CHAPTER 46

MACH-REFLECTION AS A DIFFRACTION PROBLEM

by

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and

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ABSTRACT

As under oblique wave approach water waves are reflected by a vertical wall, a wave branching effect (stem) develops normal to the reflecting wall. The waves progressing along the wall will steep up. The wave heights increase up to more than twice the incident wave height.

The study has pointed out that this effect, which is usually called MACH-REFLECTION, is not to be taken as an analogy to gasdynamics, but should be interpreted as a diffraction problem.

1. INTRODUCTION

Observations made at vertical walls in prototype as well as in experiments have shown that under certain assumptions gravity waves are apparently not reflected according to the laws of regular reflection. With small angles of wave approach ($\Theta_o < 45^o$; angle between wave crest and axis of incidence) the reflected wave will not leave the wall totally; a wave-stem normal to the wall will be developed (Fig. 1a,b). The height of the stem-wave will increase progressively along the length of the wall and will reach a value of more than double the height of the incoming wave (Fig. 1b).

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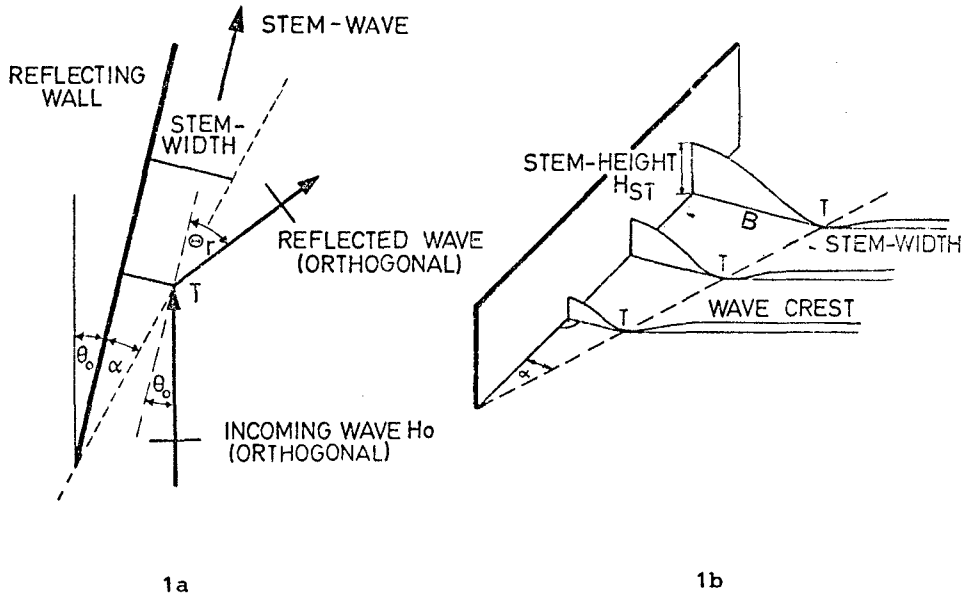

Fig. 1
Explanation of MACH-Reflection

PERROUD (7) and WIEGEL (13) explaining their investigations with solitary waves in analogy to the incoming flow of a compression shock against a re-entrant angle in qasdynamics named this effect MACH-reflection. Fig.1a shows the wave field in front of the wall using wave vectors. The wave strikes the structure with a small angle Θ_o; a stem-wave and a reflected wave are developed. The point T of intersection between the wave crests of the stem-wave and the incident wave moves on a straight line, cutting the wall with angle α.

Due to the fact that the physical problem is rather unclear and, moreover, the knowledge of the stem-height may be of considerable importance dimensioning structures against wave attack (see SFB 79 (9)) the aim of this investigation was to complete the experiments of NIELSEN and HAGER as follows:

by measurements of both stem-height and stem-width, as a function of the incoming wave-parameters and to give a theoretical explanation of this so-called MACH-effect.

The measurements were carried out at the FRANZIUS-INSTITUTE FOR HYDRAULICS AND COASTAL ENGINEERING, TECHNICAL UNIVERSITY OF HANNOVER using a three-dimensional wave basin from the SONDERFORSCHUNGSBEREICH 79 (Water research in Coastal Regions).

2. REVIEW OF PREVIOUS INVESTIGATIONS

Investigations concerning MACH-reflection of gravity waves have been conducted in the sea and in hydraulic models and with walls of different shape and slopes.

PERROUD (7), CHEN (1) and SIGURDSSON, WIEGEL (8) have studied the MACH-Effect with solitary waves at a vertical as well as at inclined walls. Bended forms also have been investigated. The measurements of NIELSEN (5) and HAGER (4) have been carried out using monochromatic waves and are restricted to straight and vertical walls.

