Non-Uniformity in the Wind Generated Gravity Waves Phase Distribution

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Abstract

Empirical results which show the existence of a relationship among the probabilistic behaviour of the sea surface elevation, the statistical properties of the phase spectral estimates and the sea surface bispectra are presented. Furthermore, some theoretical relations proposed to characterise the marginal density function of the waves phase for the linear and non-linear wave records are examined.

Introduction

The free sea surface elevation in deep waters is generally considered to be the result of the superposition of a large number of sinusoidal waves of different frequencies, amplitudes and phases. Besides, it is usually assumed that the phase angles are mutually independent and are uniformly distributed in the interval *(0, 2π]*. Hence, the free surface elevation at a given point can be represented as

$$\eta(t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t) + b_n \, sen(\omega_n t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \varepsilon_n) \qquad (1)$$

where directionality of waves has not been considered. In Eq. (1), a_n and b_n are the Fourier coefficients, assumed normally distributed, $C_n = \left(a_n^2 + b_n^2\right)^{1/2}$ are the wave amplitudes, ω_n the frequencies, ranging from *0* to infinity and $\varepsilon_n = tan^{-1}\left(-b_n / a_n\right)$ the

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phases, uniformly random.

Under these assumptions it can be shown, by invoking the central limit theorem, that $\eta(t)$ posses a Gaussian distribution,

$$p(\eta)=\frac{1}{\sigma_\eta\sqrt{2\pi}}exp\left[-\frac{(\eta^2)}{\sigma_\eta^2}\right] \tag{2}$$

For waves of low steepness there is a reasonable agreement between this linear model and observations and the free surface is very nearly Gaussian. However, various authors (e.g. Huang and Long, 1980) have reported significant deviations from the Normal distribution for steeper waves, with the free surface displaying sharper peaks and shallower troughs. The observed deviations from Gaussianity can only be explained by the existence of nonlinear interactions among wave components. The first author dealing with this problem was Longuet-Higgins (1963), who suggested that the free surface elevations should be considered as a weakly non-linear process. In other words, surface elevations can be considered as a quasi-normal process. Given that wave-wave non-linear interactions are consequence of the existence of phase relationships, or phase coupling between wave components, deviations from the Gaussian distribution for $\eta(t)$ must be reflected on the probability distribution of phase angles. It can be shown that for a stationary process with a narrow spectral bandwidth the phase distribution can be uniform only if the probability density of the free surface elevations is strictly Gaussian (e.g. Bitner, 1980).

In this paper, the relations between the bispectrum and the statistical behaviour of the sea surface elevation and the phases of the wave components are analysed. Furthermore, the effects of the phase statistical properties on wave groupiness are investigated. Finally, some features of the probability density function of the wave phases are examined.

Probability distribution of wave envelope and phase

Upon the assumption of narrow band random function, the sea surface elevation can be expressed as

$$\eta(t)=A(t)\cos\phi(t) \tag{3}$$

where $A(t)=(\eta^2+\hat{\eta}^2)^{1/2}$ is the wave envelope, $\phi(t)=tan^{-1}\left(\hat{\eta}/\eta\right)$ the phase and $\hat{\eta}$ is the Hilbert transform of $\eta(t)$. Then, introducing the normalised variable $\xi=A/A_{rms}$, where $A_{rms}=\sqrt{2}\eta_{rms}$ is the root mean square of the wave envelope amplitude, it can be shown (eg. Bitner, 1980, Tayfun, 1993) that the joint distribution of A and ϕ is given by the product of the corresponding marginal probability densities,

$$p(\xi,\phi)=p(\xi)p(\phi)=\left[2\xi\,exp\left(-\xi^2\right)\right]\left[1/2\pi\right] \tag{4}$$

This result reveals that for a linear narrow banded wave field both variables are statistically independent, with the amplitudes following a Rayleigh distribution and the phases uniformly distributed in $(0,2\pi]$.

From the above discussion it should be expected that, except for very low sea states, the probability density function of phase angles deviate from the uniformity, reflecting the existence of privileged phases. However, this fact is still controversial, with various authors presenting contradictory results.

On the other hand, the importance of phase information in the Fourier representation of wind generated waves have received very little attention because the phase spectrum contains no relevant information under the assumption of a Gaussian random process. However, phase information for non-Gaussian waves may result of vital importance in practical applications. Thus, for example, the presence of phase coupling among the various component wave frequencies results in a non-Gaussian signal which is capable of significantly alter the response of a system thought to be subject to a Gaussian input.

