_1$ yields

$$-\theta_0 m(tr\dot{\boldsymbol{e}}) + \rho_0 c_0\dot{\theta} = \rho_0 r - \nabla.\, \boldsymbol{q_0}. \tag{3.13}$$

The relations (2.7), $(3.6)_2$, (3.7) and (3.8) remain unchanged. Along with (3.12) and (3.13), they form a system of six equations governing the motion of a linear thermoelastic homogeneous isotropic body.

Substituting $(3.6)_2$ into (3.12) and (2.7), we find the displacement-temperature equation of motion

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\nabla.\boldsymbol{u}) + m\nabla\theta + \rho_0\boldsymbol{b} = \rho_0\ddot{\boldsymbol{u}}, \tag{3.14}$$

where $\nabla^2$ is the Laplacian operator.

Similarly using (3.9), (3.13) and $(3.11)_2$ yields the heat equation

$$\rho_0 c_0\dot{\theta} = \rho_0 r + k\nabla^2\theta + \theta_0\nabla.\dot{\boldsymbol{u}}. \tag{3.15}$$

Letting

$$c_1=\sqrt{\frac{\lambda+2\mu}{\rho_0}},\qquad c_2=\sqrt{\frac{\mu}{\rho_0}}\qquad (\lambda,\mu,\rho_0>0),\tag{3.16}$$

and using the Laplacian identity

$$\nabla^2\boldsymbol{u}=\nabla(\nabla.\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u}),$$

(3.14) can be rewritten as

$$c_1^2\nabla(\nabla.\boldsymbol{u})-c_2^2\nabla\times(\nabla\times\ \boldsymbol{u})+\frac{m}{\rho_0}\nabla\theta+\boldsymbol{b}=\ddot{\boldsymbol{u}}\cdot\tag{3.17}$$

We note that

$$c_1^2-c_2^2=\frac{\lambda+\mu}{\rho_0},\qquad \frac{c_1}{c_2}=\sqrt{\frac{\lambda+2\mu}{\mu}},\tag{3.18}$$

and remark that $c_1 > c_2$.
Apply the div operator to (3.17) and recall the identity div (curl) =0; we find the following form

$$\square_1\nabla.\boldsymbol{u}=-\frac{1}{\rho_0}\left(m\nabla^2\theta+\rho_0\nabla.\boldsymbol{b}\right)\tag{3.19}$$

where the wave operator $\square_\alpha$ defined by

$\square_\alpha f=c_\alpha^2\nabla^2 f-\ddot{f}\qquad (\alpha=1,2)$ has been introduced.
Similarly, applying the curl operator to (3.14) and using the identity curl (grad) =0 yields

$$\square_2\nabla\times\boldsymbol{u}=-\nabla\times\boldsymbol{b}.\tag{3.20}$$

The temperature equation is found by applying the wave operator $\square_1$ to the heat equation (3.15) along with (3.19)

$$kc_1^2\nabla^2\nabla^2\theta-\left(\frac{m^2\theta_0}{\rho_0}+\rho_0c_0c_1^2\right)\nabla^2\dot{\theta}-k\nabla^2\ddot{\theta}+\rho_0c_0\dddot{\theta}-m\theta_0\nabla.\dot{\boldsymbol{b}}+\rho_0\square_1 r=0.\tag{3.21}$$

Let us introduce the following notations

$$\sigma=\left(\frac{m^2\theta_0}{\rho_0}+\rho_0c_0c_1^2\right),\qquad \kappa_1=\frac{kc_1^2}{\sigma},\qquad \kappa_2=\frac{k}{\rho_0c_0},\tag{3.22}$$

and define the heat operator $\bigcirc_\alpha$ by

$$\bigcirc_\alpha f=\kappa_\alpha\nabla^2 f-\dot{f}.\qquad (\alpha=1,2).\tag{3.23}$$

Assuming no heat supply and a constant body force field, the temperature equation (3.21) takes the form

$$\mathfrak{M}[\theta]=0\tag{3.24}$$

where the operator $\mathfrak{M}$ has been defined by

$$\mathfrak{M}=\left(\sigma\nabla^2\bigcirc_1-\rho_0c_0\frac{\partial^2}{\partial t^2}\bigcirc_2\right).\tag{3.25}$$

## 4. Solution of Thermoelastic Dynamic Equations

We now seek the solution of the system of thermoelastic dynamic equations. We are guided by the Papkovitch (1932)-Neuber (1934) approach originally developed to solve the equations of linear isothermal elasticity.The solution is based on Helmholtz's decomposition theorem into the gradient of a scalar potential (dilatation) plus the curl of a vector potential (rotation).
We start from the displacement-temperature dynamic equation (3.14) which can be recast as

$$\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + m\left(\frac{1}{\mu}\nabla\theta\right) + \frac{\rho_0}{\mu}\boldsymbol{b} = \frac{\rho_0}{\mu}\ddot{\boldsymbol{u}}\ , \qquad \frac{\lambda+\mu}{\mu} = \frac{1}{1-2\nu} \tag{4.1}$$

where $\nu$ is the usual Poisson's ratio.
Substituting in (4.1) the Helmholtz's decomposition for the displacement

