Recent Extensions to Local Polynomial Approximation Models

Andrew B. Kennedy¹

Abstract

Local polynomial approximation models for water wave simulation are examined with the aim of improving accuracy and efficiency. Several potential improvements are considered. The first is based on a linearisation of the solution for Laplace's equation and is shown to have good theoretical characteristics. However, it is found to have a fatal instability to high wavenumber bottom disturbances. The second method uses empirical collocation adjustment to provide better agreement with exact solutions for steady waves. This method provides significant improvements in accuracy for a given level of approximation and is recommended for future use.

Introduction

For the computation of water wave evolution over varying topography, there have been two main areas of effort in recent years. One has focused on improvements to Boussinesq-type models, continually increasing the range of validity to encompass deeper and higher waves (Nwogu, 1993., Schäffer and Madsen, 1995, Wei et al., 1995, Gobbi et al., 1998). Despite increasing complexity, solution times remain reasonable enough to consider wave motion in two horizontal dimensions. This remains a very active area of research. At the other extreme from Boussinesq equations are numerical solutions of Laplace's equation using boundary element methods (BEM, e.g. Dold and Peregrine, 1980, Grilli et al., 1994). These offer unparalleled accuracy in any depth and can even compute overturning waves. However, despite significant advances (Wang et al, 1995), they are extremely computationally intensive; enough that large domains for one horizontal dimension can become problematic, and two dimensional computations are extremely limited in scope. Although research continues in this area, boundary element models are considered to be a relatively mature field.

Local polynomial approximation methods (LPA, Kennedy and Fenton, 1996, Kennedy and Fenton, 1997, Kennedy, 1997) were introduced as a way of bridging

¹Center for Applied Coastal Research, University of Delaware, Newark, DE. 19716 USA.

this gap between highly accurate, computationally intensive models (BEM), and Boussinesq models of moderate accuracy and expense. The basis of these models is a polynomial representation of arbitrary degree in the vertical coordinate combined with a weighted residual solution of the continuity equation for potential flow. For lower levels of approximation, accuracy is comparable to high end Boussinesq equations, while higher levels of approximation can provide accuracy comparable to some implementations of boundary element models. Computational expense lies in between BEM and Boussinesq schemes. A comparison of an advanced Boussinesq and LPA models is given in this volume (Gobbi et al., 1998).

This paper reports recent efforts to improve the accuracy and efficiency of LPA models. Two basic strategies are considered: one which makes further approximations which slightly degrade nonlinear properties, but potentially offers great computational savings, and a second method which improves linear and nonlinear accuracy for a given level of approximation with no increase in computational expense.

Local Polynomial Approximation Model

Local polyonomial approximation models all assume potential flow, and are essentially weighted residual solutions of Laplace's equation combined with potential flow evolution equations. The velocity potential is assumed to vary in the vertical coordinate, z, like a polynomial of some specified degree. This may be represented as

$$\phi(x, y, z) = \sum_{j=0}^{n-1} A_j(x, y) z^j$$
(1)

where $A_j(x, y)$ is an initially unspecified function of the horizontal coordinates. The level of approximation, n, controls the degree of polynomial and, hence, the accuracy of the computation. To specify these functions, boundary conditions must be given at the free surface $z = \eta(x, y)$ and bed z = -h(x, y), and continuity must be imposed over the entire flow field. The bottom boundary condition is

$$\phi_z + \phi_x h_x + \phi_y h_y = 0 \quad \text{on } z = -h(x, y) \tag{2}$$

The condition at the free surface specifies the velocity potential

$$\phi(x, y, z = \eta(x, y)) = \phi^{(s)}(x, y) \tag{3}$$

where $\phi^{(s)}(x, y)$ is the known free surface velocity potential. For all methods considered here, these two equations will remain unchanged. In a more general sense, this is not entirely necessary, and Kennedy (1997) presents a method similar to Dommermuth and Yue (1987), which expands (3) about the still water level, with some loss of accuracy. However, in this paper attention instead will be paid to Laplace's equation, which imposes continuity throughout the flow field. In its most general form, the LPA approximation to the field equation becomes

$$\int_{-\hbar}^{\eta} W_j(x,y)(\phi_{xx} + \phi_{yy} + \phi_{zz})dz = 0, \quad j = 1, \dots, n-2$$
(4)

where $W_j(x, y)$ are weighting functions, which can take almost any form. Kennedy and Fenton (1996) used polynomial weighting functions, but here, as in Kennedy and Fenton (1997), weighting functions will be taken to be Dirac delta functions. This turns (4) into a set of collocation equations

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \text{ on } z = z_j(x, y), \quad j = 1, \dots, n-2$$
 (5)

The set of collocation points in Kennedy and Fenton (1997) were given as

$$z_j = -h + (\eta + h)\alpha_j \tag{6}$$

where α_j 's were chosen so that z_j 's were the Gauss-Legendre points for N = n-2, using the free surface and the bed as limits. This set was chosen because of continuity considerations, and will be considered as the standard against which the revised methods will be judged.