The present interest on the phase information has raised mainly in relation with the wave grouping phenomenon. Also, this subject is far from solved. Thus, while some authors consider that phase information is necessary for a good description of wave groups. (e.g. Johnson et al., 1978 and Burcharth, 1978), some other authors (e.g. Rye and Lervik 1981 and Goda, 1983) support that the phases of the spectral components for wind waves can be considered as independent and uniformly distributed on the interval $(0\text{-}2\pi]$.

Following the work by Longuet-Higgins (1963), Tayfun and Lo (1989, 1990) and Tayfun (1993, 1994) have examined the representation of second-order random waves in terms of the effect of second-order nonlinearities on the wave envelope and phase. These authors assumed that, in general, in deep water nonlinearities are relatively weak so that its statistical and spectral characteristics can be adequately described by a two-term perturbational solution to the governing nonlinear equations of motion.

Tayfun (1994), expressed the sea surface displacement from the mean level as the superposition of a first order solution η_1 , given by (1), and a second order solution η_2, which includes the nonlinear nonresonant components of frequencies $\omega_m + \omega_n$ and $\omega_m - \omega_n$ generated by any two components of η_1 with frequencies ω_m and ω_n. Thus, the sea surface elevation at a given point can be expressed as

$$\eta(t) = \eta_1 + \eta_2 = \eta_1 + \left(\eta_2^+ + \eta_2^-\right) \tag{5}$$

where,

$$\eta_2^\pm = \lim_{N\to\infty} \frac{1}{4} \sum_{m=1}^{N} \sum_{n=1}^{N} C_m C_n K_{m,n}^\pm \cos[(\omega_m \pm \omega_n) + (\varepsilon_m \pm \varepsilon_n)] \tag{6}$$

and

$$K_{m,n}^\pm = K^\pm(\vec{k}_m, \vec{k}_n) = K^\pm[k_m, k_n, \cos(\theta_m - \theta_n)] \tag{7}$$

with $\vec{k}_m, \vec{k}_n$ the horizontal wave-number vectors with moduli k_m and k_n; and directions θ_m, θ_n, measured positive counterclockwise from the x axis. $K^{\pm}$ represents the interaction coefficients (e.g., Longuet-Higgins, 1963, Tayfun, 1990).

In this context, the Gaussian law is not useful to characterise the probability distribution of $\eta(t)$. For this, some authors (e.g. Longuet-Higgins, 1963; Huang et al., 1983) have suggested alternative expressions for $p(\eta)$ by considering weak interactions among wave components. Besides, in this case, the wave envelope amplitude ξ and the phase ϕ are no more statistically independent (Tayfun, 1994). The marginal density function of ξ has the same expression as in (4) whereas the marginal density of ϕ is given by

$$p(\phi) = \frac{1}{2\pi}\left(1 - \frac{1}{6}\sqrt{\frac{\pi}{2}}\lambda_3 \cos\phi\right) \tag{8}$$

and the joint distribution function of these variables is given now by

$$p(\xi,\phi) = \frac{\xi}{\pi} exp\left(-\xi^2\right)\left[1 + \frac{\sqrt{2}}{3}\lambda_3\xi\left(\xi^2 - 2\right)cos\phi\right] \tag{9}$$

where λ_3 is the coefficient of skewness. Under oceanic conditions this coefficient should be usually $\lambda_3 < 0.75$ (Srokosz and Longuet-Higgins, 1986). An asymptotic expression useful to estimate this parameter in terms of the zero m_0 and second order m_2 spectral moment is (Tayfun, 1990)

$$\lambda_3 = \frac{12\pi^2}{g}\left(\frac{m_2}{\sqrt{m_0}}\right) \tag{10}$$

From equation (4) it is easy to show (Tayfun, 1994) that in the linear case the mean value and variance of the wave phases are given by,

$$\langle\phi\rangle = \pi \quad ; \quad Var(\phi) = \frac{\pi^2}{3} \tag{11}$$

From Equation 8, in the nonlinear case the mean value of ϕ is equal to that in the linear one. However, the variance depends on the coefficient of skewness, and is given by,

$$Var(\phi) = \frac{\pi^2}{3} - \frac{1}{3}\sqrt{\frac{\pi}{2}}\lambda_3 \tag{12}$$

Measurements and Data analysis

The analysed wave records were measured, by using a Waverider buoy deployed at deep waters in the Stajfiord oil platform (Norway continental shelf). The sampling frequency was $2H_z$ and the total record length close to 20 minutes with 2048 digitised values. The overall period analysed covers from January to February 1989.