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{\psi} \tag{4.2}$$

where $\boldsymbol{u}$ is the displacement field, $\phi$ is a scalar potential and $\boldsymbol{\psi}$ is a vector potential with zero divergence, we find that

$$\nabla^2\left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right) + m\left(\frac{1}{\mu}\nabla\theta\right) = -\frac{1}{\mu}(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}}),\ (\boldsymbol{F} = \rho_o\boldsymbol{b}). \tag{4.3}$$

Applying the div operator to (4.3)

$$\nabla^2\left(\nabla.\boldsymbol{u} + \frac{1}{1-2\nu}\nabla^2\phi\right) + m\left(\frac{1}{\mu}\nabla^2\theta\right) = -\frac{1}{\mu}\nabla.(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}}) \tag{4.4}$$

and using (4.2) again, yields the expression

$$\nabla^2\left(\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right) + m\left(\frac{1}{\mu}\nabla^2\theta\right) = -\frac{1}{\mu}\nabla.(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}}). \tag{4.5}$$

We see from (4.5) that using Helmholtz's decomposition resulted in the uncoupling of the deviatoric response (curl $\boldsymbol{u}$) from the volumetric response (div $\boldsymbol{u}$) but the temperature and the deformation fields are still coupled.
A remedy to this problem is to apply the $\mathfrak{M}$ operator to (4.5) and use (3.24) i.e.

$$\nabla^2\mathfrak{M}\left[\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right] = -\frac{1}{\mu}\mathfrak{M}\left[\nabla.(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]. \tag{4.6}$$

Define a vector quantity $h$

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right). \tag{4.7}$$

Use (4.2) and the identity div (curl) =0, then

$$\nabla.\boldsymbol{h} = \frac{2(1-\nu)}{1-2\nu}\nabla^2\phi \tag{4.8}$$

and by (4.6)

$$\nabla^2\mathfrak{M}[\nabla.\boldsymbol{h}] = \nabla^2\mathfrak{M}\left[\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right] = -\frac{1}{\mu}\mathfrak{M}\left[\nabla.(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]. \tag{4.9}$$

Use the identity

$$\nabla^2(\boldsymbol{R}.\boldsymbol{h}) = \boldsymbol{R}.\nabla^2\boldsymbol{h} + 2(\nabla.\boldsymbol{h}) \tag{4.10}$$

where $\boldsymbol{R}$ is the vector position of a field point referred to the origin, then

$$(\nabla.\boldsymbol{h}) = \frac{1}{2}\left[\nabla^2(\boldsymbol{R}.\boldsymbol{h}) - \boldsymbol{R}.\nabla^2\boldsymbol{h}\right] \tag{4.11}$$

and

$$\nabla^2\mathfrak{M}[\nabla.\boldsymbol{h}] = \frac{1}{2}\mathfrak{M}\left[\nabla^4(\boldsymbol{R}.\boldsymbol{h}) - \boldsymbol{R}.\nabla^4\boldsymbol{h}\right]. \tag{4.12}$$

Comparing (4.9) and (4.12), we find the expression

$$\nabla^4\mathfrak{M}\left[\frac{2(1-\nu)}{1-2\nu}\phi - \frac{1}{2}\boldsymbol{R}.\boldsymbol{h}\right] = -\frac{1}{2}\mathfrak{M}\left[\boldsymbol{R}.\nabla^4\boldsymbol{h}\right]. \tag{4.13}$$

Let $h$ a scalar quantity be defined by

$$h = \left(\frac{2(1-\nu)}{1-2\nu}\phi - \boldsymbol{R}.\boldsymbol{h}\right), \tag{4.14}$$

then by (4.6) (see proof in Appendix 2)

$$\nabla^4\mathfrak{M}[h] = -\frac{1}{2}\mathfrak{M}\left[\boldsymbol{R}.\nabla^4\boldsymbol{h}\right] = \frac{1}{2}\mathfrak{M}\left[\boldsymbol{R}.\frac{\nabla^2(\boldsymbol{F} - \rho_o\ddot{\boldsymbol{u}})}{\mu}\right]. \tag{4.15}$$

Rearranging (4.14), the scalar potential is

$$\phi = \frac{1-2\nu}{2(1-\nu)}\left(h + \frac{1}{2}\boldsymbol{R.h}\right). \tag{4.16}$$

Then

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right) = \boldsymbol{u} + \frac{1}{2(1-\nu)}\nabla\left[h + \frac{1}{2}\boldsymbol{R.h}\right] \tag{4.17}$$

and

$$2\mu\boldsymbol{u} = 2\mu\boldsymbol{h} - \nabla\left[\frac{\mu}{(1-\nu)}\boldsymbol{h} + \frac{\mu}{2(1-\nu)}\boldsymbol{R.h}\right]. \tag{4.18}$$

or

$$2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R.A}\right], \qquad \boldsymbol{A} = 2\mu\boldsymbol{h}, \qquad B = \frac{\mu}{(1-\nu)}h \tag{4.19}$$