With the addition of conditions on the lateral boundaries, equations (2-4) form a set of linear equations which may be solved to find the flow field. Once interior velocities are known, the system may be advanced in time using the evolution equations

$$\eta_t = -\left(\int_{-h}^{\eta} \phi_x dz\right)_x + \left(\int_{-h}^{\eta} \phi_y dz\right)_y \tag{7}$$

and

$$\phi_t^{(s)} = -g\eta - \frac{1}{2} \left(\phi_x^2 + \phi_y^2 + \phi_z^2 \right) + \phi_z \eta_t \tag{8}$$

In Kennedy and Fenton (1997), an equation different from (7) was used to update the free surface elevation. This in general did not analytically guarantee conservation of mass for approximate solutions, but because the Gauss-Legendre collocation points were used, overall conservation was guaranteed for levels of approximation $n \ge 4$. However, in this paper different collocation points will be used, and to guarantee overall continuity, (7) is employed.

This system of equations defined by (1-8) may be solved in a variety of ways, for both one and two horizontal dimensions. However, for simplicity, and because of computational constraints, we will limit tests to one horizontal dimension. Here, a good method involves complex polynomials and is detailed in Kennedy and Fenton (1997). This will be the base method used here.

In the previously referenced papers, these models have been shown to provide good accuracy with a reasonable computational cost for a variety of conditions. However, two possible areas of improvement seem obvious. The first concerns the number of computational variables. According to (1), n variables define the flow field at each computational point. Each of these makes its way into the system of unknowns which must be solved at each time step. If the number of computational unknowns were reduced, computational speed would almost certainly increase. This will form one avenue of exploration. The second concerns the accuracy of any given level of approximation. The weighting functions in (4) were not specified using a strict theoretical basis and indeed none will be used now. Instead, sets of points will be found for different levels of approximation that provide some sort of best fit to a theoretical special case, such as wave celerity over a level bed. For both avenues of improvement, results will then be tested using the experiments of Ohyama et al. (1995), which measured wave transformation over a submerged breakwater.

Improved Speed: Linearised B-Splines

The first effort at improvement will be directed towards increasing computational speed. In the previous section, it was stated that one way to do this would be to reduce the number of computational unknowns that form the system of equations that must be solved at each time step. Unfortunately, despite significant effort, no such technique has been found for the general case of fully nonlinear wave motion in two horizontal dimensions. However, if one small additional approximation is made, a method exists which promises great improvements in computational efficiency. The change that must be made is to the field equation, (6), which becomes

$$z_j = h(-1 + \alpha_j) \tag{9}$$

where α_i 's are identical to those previously used. Linearly, computations will be identical but, because collocations are chosen with respect to the still water level and not the instantaneous free surface, nonlinear properties should be slightly worse. However, because collocation points are invariant with time, it is possible to use what we will call the linearised B-spline approximation to speed computations. This may only be used for one horizontal dimension, as it makes use of the complex polynomial solution method. A detailed discussion of the method (only used there for expansion techniques) is given in Kennedy (1997), but a brief description will be given here. Essentially, because equations (2,5) are homogeneous, it is possible to rewrite the basis functions used in the complex polynomial solution method so that (2,5) are satisfied and the number of equations to be solved is equal to the number of computational points (plus additional boundary conditions). As an additional bonus, the bandwith of the resulting matrix equation is reduced, further increasing speed. The new basis functions are analogous to B-splines, which may also be thought of as methods to eliminate homogeneous equations. It would be possible to rewite these basis functions even when (6) is used, but because the collocation points change with the free surface, they would need to be rederived at each time step, actually slowing computations. However, with (9), collocation points remain constant with time and the revised basis functions would only need to be computed once, at the beginning of computations.