Using a two-dimensional model, NIELSEN established the increase of the stem-height at the reflecting wall, the stem-width as a function of the angle of incidence and of the wave length. However, the number of data is too small to show the results in a functional form. For larger angles of incidence ($> 15^{\circ}$), NIELSEN assumed that the experimental results were influenced by the rather small distance to the boundary of the model at the end of the wall.

In addition to that the reflecting wall is in contact with a lateral boundary at one end which is not in accordance with the conception of a free-standing breakwater. Finally no theoretical explanation of the MACH-reflection is given by NIELSEN.

Contrarly to the investigations of NIELSEN with rather small wave heigths and small wave lengths, HAGER's experiments have been carried out under prototype conditions.

HAGER also investigated the increase of the stem-height within an extensive programme at the jetty of the Eckernförde harbour/Germany. However, the limited number of measurements and the scattering of the data only allow qualitative conclusions.

The fundamental results of the investigations of PERROUD,

author	MACH-stem-parameter				remarks concerning the experiments	remarks on theory
	stem - height	stem-width	stem-angle	behaviour of the reflecting wave		
NIELSEN (5) dimensions of the test basin a = 609,6 cm b = 111,7 cm h = 12,7 cm monochromatic wavee H (cm) / L (cm) / H/d / θ_o [°] 1.07 / 45.72 / 0.22 / 5 10 0.49 / 14.63 / 0.07 / 15 20	increase of the stem-height with increasing angle of incidence θ_o the stem height in independent of the wave length Remarks: The second MACH-stem effect is not observed in shoaling water in deep water the 2. MACH-stem increases with an increasing angle of incidence θ_o	decrease of the stem-width with increasing angle θ_o increase of the stem-width with increasing wave-height independent of the wave-height decrease of the stem-width with increasing water level Supposition: the stem-width increases linearly progressively along the length of the wall	the stem-angle decreases with an increasing angle θ_o	NIELSEN shows, that there is a reflecting wave with an angle $\theta_o < 20°$ es opposed to PERROUD	NIELSEN mentions that the stem-width at the end of the wall can be wrong because the model is too small the first part of the reflecting wall touches the wall of the test basin the experiments of the 2.MACH-stem have only been done qualitetively	there is no theoretical explenation
PERROUD (7) the same basin as NIELSEN solitary wave H (cm) / L (cm) / H/d / θ_o [°] 1.52 / / 0.38 / 5 22,5 45	the stem-height is a function of the angle of incidence and has two maxima $\theta_o = 22{,}5°$ $\theta_o = 45{,}0°$	the stem-width increases linearly along the length of the wall the stem-width of the soiltary wave is smaller than that of the periodic wavee	the stem-angle decreasss	no reflecting wave for an angle of incidence $\theta_o < 20°$ the wave height of the reflecting wave is emaller than that of the incoming wave		PERROUD describes the MACH-stem-effect for a solitary-wave with simple geometric connection
MAGER (4) Eckernförde harbour wind wave H (cm) / L (cm) / H/d / θ_o [°] 165 / 3800 / 0.05 / 5 30 10 25 15 45 20 50	the stem-height is a function of the angle of incidenceand increases progressively along the length of the wall				the dependence of the stem-height of the incoming wave on angle of incidence, changes	HAGER computes the stem-height along the reflecting wall. The equation system is eimiler to PERROUD'e

Fig. 2.1

NIELSEN and HAGER are summarized in Fig. 2.1. The stem-height increases with the increasing angle of incidence for solitary waves as well as for regular (monochromatic) waves. The stem-height and the stem-width increase in the direction of wave propagation. The stem-width decreases with the increasing angle of incidence. The decrease of the stem-angle is not mentioned very much by NIELSEN, but is readily understood from the decrease of the stem-width with the increasing angle of incidence and is similar to PERROUD's results. The observation of a second MACH-stem for periodical deep-water waves is remarkable.

Theoretical investigations about MACH-reflection of gravity waves have been conducted by PERROUD and HAGER. The analytical solution of PERROUD is connected to the problem of a solitary wave and may not be used in connection with periodical waves. The four unknown parameters - height of the stem-wave, height of the reflected wave, angle of reflection and angle of the stem - are determined by a four-equation-system, which can only be solved implicitely. Two equations of the system are found from geometrical considerations and the other two equations are deduced from the mass and energy-conversion conditions. The theoretical statement of HAGER is similar to PERROUD's and leads to the calculation of the stem-height only. This statement doesn't agree with the experimental results.