We can apply the divergence operator to (4.9) and use $\nabla^2\boldsymbol{A} = 2\mu\nabla^2\boldsymbol{h}$ , $\ddot{\boldsymbol{A}} = 2\mu\ddot{\boldsymbol{h}} = 2\mu\ddot{\boldsymbol{u}}$, with the definition on the wave operator then $\boldsymbol{A}$ and $B$ are related and satisfy the following equations

$$\begin{cases} \mathfrak{M}\left[\Box_2\nabla^2\boldsymbol{A}\right] = -2\mathfrak{M}\left[c_2^2\nabla^2\boldsymbol{F}\right] \\ \mathfrak{M}\left[\nabla^4 B\right] = \mathfrak{M}\dfrac{\boldsymbol{R}.\nabla^4\boldsymbol{A}}{4(1-\nu)}. \end{cases} \tag{4.20}$$

Substitute (4.19) in Navier equations (4.1) and apply the Laplacian operator

$$\frac{1}{2\mu}\nabla^4\boldsymbol{A} - \frac{1}{2\mu}\nabla^4\left\{\nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R.A}\right]\right\} + \frac{1}{1-2\nu}\nabla^2(\nabla.\nabla\boldsymbol{u}) + \frac{m}{\mu}\nabla^2(\nabla\theta) + \frac{\nabla^2\boldsymbol{F}}{\mu} = \rho_0\nabla^2\ddot{\boldsymbol{u}}. \tag{4.21}$$

Applying the $\mathfrak{M}$ operator to (4.21) and using $(4.20)_1$ yields the following expression

$$\mathfrak{M}\left[\nabla^4\boldsymbol{A}\right] = \mathfrak{M}\left[-2\nabla^2\boldsymbol{F} + \frac{1}{c_2^2}\nabla^2\ddot{\boldsymbol{A}}\right]. \tag{4.22}$$

Use the form of $\boldsymbol{u}$ (4.19) in (4.21),

$$-\frac{1}{2\mu}\mathfrak{M}\nabla^4\{\boldsymbol{A} - 2\mu\boldsymbol{u}\} + \mathfrak{M}\frac{1}{1-2\nu}\nabla^2(\nabla(\nabla.\boldsymbol{u})) + \frac{m}{\mu}\mathfrak{M}\nabla^2(\nabla\theta) = \mathfrak{M}[\rho_0\ddot{\boldsymbol{u}}] \tag{4.23}$$

Uisng (4.22), this becomes

$$-\frac{1}{2\mu}\mathfrak{M}\left[-2\nabla^2\boldsymbol{F} + \frac{1}{c_2^2}\nabla^2\ddot{\boldsymbol{A}}\right] + \mathfrak{M}\nabla^2\left[\nabla^2\boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{m}{\mu}(\nabla\theta)\right] = \mathfrak{M}[\rho_0\ddot{\boldsymbol{u}}] \tag{4.24}$$

then by Navier equations (4.1)

$$-\frac{1}{2\mu}\mathfrak{M}\left[\Box_2\nabla^2\boldsymbol{A}\right] + \mathfrak{M}c_2^2\nabla^2\left[\rho_0\ddot{\boldsymbol{u}} - \frac{\boldsymbol{F}}{\mu}\right] = \mathfrak{M}c_2^2[\rho_0\ddot{\boldsymbol{u}}] \tag{4.25}$$

or equivalently

$$\mathfrak{M}\left[\Box_2\nabla^2\boldsymbol{A}\right] = -2\mathfrak{M}c_2^2\nabla^2\boldsymbol{F},$$

i.e. the solution of the thermoelastic dynamic problem for homogeneous isotropic linear bodies reduces to solving a sixth order vector (wave) equation for the displacement field and a fourth order scalar equation for the temperature field (3.24). If we let $\theta=0$ in (3.24), the solution for $\boldsymbol{u}$ is identical to that of the isothermal elastodynamic case in Chandrasekharaiah and Cowin (1990) obtained by generalizing the Navier equations of equilibrium. If we let $\ddot{\boldsymbol{u}}=\boldsymbol{0}$ (via $\ddot{\boldsymbol{A}}$) in (4.23), the solution for $\boldsymbol{u}$ becomes that of the static thermoelastic case in Chandrasekharaiah and Cowin (1989) and obtained from the unified solutions of thermoelasticity and porosity. Letting both $\theta=0$ and $\ddot{\boldsymbol{u}}=\boldsymbol{0}$ (via $\ddot{\boldsymbol{A}}$) in (4.23) yields the Papkovitch-Neuber's solution of the displacement field for the isothermal elastostatic case (Mindlin 1936-also in Sadd 2009).The relevant subcases are shown in Table1 and details of the derivations are in the Appendix.