Since the pioneering paper by Hasselmann et al. (1963), bispectral analysis has been considered as a useful tool for investigating the nonlinear properties of random waves and to identify phase characteristics of non-Gaussian wind generated wave records. According to these authors, if the surface elevation $\eta(t)$ is a stationary zero mean valued random process, the spectra $S(\omega)$ and the bispectra $S(\omega_1,\omega_2)$ are defined respectively as

$$S(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(\tau)e^{-i\omega\tau}d\tau \tag{13}$$

$$S(\omega_1,\omega_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R(\tau_1,\tau_2)\,e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \tag{14}$$

where

$$R(\tau)=E[\eta(t)+\eta(t+\tau)] \tag{15}$$

$$R(\tau_1,\tau_2)=E[\eta(t)+\eta(t+\tau_1)+\eta(t+\tau_2)] \tag{16}$$

and $E[\ ]$ denotes the mathematical expectation. The bispectra has been estimated by using the method suggested by Kim and Powers (1979). That is, by segmenting the wave records into N segments to obtain the following smoothed estimation of the bispectra,

$$B(k,l)=\frac{1}{N}\sum_{n=1}^{N}X_n(k)X_n(l)X_n(k+l) \tag{17}$$

where k and l are the frequencies of the interacting wave components and X are the complex Fourier coefficients, which can be computed efficiently by means of the Fast Fourier Transform (FFT) algorithm.

Assuming the linear wave theory, the sea surface elevation should be symmetric. Thus, the asymmetry of wave record profiles implies the existence of nonlinearities. Furthermore, the vertical asymmetry of the sea surface profile with respect to the mean water level can be characterised by the coefficient of skewness λ_3 of the sea surface elevation probability distribution. It can be shown (e.g., Kim et al., 1980) that for a stationary process with zero mean λ_3 and $S(\omega_1,\omega_2)$ are related through the following relationship

$$\lambda_3=\frac{1}{\left(E[\eta^2(t)]\right)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}S(\omega_1,\omega_2)d\omega_1 d\omega_2 \tag{18}$$

Taking into account that λ_3 must be null for a Gaussian process and that it is proportional to the zero order moment of the bispectrum, a value different from zero for the bispectrum implies a non-zero value of the skewness and some deviation of the sea surface elevation probability function from the Gaussian distribution or equivalently asymmetries in the wave record. As a consequence, statistically significant deviations of $S(\omega_1,\omega_2)$ from zero can be interpreted as the existence on nonlinearities in the process being examined.

Results and Discussion

The sea surface displacement and the wave phase probability distributions have been obtained for each one of the wave time series recorded at the Stajfiord oil field. The phase spectrum was computed as a previous step to estimate the phase distributions. The Chi-square (with a significance level α=0.05) and Kolmogorov Smirnov tests were applied to examine the statistical significance of the possible deviations from the Gaussian and the uniform distributions.

From a total of 412 wave records analysed, the percentage of rejection of the hypothesis of the Gaussian distribution adequacy to describe the statistical structure of the sea surface elevation was of 64% and 59% for the χ^2 and the K-S tests, respectively. The goodness of fit of the uniform probability function to the wave phase distribution was rejected in a 60% of the cases for both tests. In addition, the probability distribution function given by equation (8) (Tayfun, 1994). Paradoxically, though this model depends on the coefficient of skewness, which represents a measure of nonlinearity, the percentage of rejection was of 66% and 62% for the χ^2 and the K-S tests, respectively. That is, greater than the obtained for the uniform distribution.

Figure 1 shows the sea surface elevation (1a) and wave phase (1b) distributions, and the bispectral density corresponding to a wave record with a very small value of skewness (-0.002) and kurtosis (fourth statistical moment) (-0.004), such as is revealed by the associated low bispectral density. In this case, the Gaussianity and the uniformity were accepted by the two statistical tests. Furthermore, the Tayfun's distribution, which tends to present a maximum at π and minima at 0 and 2π, was also accepted. In this example, the coefficient of skewnees computed by means of (10) is 0.133. Note the difference with that computed directly from the wave record which is slightly negative.

It should be remarked that, while the bispectra have been displayed in all the range of positive frequencies, it only needs to be represented between the origin, the line $f_l=f_k$ and the Nyquist frequency due to its symmetry properties, which can be easily observed in the figure.