3. DIFFRACTION THEORY

From the previous chapter it may be seen that for monochromatic waves there is no theory to calculate the wave pattern, i.e. stem-height and stem-width, with sufficient accuracy. If we suppose that the MACH-reflection has to be interpretated as a diffraction-problem in the area of reflection as opposed to how it was formerly investigated, a new theoretical concept must be examined.

From the linear partial differential equation

$$\Delta \Phi = 0$$

and using a polar coordinate system and the well-known boundary conditions of the linear wave theory, we get the scalar wave equation.

$$\Delta F + k^2 \cdot F = 0 \quad ; \quad \Delta \text{ in polar coordinates}$$

$$k = \frac{2\pi}{L}$$

As shown in (6) after some transformations it can be seen that the modulus of F (r, θ) is equal to the diffraction coefficient

$$K = \frac{\text{height of diffracted wave}}{\text{incoming wave height}} = |F(r, \theta)|$$

The solutions of the scalar wave equation aren't uniquely determined in infinity through specified sources as opposed to potential equations. The wave equation allows standing waves as a solution, which would physically mean that waves coming from infinity superimpose waves coming from the finiteness. To avoid this, the radiation condition (11) must be determined. This prevents all energy from infinity. Analytically it is enough to say that the solutions of the scalar wave equations have the following condition in infinity

$$F \longrightarrow \frac{e^{-ikr}}{\sqrt{r}}$$

SOMMERFELD (12) gives a comprehensive definition of the radiation condition

$$\lim_{r \to \infty} r \left(\frac{\delta F}{\delta r} - ikF \right) = 0$$

and shows that the solution of the scalar wave-equation is uniquely determined.

For the special case of the half-infinite breakwater, SOMMERFELD has found a solution which allows the computing of the diffraction for all wave-lengths.

SOMMERFELD's solution

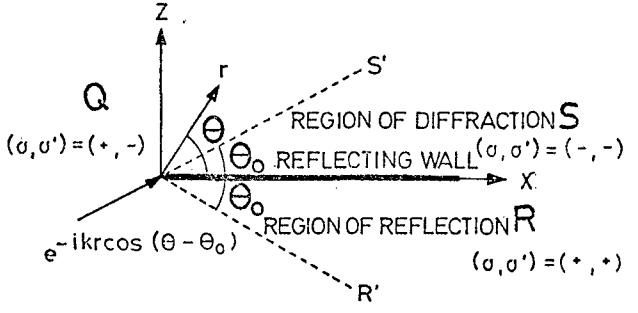

Fig. 3.1

Without mentioning the derivation of the solution function which had been handled in (12), the solution of the searched function is given with the period 4 π.

$$F = f(r,\theta-\theta_o) \pm f(r,\theta+\theta_o) \tag{3.1}$$

$$f(r,\theta-\theta_o) = F(r,\theta,\theta_o) \cdot \frac{1+i}{2} \int_{-\infty}^{\sigma} e^{-i\frac{\pi}{2}t^2} dt$$

$$\sigma = 2\sqrt{\frac{kr}{\pi}} \sin\frac{1}{2}(\theta-\theta_o)$$

Putting the result in equation (3.1) we get the solution of the diffraction problem:

$$F(r,\theta) = e^{-ikr\cos(\theta-\theta_o)} \cdot \underbrace{\left(\frac{1+i}{2} \int_{-\infty}^{\sigma} e^{-i\frac{\pi}{2}t^2} dt\right)}_{\Psi(\sigma)}$$

$$\pm e^{-ikr\cos(\theta+\theta_o)} \cdot \underbrace{\left(\frac{1+i}{2} \int_{-\infty}^{\sigma'} e^{-i\frac{\pi}{2}t^2} dt\right)}_{\psi(\sigma')} \tag{3.2}$$

$$\sigma' = -2\sqrt{\frac{kr}{\pi}} \sin\frac{1}{2}(\theta+\theta_o)$$

In this case the integral $\psi(\sigma)$ can be written in a similar way

$$\psi(\sigma) = \frac{1+i}{2}\left(\int_{-\infty}^{0} e^{-i\frac{\pi}{2}t^2} dt + \int_{0}^{\sigma} e^{-i\frac{\pi}{2}t^2} dt\right)$$

Using the LAPLACE-integral after the first integral

$$\int_0^{\infty} -e^{-\alpha\tau^2} d\tau = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

and splitting up the second integral into a real and an imaginary part, we get:

$$\psi(\sigma) = \frac{1+i}{2}\left(\left(\frac{1-i}{2}\right) + \int_0^{\sigma} \cos\frac{\pi}{2}t^2\,dt - i\int_0^{\sigma} \sin\frac{\pi}{2}t^2\,dt\right)$$

$$\psi(\sigma) = \frac{1+i}{2}\left(\frac{1-i}{2} + M - i\,N\right),$$

by which

$$M = \int_0^{\sigma} \cos\frac{\pi}{2}t^2\,dt$$

$$N = \int_0^{\sigma} \sin\frac{\pi}{2}t^2\,dt$$

M and N are the FRESNEL-integrals.