Table 1: Thermoelastic dynamic solution and subcases

| | Elastic | Thermoelastic |
|---|---|---|
| Statics | $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$<br>where $\boldsymbol{A}$ and $B$ are related and satisfy<br>$\begin{cases}\nabla^2\boldsymbol{A} = -2\boldsymbol{F}\\ \nabla^2 B = \dfrac{\boldsymbol{R}.\boldsymbol{F}}{2(1-\nu)} = -\dfrac{\boldsymbol{R}.\nabla^2\boldsymbol{A}}{4(1-\nu)}\end{cases}$ | $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$<br>where $\boldsymbol{A}$ and $B$ are related and satisfy<br>$\begin{cases}\bigcirc_1\left[\nabla^4\boldsymbol{A}\right] = -2\bigcirc_1\left[\nabla^2\boldsymbol{F}\right]\\ \bigcirc_1\left[\nabla^4 B\right] = \bigcirc_1\left[\dfrac{\boldsymbol{R}.\nabla^2\boldsymbol{F}}{2(1-\nu)}\right] = -\bigcirc_1\left[\dfrac{\boldsymbol{R}.\nabla^4\boldsymbol{A}}{4(1-\nu)}\right]\end{cases}$<br>and the temperature satisfies<br>$\bigcirc_1\left[\nabla^2\theta\right] = 0.$<br>where<br>$\bigcirc_\alpha f = \kappa_\alpha\nabla^2 f - \dot{f}\ ,\ \alpha=1,2$<br>$\kappa_1 = \dfrac{kc_1^2}{\sigma},\qquad \kappa_2 = \dfrac{k}{\rho_0 c_0}$ |
| Dynamics | $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$<br>where $\boldsymbol{A}$ and $B$ are related and satisfy<br>$\begin{cases}\Box_2\boldsymbol{A} = -2c_2{}^2\boldsymbol{F}\\ \nabla^2 B = -\dfrac{\boldsymbol{R}.\nabla^2\boldsymbol{A}}{4(1-\nu)}\end{cases}$<br>where<br>$\Box_\alpha f = c_\alpha^2\nabla^2 f - \ddot{f},\ \alpha=1,2$<br>$c_1{}^2 = \dfrac{\lambda+2\mu}{\rho_0},\qquad c_2{}^2 = \dfrac{\mu}{\rho_0}$ | $2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$<br>where $\boldsymbol{A}$ and $B$ are related and satisfy<br>$\begin{cases}\mathfrak{M}\left[\Box_2\nabla^2\boldsymbol{A}\right] = -2\mathfrak{M}\,c_2^2\left[\nabla^2\boldsymbol{F}\right]\\ \mathfrak{M}\left[\nabla^4 B\right] = -\mathfrak{M}\left[\dfrac{\boldsymbol{R}.\nabla^4\boldsymbol{A}}{4(1-\nu)}\right]\end{cases}$<br>and the temperature satisfies<br>$\mathfrak{M}\left[\theta\right] = 0.$<br>where<br>$\mathfrak{M} = \left(\sigma\nabla^2\bigcirc_1 - \rho_0 c_0\dfrac{\partial^2}{\partial t^2}\bigcirc_2\right),\ \sigma = \left(\dfrac{m^2\theta_0}{\rho_0} + \rho_0 c_0 c_1^2\right)$<br>$\bigcirc_\alpha f = \kappa_\alpha\nabla^2 f - \dot{f}\ ,\ \alpha=1,2$<br>$\kappa_1 = \dfrac{kc_1^2}{\sigma},\qquad \kappa_2 = \dfrac{k}{\rho_0 c_0}$<br>$\Box_\alpha f = c_\alpha^2\nabla^2 f - \ddot{f},\ \alpha=1,2$<br>$c_1{}^2 = \dfrac{\lambda+2\mu}{\rho_0},\qquad c_2{}^2 = \dfrac{\mu}{\rho_0}$ |

**Appendix 1**

**Case 1: Isothermal Elastic Dynamic Equations**

In the isothermal case, the counterparts of (4.1) and (4.3) are

$$\begin{aligned} &\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{\boldsymbol{F}}{\mu} = \frac{\rho_0}{\mu}\ddot{\boldsymbol{u}}, \ (\boldsymbol{F} = \rho_o \boldsymbol{b}) \\ &\nabla^2\left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right) = -\frac{1}{\mu}(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}}). \end{aligned} \tag{1}$$

Define a vector quantity $\boldsymbol{h}$ as

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right)$$

then

$$\nabla^2 \boldsymbol{h} = -\frac{1}{\mu}(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}}), \qquad \nabla.\boldsymbol{h} = \frac{2(1-\nu)}{1-2\nu}\nabla^2\phi.$$

Use identity

$$\nabla^2(\boldsymbol{R.h}) = \boldsymbol{R}.\nabla^2\boldsymbol{h} + 2(\nabla.\boldsymbol{h})$$

where $\boldsymbol{R}$ is the position vector, then

$$(\nabla.\boldsymbol{h}) = \frac{1}{2}\left[\nabla^2(\boldsymbol{R.h}) - \boldsymbol{R}.\nabla^2\boldsymbol{h}\right]$$

and

$$\nabla^2\left[\frac{2(1-\nu)}{1-2\nu}\phi - \frac{1}{2}\boldsymbol{R.h}\right] = -\frac{1}{2}\left[\boldsymbol{R}.\nabla^2\boldsymbol{h}\right].$$