Before testing the new technique, it is helpful to consider the change to the nonlinear properties caused by the approximation (9). Figure 1 shows the amplitude of the second harmonic of LPA Stokes-type steady waves compared to exact results. Two levels of approximation are shown: n = 5, which contains z^4 terms in the velocity potential, and n = 7, which contains z^6 terms. The original formulation for the level of approximation n = 5 has a second harmonic which remains quite accurate into water of depth $kh = 2\pi$, while the linearised basis functions show greater error at high wavenumbers. Still, error remains small up to a dimensionless wavenumber of kh = 5, which is quite good. Using the level of approximation n = 7, second harmonics for both the original and linearised versions are extremely accurate to wavenumbers of $kh = 3\pi$.



Figure 1: Second harmonics for a second order steady wave compared to Stokes' solution

A special case of subharmonics, second order setdowns under a steady wave were also calculated and provided a major surprise: setdowns using both the fully nonlinear and linearised basis functions were identical. Figure 2 shows setdown magnitudes relative to Stokes' solution plotted on a semi-logarithmic scale. For the level of approximation n = 5, results are very good until a dimensionless wavenumber of kh = 2.5, when error increases very quickly. For the higher level of approximation n = 7, error remains small until around kh = 4. For

COASTAL ENGINEERING 1998

high wavenumbers, both levels of approximation can show order of magnitude errors in setdown. However, since setdown decays exponentially with increasing wavenumber, for the most part the error is merely a "different flavour of zero", and should make little difference to computations. For very high wavenumbers around kh = 6, the lower level of approximation does begin to show errors which may not be negligible in terms of absolute, rather than relative, error.



Figure 2: Second order setdown for a second order steady wave compared to Stokes' solution. Symbols identical to previous figure.

These results show a small, but noticeable difference in nonlinear accuracy between the original and linearised basis functions. Computations using the level of approximation n = 7 and the linearised basis functions were then tested against the experimental results of Ohyama et al (1995). Figure 3 shows a schematic drawing of the experimental setup, where monochromatic waves propagated over a steep-sided submerged bar. All waves had an initial height H/h = 0.1, and cases 2,4, and 6 had periods $T\sqrt{g/h} = 5.94$, 8.91, 11.88 respectively

It was here that the problems with the linearised B-spline approximation became apparent. For all cases tested, and with all spatial and temporal resolutions used, computations became unstable as waves passed from the crest of the shelf onto the back slope. The reason for this was not readily obvious, but became apparent after some additional analysis. The culprit was a nonlinear instability to short wavelength bottom disturbances. Consider a small amplitude



Figure 3: Setup for Experiments of Ohyama et al. (1995). All measurements in m.

sinusoidal bottom disturbance in otherwise quiescent water. The water level is assumed to be at elevation $\eta = 0.5h$, similar to that under the crest of a very large wave. Although this is somewhat different from what happens when the wave crest passes over the sharp corner of the submerged bar, it provides an illuminating demonstration of model response to short wavelength disturbances in topography.



Figure 4: Vertical velocity due to a bottom disturbance, n = 7; -, exact; ---, fully nonlinear (obscured); - -, linearised basis functions

Figure 4 plots the structure of the resulting vertical velocity field, showing the exact solution using hyperbolic functions, the fully nonlinear LPA solution, and the LPA solution using linearised basis functions, both with the level of approximation n = 7. The wavenumbers shown straddle the Nyquist wavenumber used in computations. The fully nonlinear LPA solution closely resembles the exact decaying relation for both short wavenumbers tested, but results using the linearised functions show wild oscillations and large amplitudes above the still water level. This is because the linearised functions have collocation points which extend only from the bed to the still water level. Above this, the linearised basis functions essentially extrapolate the interpolated solution of Laplace's equation. In contrast, the fully nonlinear solution has collocation points which move with the instantaneous water surface (Eq. (6)). This ensures that solutions of the field equation are interpolations rather than extrapolations, as with the linearised solution. Possible solutions to this dilemma would be to increase the range of collocation points using the linearised functions to above the still water level, but this has its own problems. Both linear and nonlinear performance would deteriorate for any given level of approximation, perhaps enough to cancel out any increase in efficiency. No completely satisfactory solution has yet been found.

This instability to bottom disturbances is also of concern for higher order Boussinesq-type approximations, which extrapolate solutions of Laplace's equation using Taylor series. In fact, the higher order Boussinesq model of Gobbi et al. (1998) has also exhibited instabilities in computational tests with sharp corners (M. Gobbi, pers. commun.). Clearly, the extrapolation of a decaying function with a polynomial remains an area of difficulty.