To discuss the physical problems, in fig.3.1 the lines OS' and OR' are marked as are the shadow borders that are generated through the incoming and reflected wave.

We get three regions S,Q,R. σ and σ' possess a special sign in each of these parts.

That is, for e.g., in the area S:

$$\sigma^{(')} = (-)\ 2\sqrt{\frac{k\cdot r}{\pi}}\ \sin\left(\frac{1}{2}\left(\theta \overset{(+)}{-} \theta_o\right)\right)$$

$$\theta - \theta_o < 0 \Rightarrow \sigma < 0$$

$$\theta + \theta_o > 0 \Rightarrow \sigma' < 0$$

That is why the regions (fig.3.1) have the following arrangements of signs

$$\begin{aligned} &S \text{ - region} && (\sigma,\sigma') = (-,-) \\ &Q \text{ - region} && (\sigma,\sigma') = (+,-) \\ &R \text{ - region} && (\sigma,\sigma') = (+,+) \end{aligned}$$

The wave-heights in the three regions finally result in the following solutions:

a) In the region of the geometrical shadow $\sigma < o$, $\sigma' < o$

$$F(r,\theta) = \psi(-\sigma)\, e^{-ikr\cos(\theta-\theta_o)} + \psi(-\sigma')e^{-ikr\cos(\theta+\theta_o)}$$

b) in the unshadowed region $\sigma > o$, $\sigma' < o$

$$F(r,\theta) = e^{-ikr\cos(\theta-\theta_o)} - \left\{ \begin{matrix} \psi(\sigma)\, e^{-ikr\cos(\theta-\theta_o)} + \\ +\psi(-\sigma')e^{-ikr\cos(\theta+\theta_o)} \end{matrix} \right\}$$

c) in the reflecting region $\sigma > o$, $\sigma' > o$

$$\begin{aligned} F(r,\theta) = {} & \underbrace{e^{-ikr\cos(\theta-\theta_o)}}_{\text{incident wave}} + \underbrace{e^{-ikr\cos(\theta+\theta_o)}}_{\text{reflected wave}} \\ & - \Big(\psi(-\sigma)e^{-ikr\cos(\theta-\theta_o)} + \psi(-\sigma')e^{-ikr\cos(\theta+\theta_o)} \Big) \\ & \qquad \text{diffracted wave} \end{aligned} \tag{3.3}$$

This solution in the reflecting region describes the MACH-reflection of gravity waves. Computing the wave-height in front of the wall (stem-height) it is

$$\frac{H_s}{H} = F(r,\theta)$$

The equation (3.3) was computed in Fortran IV.

4. HYDRAULIC MODEL AND TEST CONDITIONS

The wave basin mentioned before has the dimensions 18.3 x 45.0 m^2, the test area was 11.0 x 45 m^2. The lateral limitations (guide vanes) in the direction of the wave orthogonals have been installed to control the energy entry of the waves. The length of the reflecting wall has been 7,32 m or 9.80 m respectively. Opposite to the wavemakers, a wave absorber has been installed in the basin consisting of a 7.6°-slope and specially-designed wave absorbing elements which have been tested in some pilot tests (2) before.

The basin is covered by an electrical driven measuring bridge mouvable with constant speeds of 5 m/min or 20 m/min for measuring the wave field. Resistance-type wave gauges (FÜHRBÖTER (3)) were used for all tests. The wave heights were registered on a thermosensitive recording instrument.

Mechanical and electrically controlled wavemakers were used for the tests. The movement of the wave paddles was adjustible corresponding to the chosen wave parameters. The combined motion components (translatory + rotary) have been optimized using special tests for these machines at the FRANZIUS-INSTITUT.

Despite this, some model-caused inaccuracies in the experiments should be noted. The reflection-coefficient of the wave-absorber was in the order of 9%, but the error in reproducibility of single tests (measuring time 30 min.) was in the order of 2% only. In addition to the reflection at the wave absorber and re-reflection at the wavemaker-paddle, transverse oscillations in the wave field could be seen leading, to some extent, to disturbances in the wave field. But model-caused reflections have not been investigated in more detail within this programme and the test results reported in this paper show the original (unfiltered) data.