Let $h$ a scalar quantity be defined by

$$h = \left(\frac{2(1-\nu)}{1-2\nu}\phi - \boldsymbol{R.h}\right),$$

then

$$\nabla^2[h] = -\frac{1}{2}\left[\boldsymbol{R}.\nabla^2\boldsymbol{h}\right] = \frac{1}{2}\left(\boldsymbol{R}.\frac{\boldsymbol{F} - \rho_o\ddot{\boldsymbol{u}}}{\mu}\right)$$

and

$$\phi = \frac{1-2\nu}{2(1-\nu)}\left(h + \frac{1}{2}\boldsymbol{R.h}\right).$$

Thus,

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right) = \boldsymbol{u} + \frac{1}{2(1-\nu)}\nabla\left[h + \frac{1}{2}\boldsymbol{R.h}\right]$$

and

$$2\mu\boldsymbol{u} = 2\mu\boldsymbol{h} - \nabla\left[\frac{\mu}{(1-\nu)}h + \frac{\mu}{2(1-\nu)}\boldsymbol{R.h}\right].$$

Letting

$$\boldsymbol{A} = 2\mu\boldsymbol{h}, \qquad B = \frac{\mu}{(1-\nu)}h,$$

the displacement field can be written as

$$2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{1}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right].$$

Use the expression of $\boldsymbol{A}$ in terms of $\boldsymbol{h}$

$$\boldsymbol{A} = 2\mu\boldsymbol{h} = 2\mu\left\{\boldsymbol{u} + \frac{1}{2(1-\nu)}\nabla\left[h + \frac{1}{2}\boldsymbol{R}.\boldsymbol{h}\right]\right\}$$

then

$$\ddot{\boldsymbol{A}} = 2\mu\ddot{\boldsymbol{h}} = 2\mu\ddot{\boldsymbol{u}}, \quad \nabla^2\boldsymbol{A} = 2\mu\nabla^2\boldsymbol{h} = -2\mu\left(\frac{\boldsymbol{F} - \rho_o\ddot{\boldsymbol{u}}}{\mu}\right) = -2\boldsymbol{F} + \frac{1}{c_2{}^2}\ddot{\boldsymbol{A}}, \quad \nabla^2 B = \frac{\mu}{1-\nu}\nabla^2 h = \frac{\mu}{1-\nu}\left(\boldsymbol{R}.\frac{\boldsymbol{F} - \rho_o\ddot{\boldsymbol{u}}}{\mu}\right)$$

and $\boldsymbol{A}$ and $B$ are related and satisfy the following equations

$$\begin{cases} \square_2\boldsymbol{A} = -2c_2{}^2\boldsymbol{F} \\ \nabla^2 B = -\dfrac{\boldsymbol{R}.\nabla^2\boldsymbol{A}}{4(1-\nu)}. \end{cases}$$

If we substitute the form of $\boldsymbol{u}$ in the Navier equations, we find

$$\frac{1}{2\mu}\nabla^2\boldsymbol{A} - \frac{1}{2\mu}\nabla^2\left\{\nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]\right\} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{\boldsymbol{F}}{\mu} = \rho_0\ddot{\boldsymbol{u}},$$

then

$$\frac{1}{2\mu}\left[-2\boldsymbol{F} + \frac{1}{c_2^2}\ddot{\boldsymbol{A}}\right] - \frac{1}{2\mu}\nabla^2\{\boldsymbol{A} - 2\mu\boldsymbol{u}\} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{\boldsymbol{F}}{\mu} = \rho_0\ddot{\boldsymbol{u}}$$

$$\frac{1}{2\mu c_2^2}\ddot{\boldsymbol{A}} - \frac{1}{2\mu}\nabla^2\boldsymbol{A} + \left[\nabla^2\boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u})\right] = \rho_0\ddot{\boldsymbol{u}}$$

By Navier equations, this is

$$\frac{1}{2\mu c_2^2}\ddot{\boldsymbol{A}} - \frac{1}{2\mu}\nabla^2\boldsymbol{A} + \left[\rho_0\ddot{\boldsymbol{u}} - \frac{\boldsymbol{F}}{\mu}\right] = \rho_0\ddot{\boldsymbol{u}}.$$

or equivalently,

$$\square_2\boldsymbol{A} = -2c_2^2\boldsymbol{F},$$

i.e. the solution of the isothermal dynamic problem reduces to solving a second order (wave) vector equation.

