Wave Crest Interaction in Water of Intermediate Depth

C.C.Bird † and D.H.Peregrine ‡

Abstract

Numerical computations using a fully nonlinear potential flow solver are carried out on water of intermediate depth. Properties of single and multiple wave groups are examined on horizontal and sloping beaches. Comparisons are made with linear and weakly nonlinear theories. An important result is that the increase in wave amplitude due to wave shoaling is likely to be much less than the linear theory result for short steep wave groups because of the tendency for the group to spread.

Introduction

This paper is part of a series of studies of waves on a beach, which aims to learn more of finite amplitude effects on irregular waves. It follows Bird & Peregrine (1997) where attention is directed towards the long waves associated with and generated by a single wave group. Here the waves within one or more wave groups are studied in water of intermediate depth, that is for $1.36 > kh >\approx 0.7$, where h is the water depth and k is the wavenumber.

The characteristic behaviour of wave groups on deep water differs greatly from that on shallow water. On deep water the waves are dispersive, and thus their energy travels at a speed different from that of the individual waves. Typically the group and its energy travel at half the speed of the individual waves in that group (the group velocity concept). In wave groups where the wave number is constant, the individual waves do not meet, and their interaction is in the form of energy 'slipping' back through the group relative to the wave crests.

†Research Student, School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK (*C.C.Bird@bristol.ac.uk*)

‡Professor of Applied Mathematics, School of Mathematics, University of Bristol (*D.H.Peregrine@bristol.ac.uk*)

In shallow water, energy flows with the wave crests. In the Boussinesq region, which often occurs before the waves break, the waves are typically solitary type waves, and as is described by two-soliton solutions of the Korteweg-de-Vries equation, they may merge temporarily if a wave of high amplitude catches up a smaller one ahead of it. However, unless the waves break at the point at which they merge, they separate again, suffering only a change of phase. The case of two waves of almost equal amplitude is slightly different; the solution of the KdV equation indicates if such waves are close, then they do not merge, but exchange amplitudes and shift phases as if they had. Figure 1 shows an example of the first case, computed using a fully nonlinear potential flow solver (described in a later section). Successive surface elevation profiles are plotted, shifted with time. The initial conditions consist of a single group of 5 modulated solitary waves, a large central one and much smaller outer waves. A small amount of set down is added, but not enough to balance the mass of the waves. They are propagated on a beach of slope 1:60. It is clear to see that the largest wave merges with the one ahead of it. The waves have almost passed through each other when the largest one breaks.

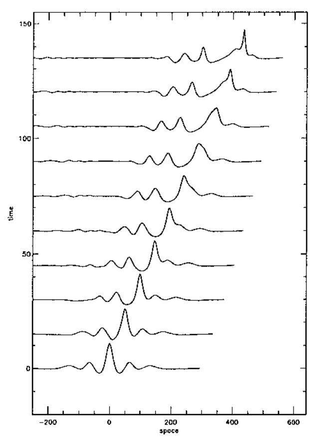

Figure 1: Surface elevation profiles of a single wave group progressing up a 1:60 beach.

After breaking, waves can be modelled by bores (discontinuites of the shallow water equations). Bores frequently catch up one another and merge. This has been observed in the field study carried out by Packwood and Peregrine in 1980; some details are given in Peregrine (1998). Unlike solitary waves however, a merged bore retains its unity. Peregrine (1974) uses invariants of the nonlinear shallow water equations to show that in this situation, a small backward travelling wave of depression is formed at the merger.

However, the present topic of interest is the region between deep water and the Boussinesq regime, ie. intermediate depth water. Barnes & Peregrine (1995) used a fully nonlinear potential flow model to study the progress of individual wave groups travelling up a plane beach in the context of low frequency waves. Instead of the converging type behaviour seen in figure 1, they noted that the group and its energy appeared to spread significantly, and also that the waves shoaled far less than is predicted by (regular wave) linear theory. We address these last points by carrying out computations using the same nonlinear potential flow solver. Both single and multiple wave groups are considered, on horizontal and sloping beaches. It is useful to examine the behaviour of groups over a horizontal bed in order to clarify the processes at work since then the effect of the slope is separated out. But first we briefly review the relevant theory for wave groups.