Improved Accuracy: Revised Collocation

Although the efforts to improve efficiency in the previous section ultimately proved unworkable, results here were much more successful. The question was quite simple: for a given level of approximation, n, find the set of collocation points which best optimises certain analytic properties, and see if this translates to better performance for real computations.

To begin with, the analytic property chosen for comparison was small amplitude phase celerity, as it greatly influences accuracy in comparison with experiments such as Ohyama et al. (1995). The exact celerity was expanded in a Taylor series about the long wavelength limit kh = 0, and LPA celerities were made to match as many terms as possible. First of all, we will start with the low level of approximation, n = 3, although this level is of little practical use. One free collocation point is available for manipulation, and for any choice of collocation points, the LPA dispersion relationship is accurate to $O(kh^2)$. However, for the choice $\alpha_1 = 1/\sqrt{5}$, dispersion is accurate to $O(kh^4)$. In fact, the resulting dispersion relationship is the [2,2] Padé approximant found in Nwogu (1993) and Wei et al. (1995). For the level of approximation n = 5, three collocation points were available, while the base level of accuracy was $O(kh^4)$. With the choice of $\alpha_1 = 0.1255280883$, $\alpha_2 = 0.5008959415$, $\alpha_3 = 0.8419853513$, accuracy is increased to $O(kh^{10})$. For the level of approximation n = 7, however, no optimised approximation was found. This was due to the lack of a general solution for dispersion; i.e. dispersion relationships could be found for any particular set of α_i 's, but not for the general set. Solutions using the Gauss-Legendre points were close to optimal, but the possibility for improvements remains.

Figure 5 shows the original and improved dispersion relationships compared to the exact small amplitude solution. For the level n = 3, the resulting relationship is usable up to the nominal deep water limit of $kh = \pi$, which is a significant improvement. However, in contrast to the Gauss-Legendre relationship, phase speeds are greatly overpredicted for high wave numbers. For the level n = 5 errors in improved phase speed remain less than 1 percent until a dimensionless depth of approximately kh = 9. However, this level is still somewhat less accurate than the level of approximation n = 7, which has excellent accuracy. Clearly, dispersion relationships can be much improved.



Figure 5: LPA celerities using Gauss-Legendre (GLE) and new collocation points. Squares, GLE n = 3; stars, new collocation n = 3; +, GLE n = 5; triangles new collocation n = 5; x GLE n = 7

Nonlinear performance is improved as well. Figure 6 shows amplitudes of second harmonics for a steady wave relative to Stokes' solution. For all levels of approximation, results are much asymptotically much improved, although at higher wave numbers, the revised collocation points tend to show greater error. Subharmonics show a similar trend. Figure 7 shows a significant improvement in the accuracy of the steady wave setdown for the levels of approximation n = 3 and n = 5. Notably, for the level of approximation n = 5 using the new collocation points, absolute errors remains small over the entire range considered. Accuracy for the level of approximation n = 7, however, remains significantly better.

We will now consider computational results over the topography of Ohyama et al. (1995) which, as mentioned previously, is shown in Figure 3. Fortunately, computations with the revised collocation points showed none of the instabilities



Figure 6: Second harmonics for a second order steady wave compared to Stokes' solution

of the previous section. Full experimental results are available for comparison at both Stations 3 and 5. However, almost any nonlinear model will give good results at Station 3, while good agreement at Station 5 is much more difficult. Therefore, instead of showing results at Station 3, we will merely say that all comparisons were uniformly excellent, no matter what levels of approximation were used. Figure 8 shows results at Station 5 for the level of approximation n = 5, for both the original and revised collocation points. For case 2, the shortest wave, computations using the original collocation show some error, while this error is greatly reduced with the revised collocation. For case 4, computations show a similar trend, although the reduction in error is not as large. For case 6, agreement using the improved collocation is quite good, while results using the original formulation show higher harmonics which are visibly too large.

Figure 9 shows results for the level of approximation n = 7. For all cases considered, results are excellent, with experimental and computational results nearly identical. Once again, the higher level of approximation gives results somewhat better than the improved lower level.

Discussion and Conclusions

Analytic results and computational tests using the revised collocation points show significant improvements in accuracy for all levels of approximation. Since



Figure 7: Setdown under a second order wave compared to Stokes' solution. Symbols identical to previous figure.

a change in collocation points provides no increase in computational expense, there seems to be no reason why this new feature should not be adopted as a matter of routine.