**Case 2: Static Thermoelastic Equations**

In the static case, the counterparts of (4.1) and (4.5) are

$$\nabla^2\boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{m}{\mu}\nabla\theta + \frac{\boldsymbol{F}}{\mu} = \boldsymbol{0}, \quad (\boldsymbol{F} = \rho_0\boldsymbol{b}) \tag{2}$$

$$\nabla^2\left(\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right) + m\left(\frac{1}{\mu}\nabla^2\theta\right) = -\frac{1}{\mu}\nabla.\boldsymbol{F} \tag{3}$$

and the heat equation is still valid

$$\rho_0 c_0\dot{\theta} = \rho_0 r + k\nabla^2\theta + \theta_0 m(\nabla.\dot{\boldsymbol{u}}), \tag{4}$$

i.e. Helmholtz's decomposition uncouples the deviatoric response (curl ***u)*** from the volumetric response (div ***u***) in (3) but the temperature and the deformation fields in (4) remain coupled.
In the static case, the counterpart of (4.1) (see 3.19) is

$$\nabla^2(\nabla.\boldsymbol{u}) = -\frac{1}{\rho_0 c_1^2}\left(m\nabla^2\theta + \nabla.\boldsymbol{F}\right)$$

and its time derivative is

$$\nabla^2(\nabla.\dot{\boldsymbol{u}}) = -\frac{1}{\rho_0 c_1^2}\left(m\nabla^2\dot{\theta} + \nabla.\dot{\boldsymbol{F}}\right).$$

Assume no heat supply and a constant body force field and substitute the expression in (4) for $(\nabla.\,\dot{\boldsymbol{u}})$

$$\nabla^2\left(\frac{\rho_0 c_0\dot{\theta} - k\nabla^2\theta}{\theta_0 m}\right) = -\frac{1}{\rho_0 c_1^2}\left(m\nabla^2\dot{\theta}\right).$$

This can be recast as

$$\kappa_1\nabla^4\theta - \nabla^2\dot{\theta} = 0, \qquad \kappa_1 = \frac{kc_1^2}{\sigma}, \qquad \sigma = \left(\frac{m^2\theta_0}{\rho_0} + \rho_0 c_0 c_1^2\right)$$

or using the heat operator,

$$\bigcirc_1\left[\nabla^2\theta\right] = 0. \tag{5}$$

Apply the heat operator to (3) and use (5)

$$\bigcirc_1\nabla^2\left(\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right) = \bigcirc_1\frac{1}{\mu}(\nabla.\boldsymbol{F}),$$

i.e. the volume change and temperature fields have been uncoupled but at the expense of involving higher derivatives.
Define a vector quantity ***h*** by

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right),$$

then using Helmholtz's decomposition

$$\nabla.\boldsymbol{h} = \left(\nabla.\boldsymbol{u} + \frac{1}{1-2\nu}\nabla.\nabla\phi\right) = \frac{2(1-\nu)}{1-2\nu}\nabla^2\phi$$

and

$$\nabla^2\bigcirc_1(\nabla.\boldsymbol{h}) = \nabla^2\left(\bigcirc_1\frac{2(1-\nu)}{1-2\nu}\nabla^2\phi\right) = -\bigcirc_1\frac{(\nabla.\boldsymbol{F})}{\mu}$$

$$\bigcirc_1\nabla^4\boldsymbol{h} = -\bigcirc_1\frac{\left(\nabla^2\boldsymbol{F}\right)}{\mu}. \tag{6}$$

Recall the identity

$$\nabla^2(\boldsymbol{R}.\boldsymbol{h}) = \boldsymbol{R}.\nabla^2\boldsymbol{h} + 2(\nabla.\boldsymbol{h}),$$

then

$$\nabla^2\bigcirc_1[\nabla.\boldsymbol{h}] = \frac{1}{2}\bigcirc_1\left[\nabla^4(\boldsymbol{R}.\boldsymbol{h}) - \boldsymbol{R}.\nabla^4\boldsymbol{h}\right]$$

and

$$\nabla^4\bigcirc_1\left(\frac{2(1-\nu)}{1-2\nu}\phi - \boldsymbol{R}.\boldsymbol{h}\right) = -\frac{1}{2}\bigcirc_1\left(\boldsymbol{R}.\nabla^4\boldsymbol{h}\right).$$

Let a scalar quantity $h$ be defined as

$$h = \left(\frac{2(1-\nu)}{1-2\nu}\phi - \boldsymbol{R}.\boldsymbol{h}\right),$$

then

$$\nabla^4 \bigcirc_1 h = -\frac{1}{2}\bigcirc_1\left(\boldsymbol{R}.\nabla^4 \boldsymbol{h}\right) = \frac{1}{2}\bigcirc_1\left(\boldsymbol{R}.\frac{\nabla^2 \boldsymbol{F}}{\mu}\right)$$

and

$$\boldsymbol{h} = \left(\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\phi\right) = \boldsymbol{u} + \frac{1}{2(1-\nu)}\nabla\left(h + \boldsymbol{R}.\boldsymbol{h}\right).$$

Upon rearranging

$$2\mu\boldsymbol{u} = 2\mu\boldsymbol{h} - \nabla\left[\frac{\mu}{(1-\nu)}h + \frac{\mu}{2(1-\nu)}\boldsymbol{R}.\boldsymbol{h}\right]$$