Brief review of applicable linear and weakly nonlinear theory

To model modulated waves it is useful to consider the wave elevation to be described by

$$\eta(x,t) = A(X,T)e^{i(kx-\omega t)} \;+\; complex\;conjugate \tag{1}$$

(eg. see Mei, 1983). Here X and T represent the 'slow' space and time coordinates, and ω and k are related by the dispersion equation

$$\omega^2 = gk\tanh(kh) \tag{2}$$

The water depth h is a function of the slow space variable, ie. $h(X)$, and g is the gravitational acceleration.

From the assumption of slow variations of the wave amplitude A, linear theory gives a first approximation:

$$\frac{\partial A}{\partial T} + c_g\frac{\partial A}{\partial X} = 0; \qquad c_g = \frac{\partial \omega}{\partial k} \tag{3}$$

This result corresponds to geometrical ray theory if extended to two dimensions. It gives the usual linear shoaling theory for varying depth, and indicates that a wave group simply translates with the group velocity c_g. As already noted, unless kh is small, the wave speed $c = \omega/k \neq c_g$.

Including the next linear terms in the slow modulation approximation gives

$$\frac{\partial A}{\partial T} + c_g\frac{\partial A}{\partial X} = \frac{i}{2}\omega''\frac{\partial^2 A}{\partial X^2}; \qquad \omega'' = \frac{\partial^2 \omega}{\partial k^2} \tag{4}$$

which is a Schrödinger equation, or a linear parabolic wave equation. The extra term on the right hand side gives an extra dispersion to the wave groups.

Including weakly nonlinear terms in the above approximation gives

$$i\frac{\partial A}{\partial T} + ic_g\frac{\partial A}{\partial X} + \frac{1}{2}\omega''\frac{\partial^2 A}{\partial X^2} = \lambda|A|^2 A \tag{5}$$

a nonlinear Schrödinger (NLS) equation. Here λ is a function of kh. In deep water, $\omega'' < 0$ and the equation has soliton solutions corresponding to special groups that propagate unchanged. This equation then gives the Benjamin-Feir modulational instability, which can also be described as 'self-focussing' of the waves. However for intermediate water depths, $kh < 1.36$, ω'' has the opposite sign, and the equation has a more stable 'defocussing' character. This aspect of wave modulations can be seen in the following computations with an accurate flow solver.

The numerical model

The computations were carried out using an accurate fully nonlinear irrotational flow solver. This is able to solve from deep water up to the first occurrence of wave breaking. The spatial domain is of finite length, bounded by uniform conditions.

The fluid is considered to be two-dimensional, inviscid, incompressible and irrotational, and surface tension is neglected. Defining points (x, y) on the free surface by $(X(s,t), Y(s,t)) = \mathbf{R}(s,t)$, where s is a particle-following (Lagrangian) coordinate, we have:

$$\nabla^2\phi = 0 \quad \text{in fluid body} \tag{6}$$

$$\frac{D\mathbf{R}}{Dt} = \nabla\phi \quad \text{on free surface} \tag{7}$$

$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla\phi|^2 - gY \quad \text{on free surface} \tag{8}$$

with the appropriate bed boundary conditions. The velocity potential is represented by $\phi(x, y, t)$.

If the spatial domain is a plane sloping beach, then the model first transforms this into a horizontal plane using a conformal mapping. The above system of equations is solved using a Cauchy theorem boundary integral, and hence the velocity at the free surface is computed. The surface is then stepped in time using a Taylor series truncated at the sixth order. Further details may be found in Tanaka *et al.* (1987) and Cooker (1990).

A single wave group propagated on a horizontal bed

Once one departs from considering a regular periodic wave train there is a huge range of incident waves that may be considered. For simplicity and uniformity we have chosen initial conditions in the form of a wave group with the surface properties of the deep water soliton of the same steepness, ie. to linear approximation

$$\eta(x,0) = \frac{1}{2}\left(a \operatorname{sech}\left(\sqrt{2}ak^2x\right)e^{ikx} + c.c.\right) \tag{9}$$

$$\phi(x,0) = -\frac{g}{2\omega}\left(ia \operatorname{sech}\left(\sqrt{2}ak^2x\right)e^{ikx} + c.c.\right). \tag{10}$$

The subsurface velocities adjust appropriately in the computation when the frequency ω comes from the linear dispersion relation given above (equation 2), chosen to be appropriate to the depth.