In contrast with the low values observed for the bispectrum of the first example, the bispectral density represented in figure 2 presents a large peak revealing an important contribution of nonlinearities, mainly from the auto-interactions of the peak frequency. This effect is reflected in the deviations of the sea surface elevation from the Gaussian distribution, which was rejected by the two statistical goodness of fit tests applied. Such as expected, the probability distribution of wave phases deviate significantly from the uniformity, displaying a maximum at π and minima at 0 and 2π (the coefficient of skewness is 0.356). Hence, the goodness of fit is rejected for the uniform distribution and accepted for the Tayfun's model.

In various cases the wave phase distribution presents an inverse pattern to that expected. That is, the phase probability density presents a maximum at $\phi=0$ and another at $\phi=2\pi$ and a minimum near to $\phi=\pi$. An example of this kind of phase distribution is shown in figure 3. In this example, the Gaussian hypothesis was accepted but the goodness of fit of the wave phase distribution to the uniform distribution and to the Tayfun's model was rejected. In fact, the relatively large percentage of this kind of phase distribution is the motive for the above mentioned large quantity of rejections for the Tayfun's model, greater than the obtained for the uniform distribution. In some of these cases, the deviation from uniformity is relatively small but the Tayfun's model tend to fit a distribution with an inverse structure to that displayed by the measurements and, as a consequence, it is rejected. Note that in this case the bispectrum presents a moderate amplitude and the coefficient of skewness is 0.356.

Wave phases distributions with a similar structure have been presented (but not discussed) by some authors (eg. Bitner, 1980; Mase et al., 1983). The properties of this wave records are being studied now in detail.

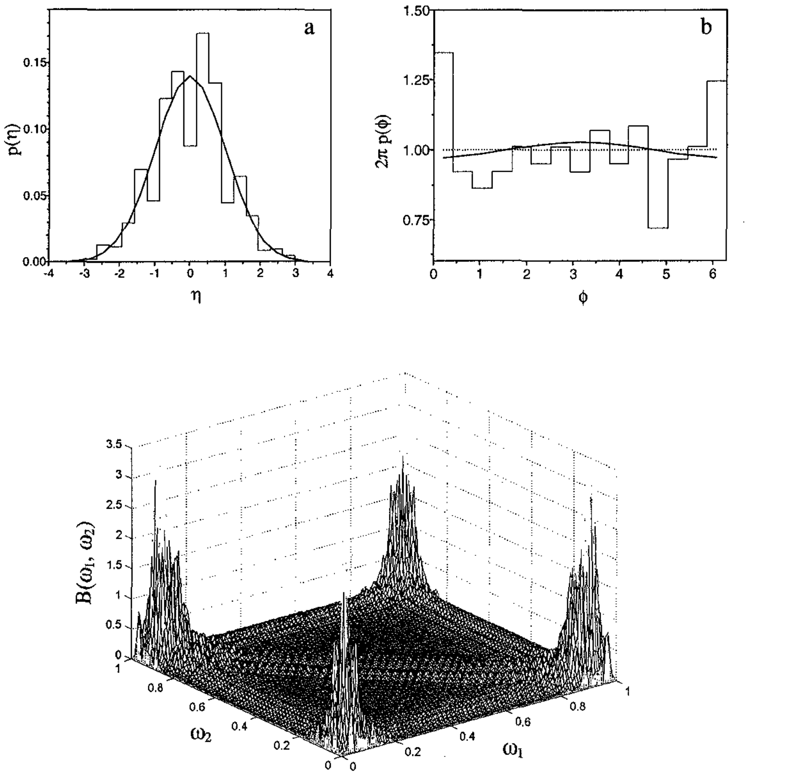

Figure 1. Sea surface elevation (a) and wave phase (b) probability distributions and bispectrum associated to a Gaussian and uniform wave phases distribution wave record.