Letting

$$\boldsymbol{A} = 2\mu\boldsymbol{h}, \qquad B = \frac{\mu}{(1-\nu)}h,$$

the displacement field can be expressed as

$$2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{1}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$$

Then

$$\bigcirc_1\nabla^4\boldsymbol{A} = \bigcirc_1\left(2\mu\nabla^4\boldsymbol{h}\right)$$

$$\bigcirc_1\nabla^4 B = \frac{\mu}{1-\nu}\bigcirc_1\nabla^4 h = -\frac{\mu}{2(1-\nu)}\bigcirc_1\left(\boldsymbol{R}.\nabla^4\boldsymbol{h}\right) = \frac{\mu}{2(1-\nu)}\left(\boldsymbol{R}.\frac{\nabla^2\boldsymbol{F}}{\mu}\right)$$

so $\boldsymbol{A}$ and $B$ are related and satisfy the following

$$\begin{cases} \bigcirc_1\nabla^4\boldsymbol{A} = -2\bigcirc_1\nabla^2\boldsymbol{F} \\ \bigcirc_1\nabla^4 B = -\bigcirc_1\dfrac{\boldsymbol{R}.\nabla^4\boldsymbol{A}}{4(1-\nu)}, \end{cases}$$

while the temperature field satisfies

$$\bigcirc_1\left[\nabla^2\theta\right] = 0.$$

Substitute the form of $\boldsymbol{u}$ in Navier equations (2)

$$\frac{1}{2\mu}\nabla^2\boldsymbol{A} - \frac{1}{2\mu}\nabla^2\left\{\nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]\right\} + \frac{1}{1-2\nu}\nabla\left(\nabla.\boldsymbol{u}\right) + \frac{m}{\mu}\nabla\theta + \frac{\boldsymbol{F}}{\mu} = \boldsymbol{0},$$

and apply successively the Laplacian and the heat operator

$$\frac{1}{2\mu}\bigcirc_1\nabla^4\boldsymbol{A} - \frac{1}{2\mu}\bigcirc_1\nabla^4\left\{\nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]\right\} + \frac{1}{1-2\nu}\bigcirc_1\nabla^4\boldsymbol{u} + \frac{m}{\mu}\bigcirc_1\nabla^2(\nabla\theta) + \bigcirc_1\frac{\nabla^2\boldsymbol{F}}{\mu} = \boldsymbol{0}.$$

then

$$-\frac{1}{2\mu}\bigcirc_1\nabla^4\left\{\boldsymbol{A} - 2\mu\boldsymbol{u}\right\} + \bigcirc_1\frac{1}{1-2\nu}\nabla^4\boldsymbol{u} + \frac{m}{\mu}\bigcirc_1\nabla^2(\nabla\theta) = 0$$

$$-\frac{1}{2\mu}\bigcirc_1\nabla^4\boldsymbol{A} + \bigcirc_1\nabla^2\left[\nabla^2\boldsymbol{u} + \frac{1}{1-2\nu}\nabla\left(\nabla.\boldsymbol{u}\right) + \frac{m}{\mu}(\nabla\theta)\right] = 0$$

Appeling to Navier equations for the term in the paranthesis, we get

$$\bigcirc_1\nabla^4\boldsymbol{A} = -2\bigcirc_1\nabla^2\boldsymbol{F},$$

i.e. the solution of the thermoelastic static problem reduces to solving a sixth order vector equation for the displacement field and a fourth order scalar (heat) equation for the temperature field (equation 5).

**Case 3: Isothermal Elastic Static Equations**

For the isothermal case, the counterpart of (4.1) is

$$\nabla^2 \boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{\boldsymbol{F}}{\mu} = \boldsymbol{0}$$

and following the same procedure as above, the form of $\boldsymbol{u}$ is found to be

$$2\mu\boldsymbol{u} = \boldsymbol{A} - \nabla\left[B + \frac{1}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]$$

where $\boldsymbol{A}$ and $B$ are related and satisfy

$$\begin{cases} \nabla^2\boldsymbol{A} = -2\boldsymbol{F} \\ \nabla^2 B = \dfrac{\boldsymbol{R}.\boldsymbol{F}}{2(1-\nu)}. \end{cases}$$

Substitute the form of $\boldsymbol{u}$ in Navier equations

$$\frac{1}{2\mu}\nabla^2\boldsymbol{A} - \frac{1}{2\mu}\nabla^2\left\{\nabla\left[B + \frac{\mu}{4(1-\nu)}\boldsymbol{R}.\boldsymbol{A}\right]\right\} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) + \frac{\boldsymbol{F}}{\mu} = \boldsymbol{0}$$

then

$$-\frac{1}{2\mu}\nabla^2\{\boldsymbol{A} - 2\mu\boldsymbol{u}\} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u}) = \boldsymbol{0}$$

$$-\frac{1}{2\mu}\nabla^2\boldsymbol{A} + \left[\nabla^2\boldsymbol{u} + \frac{1}{1-2\nu}\nabla(\nabla.\boldsymbol{u})\right] = \boldsymbol{0}$$

$$-\frac{1}{2\mu}\nabla^2\boldsymbol{A} + \left[-\frac{\boldsymbol{F}}{\mu}\right] = \boldsymbol{0} \textit{ by Navier Equations}$$

$$\nabla^2\boldsymbol{A} = -2\boldsymbol{F}$$

i.e. Navier equations reduce to solving a simple second order vector equation (Mindlin 1936-also in Sadd 2009).