In the initial conditions the higher order corrections to equations (9) and (10) were not fully included. Usually only the second harmonic terms were added, thus generation of associated long waves occurs. See Bird & Peregrine (1997) for discussion of this feature.

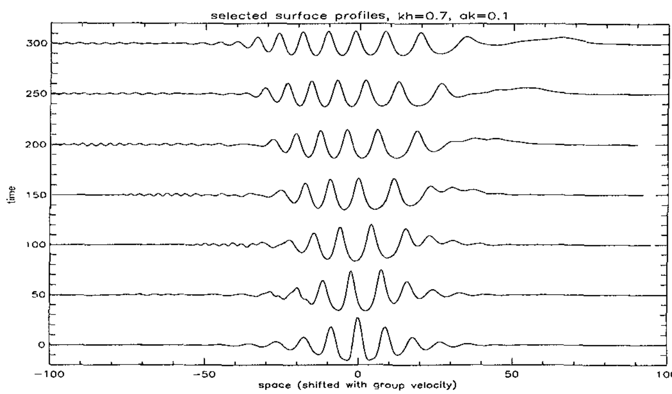

Figure 2: Selected surface profiles for single wave group progressing over a horizontal bed, in a frame of reference moving with the group velocity.

Results from a typical computation over a horizontal bed are given in figure 2. Selected surface profiles illustrate the progress of the free surface with time. Time in non-dimensionalised units ($t^* = t/\sqrt{h/g}$) is shown on the relevant axes. We summarize results from several such computations.

In every case the expected defocussing, or spreading out of the group is evident. Such behaviour might be expected to scale with the Ursell number, $Ur = a/k^2h^3$, but it depends most strongly on wave steepness ak, and less so on Ur or kh. As Peregrine (1983) notes, all NLS equations can be transformed to the same canonical form, with either a $+$ or $-$ sign for the nonlinear terms. The canonical equation can then have any solution transformed in the wave amplitude, space and time to give a whole family of solutions, depending on amplitude as a parameter. Here, because of the dependence of ω'' and λ on kh such transformations do not give any particularly simple result. However, they do show that if the waves are not too steep or long, e.g. $Ur < 0.3$ then this defocussing behaviour is to be expected from our knowledge of solutions of NLS equations.

Linear theory is appropriate for small waves at the margins of a wave group, and hence, as we can expect, for a lengthening group the waves at the front of each group are longer than those at the rear. The additional trailing group of very short waves comes from the waves' requirement for bound high harmonics that were not included in the initial conditions. Corresponding short free waves are emitted. Similarly, the long set-down wave was not included and a free long wave can be seen emerging from the front of the group, see Bird & Peregrine (1997) for more details.

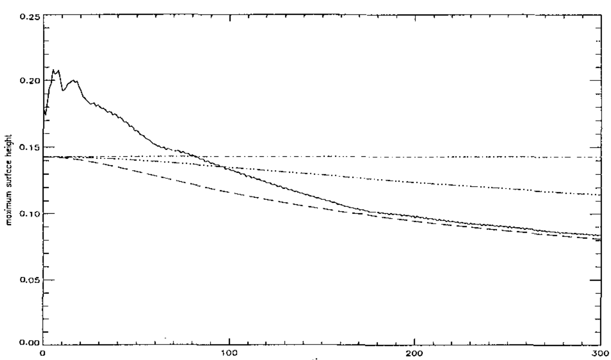

Figure 3: Maximum wave amplitude of a single wave group progressing over a horizontal bed; solid = fully nonlinear method, dash dot = basic linear theory, dash dot dot dot = linear theory including higher order dispersion, dash = weakly nonlinear theory (NLS).