5. RESULTS OF MEASUREMENTS AND COMPARISON WITH DIFFRACTION THEORY

The stem-height has been measured for different wave heights and wave lengths as a function of the direction of the incident wave as shown in fig. 5.1. The water depth was constant in all tests.

For cxample, the stem-heigths for a wave length of L = 100 cm are plotted in fig. 5.1 to 5.4. The full lines show the theorctical development of the stem according to the diffraction theory. The stem-height increases with an incrcasing angle Θ_o of the incoming wave, as demonstrated by NIELSEN and HAGER (see fig.2.1). Because the theoretical and experimental results werc in good agreement, the thcoretical development of thc stem as a function of the wall-length has been summarized for different angles Θ_o in fig. 5.5.

Thc oscillating data in fig. 5.1 to 5.4 may be explained by model-caused disturbances (see chapter 4). After about 2/3 of the wall lcngth, the differences between the measured results and the theoretical curvc are somewhat greater than at the beginning of the wall. These deviations may be explained by the fact that in addition, a diffraction wave is caused by the end of the wall which is superimposcd with the wave field.

For the test conditions given in Tab.1 the stem-widths in front of the wall have also been mcasured (x/L spaced equidistantly). As an example fig. 5.6 shows the experimental and corresponding theoretical results for an angle of incidence $\Theta_o = 20^o$.

The agreement between theory and measuremcnt can be seen rather well. The trough bounding thc stem-width becomes steepcr and narrower in proportion to the stem-wave's propagation along the wall.

Thc scattering of the data may be cxplained by model-caused disturbances (see chapter 4) as mentioned beforc.

Test conditions

H_I (cm)	L (cm)	Θ_o (^{o})	d (cm)
2,2	100	10	25
3,6	150	15	
4,3	200	20	
		25	

Tab. 1

The examples of fig. 5.1 to 5.6 have shown that the diffraction theory describes totally the development of the stem-height at the reflecting wall as well as the wave pattern in front of the wall (stem-width). A comparison with the measurements of NIELSEN and HAGER (see fig. 2.1) although qualitative, confirms both theoretical and experimental results.

The development of a second stem-wave as observed by NIELSEN with larger angles Θ_o of incidence and with a height also greater than double the incident-wave height is approved by the diffraction theory, too. Fig.5.7 shows as an example, the water level normal to the wall at a distance of 5 wave lengths from the wall edge for wave-approach angles $\Theta_o = 20^o$, 25^o and 30^o. A second stem can be seen clearly.

6. CONCLUSION

Investigations with regular waves in connection with the so-called MACH-effect have only been carried out by NIELSEN and HAGER.

From supplementary measurements and by comparing the results with the diffraction theory, it has been proved that the MACH-reflection, i.e. the increase of a wave up to more than double the height of the incoming wave striking a wall with an acute angle, should not to be seen as an analogy to gas-dynamics.

On the contrary, the increase of the wave and the wave pattern before the wall is to be interpreted as a diffraction problem within a region of reflection.

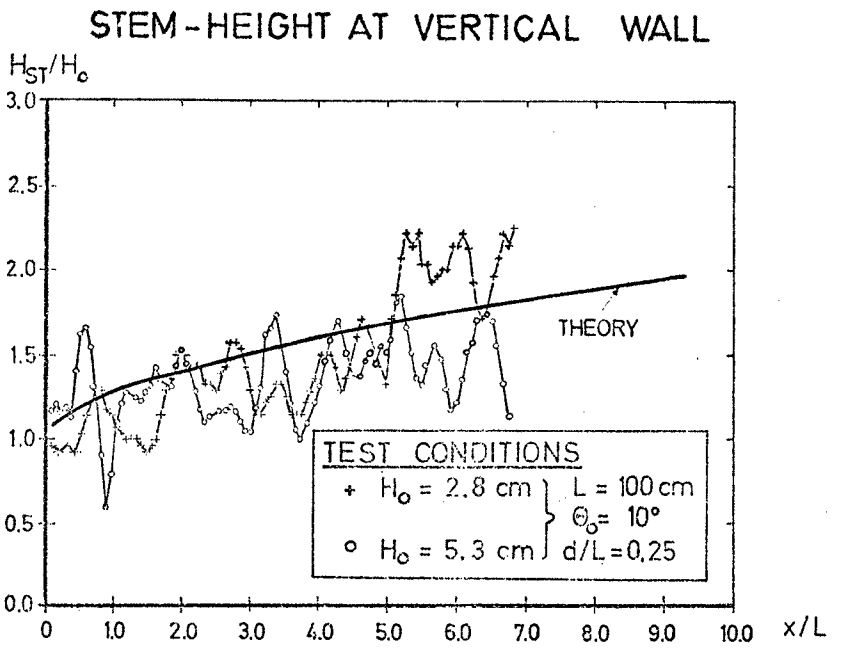
STEM-HEIGHT AT VERTICAL WALL
H_{ST}/H_o
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
x/L
THEORY
TEST CONDITIONS
+ H_o = 2.8 cm
o H_o = 5.3 cm
L = 100 cm
Θ_o = 10°
d/L = 0.25