**Appendix 2**

We show that

$$\mathfrak{M}\left[\nabla^4\boldsymbol{h}\right] = -\mathfrak{M}\left[\frac{\nabla^2(\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})}{\mu}\right].$$

Start with

$$\nabla^2\mathfrak{M}[\nabla.\boldsymbol{h}] = -\frac{1}{\mu}\mathfrak{M}\left[\nabla.\ (\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]$$

and apply the del operator to both sides

$$\nabla\left\{\nabla^2\mathfrak{M}[\nabla.\boldsymbol{h}]\right\} = -\nabla\left\{\frac{1}{\mu}\mathfrak{M}\left[\nabla.\ (\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]\right\}$$

$$\mathfrak{M}\left\{\nabla\left\{\nabla^2[\nabla.\boldsymbol{h}]\right\}\right\} = -\mathfrak{M}\left\{\nabla\left\{\frac{1}{\mu}\left[\nabla.\ (\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]\right\}\right\}$$

$$\mathfrak{M}\left\{\nabla^2\left\{\nabla[\nabla.\boldsymbol{h}]\right\}\right\} = -\mathfrak{M}\left\{\nabla\left\{\frac{1}{\mu}\left[\nabla.\ (\boldsymbol{F} - \rho_0\ddot{\boldsymbol{u}})\right]\right\}\right\}.$$

Using the identity for the Laplacian

$$\mathfrak{M}\left\{\nabla^2\left\{\nabla\times(\nabla\times\boldsymbol{h}) + \nabla^2\boldsymbol{h}\right\}\right\} = -\mathfrak{M}\left\{\frac{1}{\mu}\left[\nabla\times(\nabla\times\boldsymbol{F}) + \nabla^2\boldsymbol{F} - \nabla\times(\nabla\times\rho_0\ddot{\boldsymbol{u}}) - \nabla^2(\rho_0\ddot{\boldsymbol{u}})\right]\right\}$$

Use

$$\boldsymbol{h} = \left( \boldsymbol{u} + \frac{1}{1-2\nu} \nabla \phi \right),$$

then

$$\nabla \times \left( \nabla \times \boldsymbol{h} \right) = \nabla \times \left( \nabla \times \boldsymbol{u} \right) + \nabla \times \left( \nabla \times \frac{1}{1-2\nu} \nabla \phi \right).$$

If we assume that the displacement is irrotational, i.e. curl $\boldsymbol{u}=\boldsymbol{0}$ and use the identity curl(grad)=0, then

$$\nabla \times \left( \nabla \times \boldsymbol{h} \right) = \boldsymbol{0}$$

and

$$\nabla \left\{ \nabla^2 \mathfrak{M} \left[ \nabla . \boldsymbol{h} \right] \right\} = -\nabla \left\{ \frac{1}{\mu} \mathfrak{M} \left[ \nabla . \left( \boldsymbol{F} - \rho_0 \ddot{\boldsymbol{u}} \right) \right] \right\}$$

is verified.

If the requirement of irrotationality is too strict, we can use the Helmholtz's decomposition of $\boldsymbol{u}$, where $\psi$ is arbitrary and commonly chosen with zero divergence (Sadd 2009, page 266)

$$\left( \nabla \times \boldsymbol{u} \right) = \left( \nabla \times (\nabla \phi + \nabla \times \boldsymbol{\psi}) \right) = \left( \nabla \times \nabla \phi \right) + \nabla \times \left( \nabla \times \boldsymbol{\psi} \right) = \left( \nabla . \boldsymbol{\psi} \right) \nabla - \left( \nabla . \nabla \right) \boldsymbol{\psi},$$

then

$$\left( \nabla \times \boldsymbol{u} \right) = \left( \nabla . \boldsymbol{\psi} \right) \nabla - \left( \nabla . \nabla \right) \boldsymbol{\psi} = \boldsymbol{0} - \nabla^2 \boldsymbol{\psi}.$$

If we assume

$$\nabla^2 \boldsymbol{\psi} = \boldsymbol{0}, \qquad \nabla \times \boldsymbol{F} = \boldsymbol{0},$$

then

$$\left( \nabla \times \boldsymbol{u} \right) = \boldsymbol{0}, \qquad \mathfrak{M} \left\{ \nabla^2 \left\{ \nabla^2 \boldsymbol{h} \right\} \right\} = \mathfrak{M} \left\{ \nabla^4 \boldsymbol{h} \right\} = -\mathfrak{M} \left\{ \frac{\nabla^2 \left( \boldsymbol{F} - \rho_o \ddot{\boldsymbol{u}} \right)}{\mu} \right\}.$$

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