The maximum surface height as computed by the fully nonlinear potential flow solver, is compared with the the theories described in the previous section (linear, linear with higher order dispersion, weakly nonlinear) in figure 3. A method by Taha & Ablowitz (1984) is used to compute solutions of the nonlinear Schrödinger equation; this is easily adjusted to compute the solutions to the linear equations. Since no higher order correction terms were added to the initial conditions when computing the solutions to the approximate theories, there is a difference in the initial heights of the surfaces.

A more accurate description of the value plotted for the fully nonlinear method is that it is of the highest grid point, and since no interpolation is carried out, this differs slightly from the maximum height of the surface. This results in the small oscillations evident on this plot. The larger oscillations are due to the difference in the phase and group velocities of the waves. Individual waves still pass through the group in the manner described above for deep water (although they no longer travel at twice the speed of the group), and thus the largest wave persists only for a limited time, marked by the length of these oscillations. These oscillations are not seen in the solutions to the theories since these give only details of the wave envelope rather than the individual waves.

We can see that the NLS equation gives a good approximation after the initial transients whereas the linear theories are poor. From consideration of other examples we find that the basic linear theory is adequate only for very gentle waves which, because of our special choice of initial conditions, also corresponds to very long groups. The linear theory with higher-order dispersion is noticeably better, and in a number of cases is almost as good as the NLS.

Three wave groups propagated on a horizontal bed

Here the initial conditions consist of three envelope soliton groups alongside each other, centred a distance of 6 or more wavelengths apart. These distances were chosen to ensure that the groups were sufficiently far enough apart to have initial identities, but not so far apart as to prevent significant interaction between them. The example presented has the same initial wave steepness and depth as that in the section on single groups above. Again terms of the second harmonics are added. This example is illustrated in figures 4 and 5.

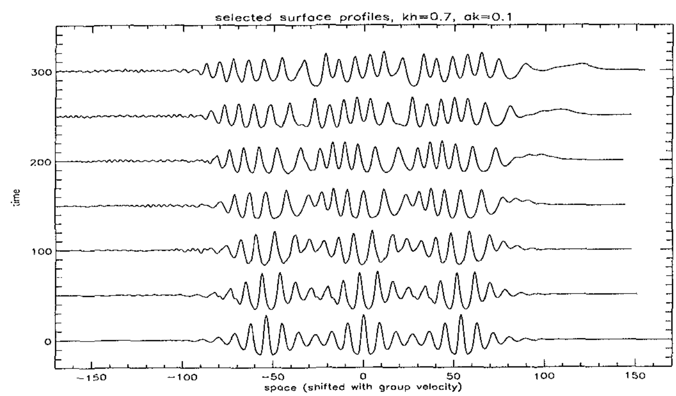

Figure 4: Selected surface profiles for three wave groups progressing over a horizontal bed in a reference frame moving with the group velocity.

Selected profiles illustrating the progression of the surface elevation in time are given in figure 4. Again the groups spread to varying degrees depending on their initial steepnesses, but the interactions between the waves slightly limit this effect. The individual waves interact with each other. Initially waves in each group are in phase, but the phase changes involved in the spreading out of the groups lead to significant irregularities as the groups overlap.

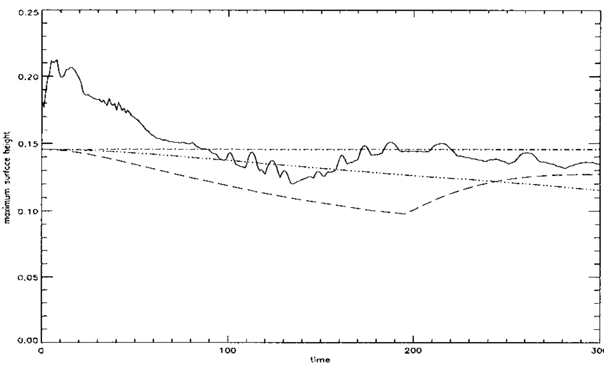

Figure 5: Maximum wave amplitude of three wave groups progressing over a horizontal bed; solid = fully nonlinear method, dash dot = basic linear theory, dash dot dot dot = linear theory including higher order dispersion, dash = weakly nonlinear theory (NLS).