Fig. 5.1

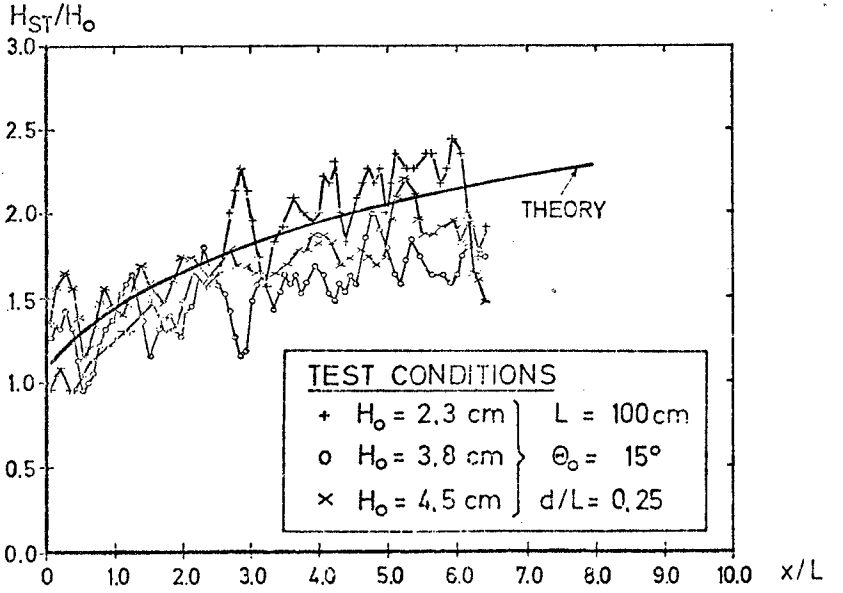
H_{ST}/H_o
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
x/L
THEORY
TEST CONDITIONS
+ H_o = 2.3 cm
o H_o = 3.8 cm
× H_o = 4.5 cm
L = 100 cm
Θ_o = 15°
d/L = 0.25

Fig. 5.2

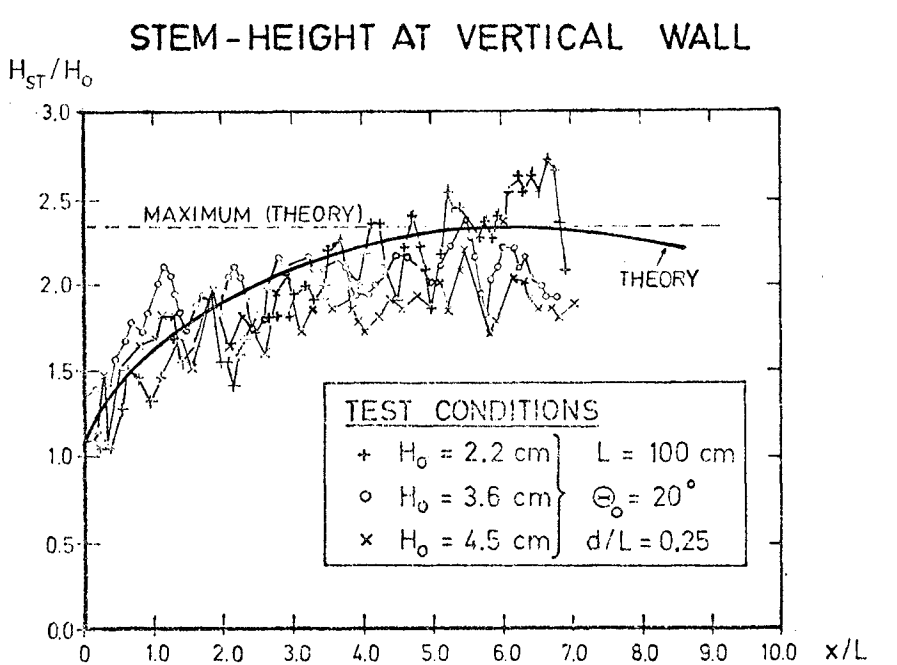
STEM-HEIGHT AT VERTICAL WALL
H_{ST}/H_o
3.0
2.5
2.0
1.5
1.0
0.5
0.0
MAXIMUM (THEORY)
THEORY
TEST CONDITIONS
+ H_o = 2.2 cm
o H_o = 3.6 cm
× H_o = 4.5 cm
L = 100 cm
Θ_o = 20°
d/L = 0.25
0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0 x/L