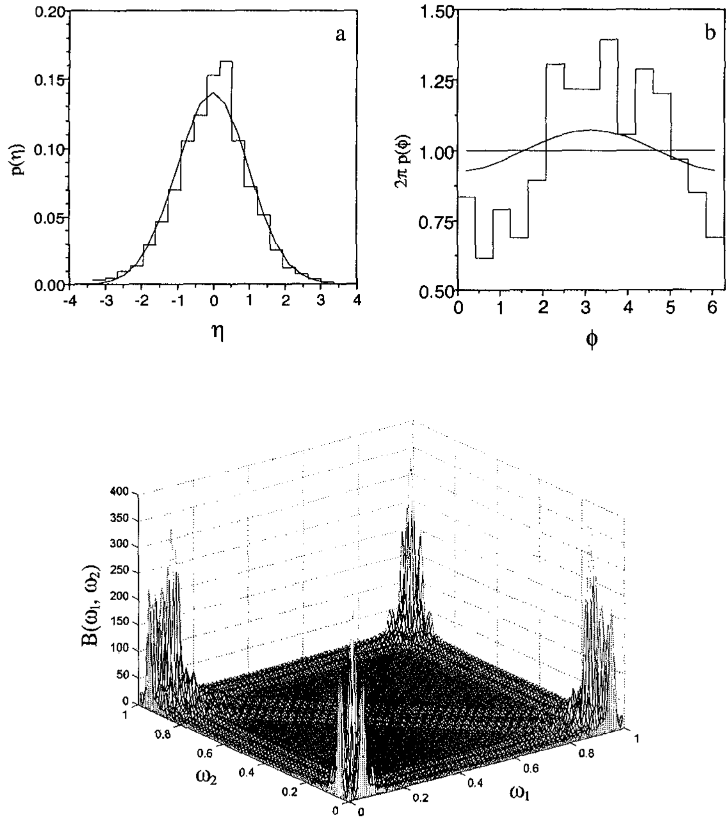

Figure 2. Sea surface elevation (a) and wave phase (b) probability distributions and bispectrum associated to a non-Gaussian and non-uniform wave phases distribution wave record.

In general, there is a trend of the coefficient of skewness to increase with the significant wave height (see Fig. 4a), such as expected according to the comments above and in agreement with results reported by various authors. On the other hand, according to Mase et al., (1983) if the groupiness factor GF, computed in terms of the SIWEH (smoothed instantaneous wave energy history) and proposed by Funke and Mansard (1980), is larger than 0.7, the phase distribution is not uniform, but the appearance frequency becomes large at the phases close to π. Similarly, Yuxiu and Manhai (1996) used a different groupiness parameter named group height factor (GFH) and expressed in terms of the standard deviation and the mean value of the wave envelope, which is estimated directly from the wave records with the Hilbert transform technique. These authors analysed three sets of field wave data and observed that when GFH is large, the phase distribution is not uniform, but concentrated around π.

The values of the groupiness factor are represented versus the coefficient of skewness in Figure 4b. It can be observed that GF presents a large scatter and that there is not a clear trend of GF to increase with the coefficient of skewness. Thus, the observed results does not support the idea of an increase of the wave groupiness as the nonlinear interactions becomes more significant.

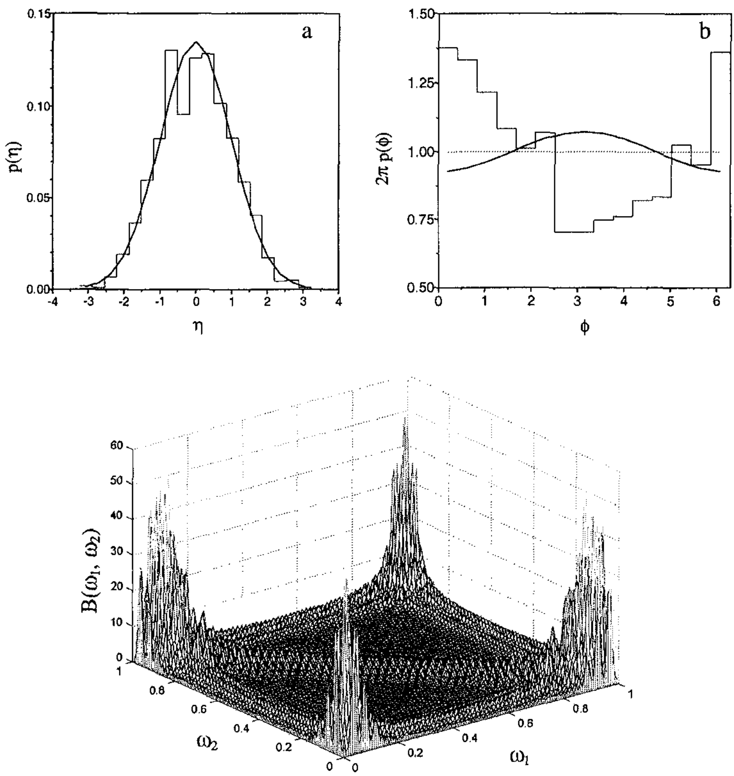

Figure 3. Sea surface elevation (a) and wave phase (b) probability distributions and bispectrum associated to a Gaussian and non-uniform wave phases distribution wave record.