The maximum surface height of the wave group as computed by the fully nonlinear potential flow solver is compared with the basic linear, linear with higher order dispersion and weakly nonlinear theories in figure 5. In the fully nonlinear plots we observe additional oscillations on different (and varying) lengthscales from the short and long ones discussed above for single wave groups. These are due to both the forward travelling free long waves passing through the groups and elevating individual waves for short periods of time, wave interference, and differing modulations giving the highest wave. This latter behaviour is the source of the large kink in each of the nonlinear curves in figure 5.

The basic linear theory is slightly more successful in predicting the maximum wave amplitude, as it is generally slightly higher than for an isolated group. However, the envelopes of the steeper groups do not progress without change of form, thus basic linear theory is still not appropriate for changes on these length scales. Again the weakly nonlinear theory and the linear higher order dispersion theory perform better. One of the reasons for this is the inability of the theories to model the free long waves and harmonics which emerge from the group. We have already noted that the long waves computed in the fully nonlinear computation pass through the group and elevate individual waves, thus effectively increasing the maximum surface elevation. The long waves are higher for the steeper groups, and so it is for these groups that we see the greater difference in maximum amplitude. However, modulation equations

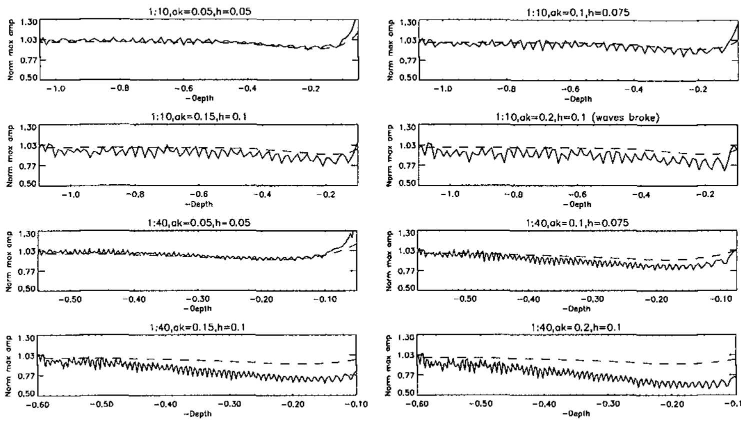

Figure 6: Normalised maximum amplitude of single wave groups progressing on a plane beach (solid line), shoaling coefficient (dashed line).

corresponding to the NLS level of approximation can model the long waves. Such free long waves and very short waves are also generated in real situations of depth variation.

So in summary, the multiple groups spread out slightly less than isolated groups of the same steepness on the same depth. The maximum surface elevation is higher, due to the interactions between the waves. Basic linear theory is again the least appropriate for the steeper groups. Both linear theory which includes higher order dispersion and the weakly nonlinear theory are more successful, but the free long waves formed by the group are important.

A single wave group propagated on a plane sloping beach

In this section we repeat and extend some of the computations of Barnes & Peregrine (1985). The bed topography is a sloping beach leading onto a shelf of constant depth. The initial condition is a single deep water envelope soliton, with no added higher order terms. Beach slopes of 1:10 and 1:40 are used. The beach corners are located at $x = 10$ for the 1:10 beach and $x = 20$ for the 1:40 beach. The depth of the water at the centre of the group's initial position may be computed by adding the shelf depth h (indicated in figures 6 and 9) to either 1.0 for the 1:10 beaches or 0.5 for the 1:40 beaches. This means that the groups start in a water depth of at least half a wavelength.

Figure 6 shows a comparison between the linear shoaling coefficient (dashed line) and the shoaling of the computed wave groups (solid line), calculated as the maximum surface elevation at each depth. The irregularity of the computational results is due both to the irregularities explained earlier, and to the finite intervals at which the computation is sampled. We note that the maximum surface elevation increases quite suddenly and significantly as the waves propagate on the constant depth shelf. It is easy to verify the conclusions of Barnes & Peregrine (1995), that shorter, steeper groups spread out and thus shoal much less than longer, less steep groups, and that this effect is more marked on gentle beaches. After considering the horizontal bed case, this conclusion is not surprising. The group does not encounter other waves around it as it spreads and thus cannot interact with any other waves. On a gentler beach the group spreads more since it has more time in which to spread.