Fig. 5.3

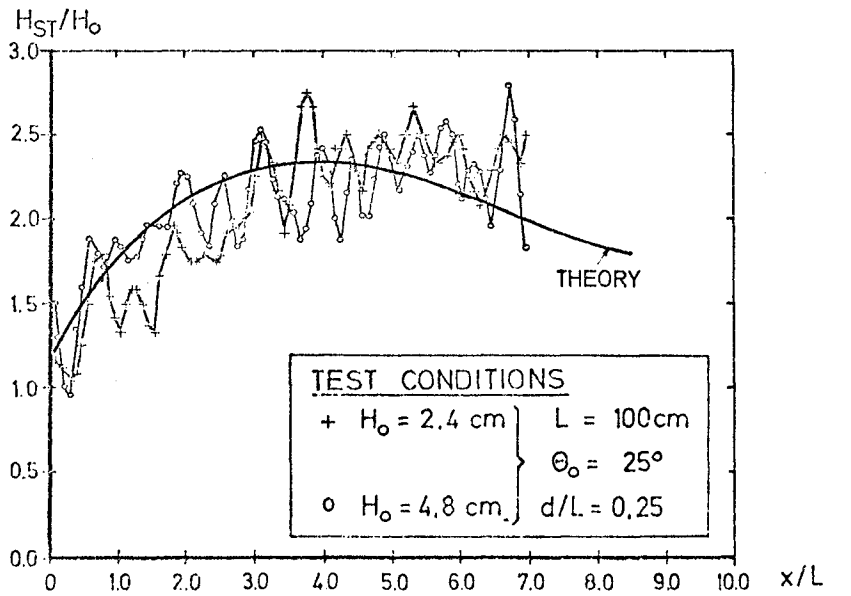
H_{ST}/H_o
3.0
2.5
2.0
1.5
1.0
0.5
0.0
THEORY
TEST CONDITIONS
+ H_o = 2.4 cm
o H_o = 4.8 cm
L = 100cm
Θ_o = 25°
d/L = 0.25
0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0 x/L