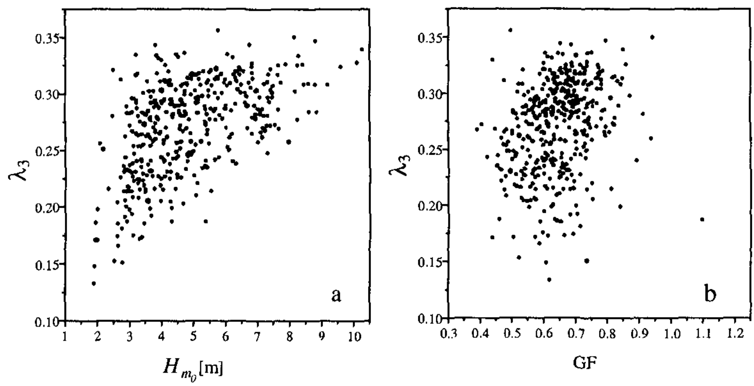

Figure 4. Coefficient of skewness computed by using Eq. 10 versus significant wave height (a) and the groupiness factor (Funke and Mansard, 1980), for the whole data set analysed.

In relation to the mean and variance of the wave phase distribution, Figure 5 displays the values of these parameters obtained for each one of the 412 wave records analysed. Figure 5a represents the observed values for the mean, together with the theoretical value , π, derived for the linear and nonlinear wave models. It can be observed that the theoretical mean value is coincident with that empirically observed only for very low values of the coefficient of skewness, but there is an increasing scatter as λ_3 increases. Similar results are observed for the variance of the wave phase distribution, which are represented together with the theoretical values predicted for the linear and the nonlinear wave model by Tayfun (1994).

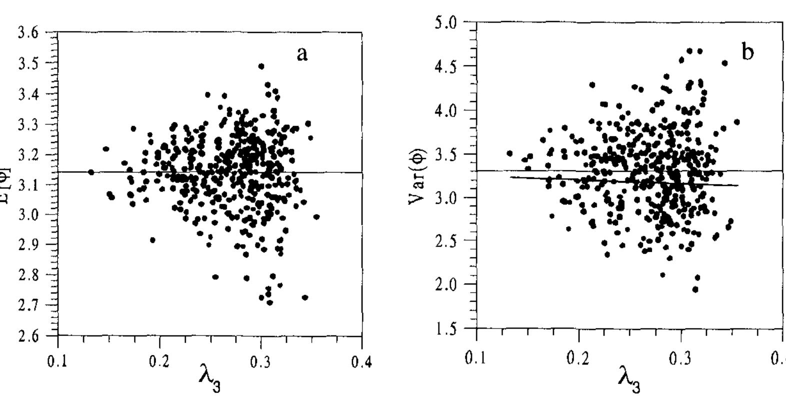

Figure 5. Variability of the wave phase mean value and variance with the coefficient of skewness for the whole data set analysed, and theoretical values for thc linear and nonlinear cases, according to Tayfun (1994).

Conclusions

Bispectral analysis of measured wave records has been used to detect the presence of phase coupling among the various component wave frequencies. It is shown that when bispectral density function is statistically insignificant, sea surface displacements are distributed according to the Gaussian law and the angle phases are uniformly distributed. On the another hand, a non zero bispectrum is related to non Gaussian sea surface displacements and non uniformly distributed phases.

It is observed that, in most cases, the phase distribution deviations from uniformity are characterised by a higher relative frequency of angles in the range from 0.5π to 1.5π. That is, it presents a maximum near $\phi=\pi$ and minima at $\phi=0$ and $\phi=2\pi$.

These results agree with those reported by Tayfun (1994) who analysed data from hurricane Camille and observed a similar behaviour in phase distributions. However, in same cases, our results display an inverse structure to that expected according to this author. In other words, phase probability density presents a maximum at $\phi=0$ and another at $\phi=2\pi$ and a minimum near to $\phi=\pi$.

There is not a clear trend of the wave groupiness to increase with the deviation of phase angles from uniformity, in contrast with the results previously reported by some authors.

The theoretical values of the mean and the variance of the wave phase distribution agree with the empirically observed values only for low values of the coefficient of skewness.

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