It should be possible to increase nonlinear accuracy further by changing the collocation technique itself. If the definition for collocation points (6) is changed to

$$z_j = h(\alpha_j - 1) + \beta_j \eta \tag{10}$$

where α_j 's are unchanged, linear dispersion will be the same, but an additional parameter will be available to tune second or third order nonlinear characteristics. Although no comprehensive effort has yet been made to optimise β_k parameters, ad hoc tests show that significant improvement in nonlinear parameters is indeed possible. Figure 10 shows second harmonics computed using (10) using the set $\beta_1 = 0.0319$, $\beta_2 = 1.4018$, $\beta_3 = 1.4297$. Although this set is not fully optimised, significant improvement can be seen. However, manipulation of this sort introduces a dependency on the definition of the still water level which was not present in the previous fully nonlinear LPA solutions. Slight changes in the still water level from what was assumed could lead to significant changes in the model properties if (10) were used, negating any advantage that might be gained. Still, this idea may deserve further consideration.

Another possible area of investigation lies in what quantities should be op-



Figure 8: Experimental (—) and computed (---) time series at Station 5 for experiment of Ohyama et al., n = 5

timised for best results. Previously, linear dispersion was the only analytic quantity used, but the experience with the linearised B-splines suggests that it might be wise to include response to bottom disturbances in the optimisation criteria. Yet another candidate is linear shoaling, although this would involve a significantly more complicated analyis.

Practical performance of the linearised B-spline approximation was quite disappointing, and the solution does not appear to be simple. The basic problem of extrapolating a decaying function is fundamental, and requires careful thought. Several possible solutions were proposed in an earlier section, but all will require testing.

So in conclusion, it can be stated that revised collocation offers significant improvements in accuracy for all levels of approximation considered and should be used on a regular basis. However, the linearised B-spline approximation exhibits a strong nonlinear instability to short wavenumber bottom disturbances, and presently remains unworkable.

References

- Dold, J., and Peregrine, D., 1980 "Steep, unsteady water waves: an efficient computational scheme", Proc. Int. Conf. Coastal Eng., Houston, 955-967.
- Dommermuth, D.G, and Yue, D.K.P., 1987, "A high-order spectral method for the study of nonlinear gravity waves", J. Fluid Mech., 184, 267-288.



Figure 9: Experimental (--) and computed (--) time series at Station 5 for experiment of Ohyama et al., n = 7

- Gobbi, M.F., Kennedy, A.B., and Kirby, J.T., 1998, "A comparison of higher order Boussinesq and local polynomial approximation models", Proc. 26th Int. Conf. Coastal, Eng., Copenhagen.
- Gobbi, M. F., Kirby, J. T., and Wei, G., 1998 "A fully nonlinear Boussinesq model for surface waves: II. Extension to O(kh)⁴", J. Fluid Mech., Submitted.
- Grilli, S., Subramanya, R., Svendsen, I., and Veeramony, J., 1994, "Shoaling of solitary waves on plane beaches", J. Waterway, Port, Coastal and Ocean Eng. 120(6), 609-628.
- Kennedy, A. B., 1997 "The propagation of water waves over varying topography", Ph.D. Thesis, Monash University, Australia, 163pp..
- Kennedy, A. B. and Fenton, J. D., 1996 "A fully nonlinear 3D method for the computation of wave propagation", Proc. 25th Int. Conf. Coastal Eng., Orlando, 1102-1115.
- Kennedy, A. B. and Fenton, J. D., 1997 "A fully-nonlinear computational method for wave propagation over topography", *Coastal Eng.*, 32, 137-161.
- Nwogu, O., 1993 "An alternative form of Boussinesq equations for nearshore wave propagation", J. Waterway, Port, Coast., Ocean Engng, 119, 618-638.



Figure 10: Second harmonics for a second order steady wave compared to Stokes' solution

- Ohyama, T., Kiota, W., and Tada, A., 1995 "Applicability of numerical models to nonlinear dispersive waves", *Coastal Eng.*, 24, 297-213.
- Schäffer, H. A. and Madsen, P. A., 1995 "Further enhencements of Boussinesqtype equations", *Coastal Eng.*, 26, 1-14.
- Wang, N.T., Yao, Y., and Tulin, M., 1995, "An efficient numerical tank for nonlinear water waves, based on the multi-subdomain approach with BEM", Int. J. Numer. Meth. Fluids 20, 1315-1336.
- Wei, G., Kirby, J. T., Grilli, S. T., and Subramanya, R., 1995 " A fully nonlinear Boussinesq model for surface waves. Part 1. Highly nonlinear unsteady waves. J. Fluid Mech., 294, 71-92.