Fig. 5.4

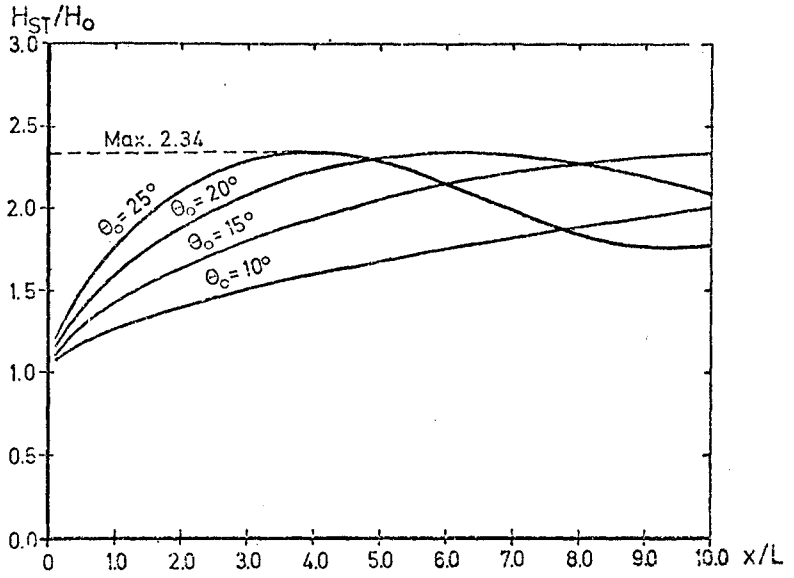

Fig. 5.5

stem-height at vertical wall
diffraction theory

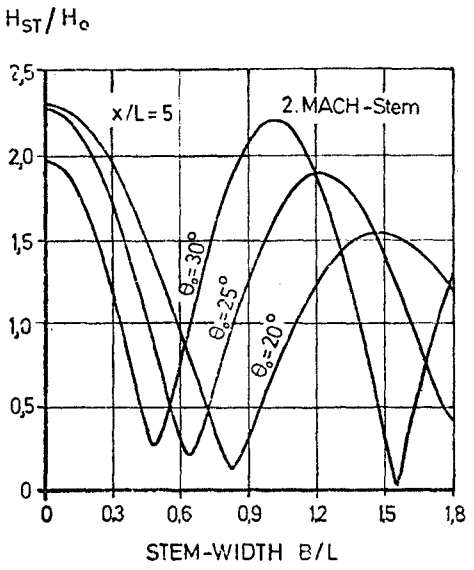

Fig. 5.7
Explanation of a 2.MACH-stem

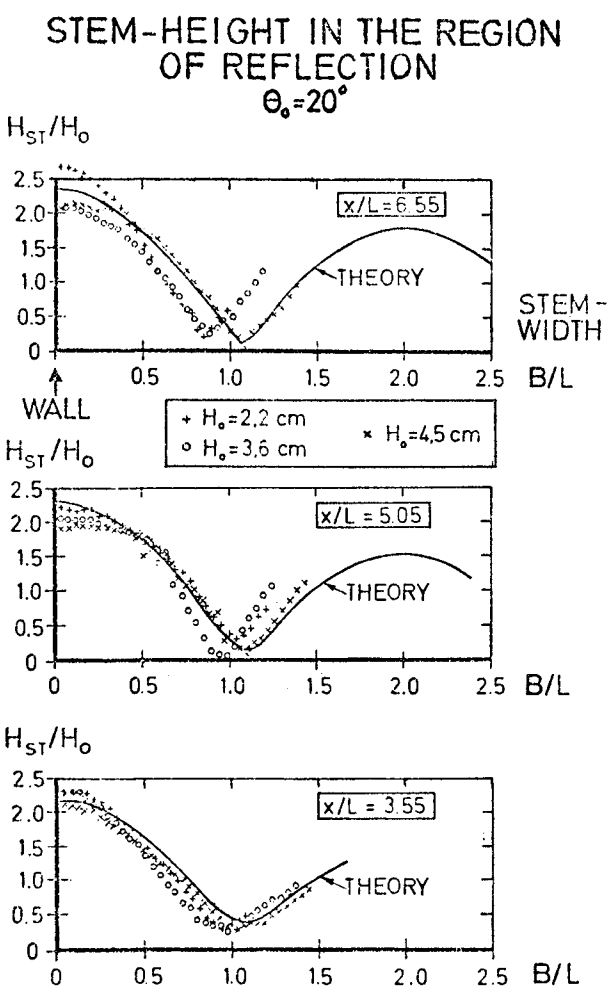
STEM-HEIGHT IN THE REGION OF REFLECTION
$\Theta_0=20°$
H_{ST}/H_o
x/L=6.55
THEORY
STEM-WIDTH
WALL
B/L
+ H_o=2,2 cm
o H_o=3,6 cm
× H_o=4,5 cm
x/L = 5.05
x/L = 3.55

Fig. 5.6

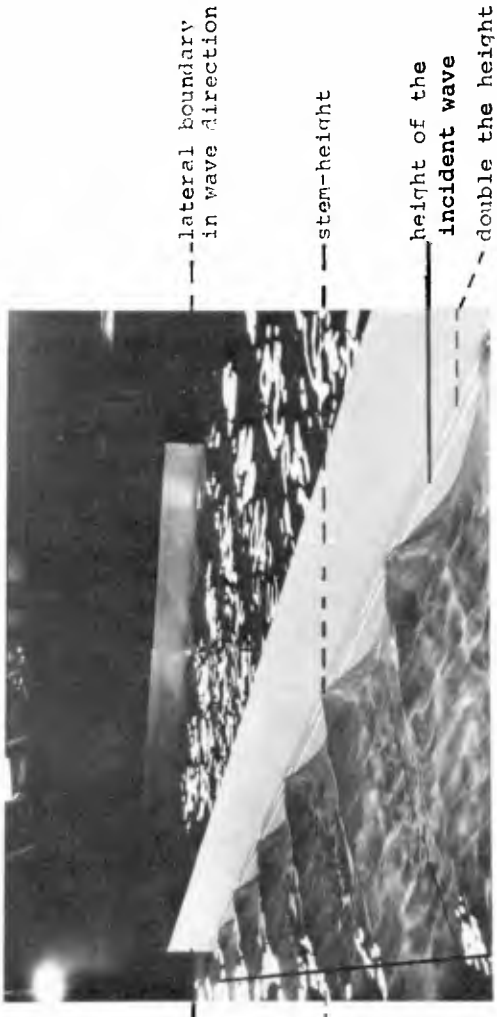

Fig. 5.8
MACH-reflection in the test-basin with monochromatic waves
$\Theta_1 = 20^\circ$; H= 5cm ; L= 100cm

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