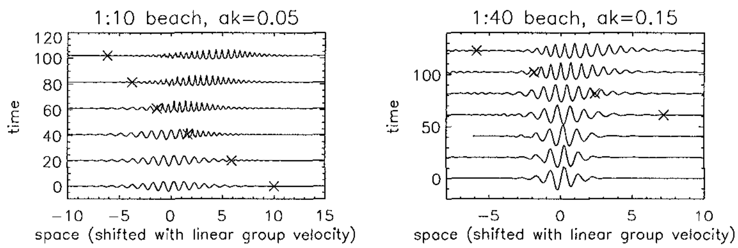

Figure 7: Selected surface profiles for single wave groups progressing on a plane beach. 'x' marks the beach corner position.

Selected surface profiles from two of the examples are shown in figure 7. The space coordinate has been shifted with linear group velocity, and thus we mark the beach corner position (ie. that point where the slope changes to constant depth shelf) by crosses. It is easy to see the transformation of the sinusoidal waves to near solitary type waves in the shallowest water.

Many of the features of the groups observed on the horizontal bed topography may be seen here too in our full range of examples. The steeper groups spread more than the shallower ones; again the wave groups formed of waves of steepness $ak = 0.05$ barely spread at all. Higher-order harmonic corrections are emitted from the back of the group, as before, but there is no easily visible evidence of free long waves until the group passes onto the shelf. Longer waves are towards the front and shorter waves towards the back of the group while it is on the sloping section, and the envelope becomes asymmetric. However as the waves pass onto the shelf, their wavelength reduces (in addition to their amplitude suddenly increasing). This is quite dramatic in the cases of the wave groups of steepness $ak = 0.05$, due to particularly shallow depth chosen for those shelves. On a plane beach without a shelf, breaking would occur just after the point where the corner is placed.

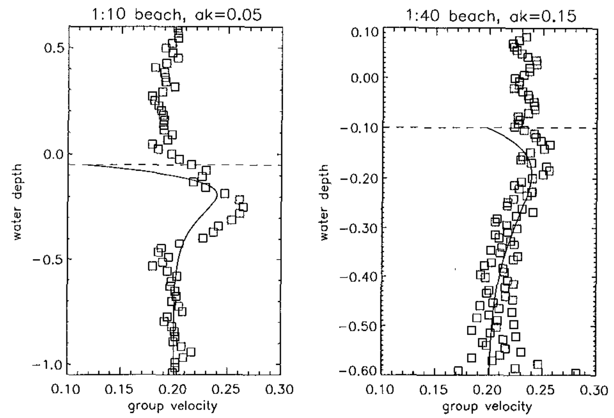

Figure 8: Speed of single wave group progressing on a plane beach (squares), linear group velocity (solid line).

It is easy to observe that the group does not always travel with the linear group velocity. The trend is for the group to travel a little slower than c_g shortly before it reaches the shelf, and faster as it passes onto and travels along the shelf. This may be observed more clearly in figure 8, which compares the speed of the computed groups (squares) with the linear group velocity (solid line). The dotted line represents the start of the shelf; above it the depth is constant (the markings on the vertical axis are inappropriate there).

The speed of the computed groups is estimated as follows: at each time, a Hilbert transform of the surface elevation is taken to give the group envelope. Since the waves are nonlinear, this envelope is very 'noisy', and so we use a Fourier filter to remove the higher harmonics. We denote the position at which this smoothed envelope has a maximum to be the centre of the group. Numerically differentiating the positions of the group centres with respect to time gives the group velocity.

It is not very surprising that the comparison is poor at the shallower depths; with the waves being more nonlinear there (behaving more as individual solitary-type waves) we can expect the linear group velocity concept to fail.

In summary, the maximum amplitude of a single wave group propagated up a sloping beach is almost always less than that predicted by the linear shoaling coefficient. This effect is most marked for steep groups on gentle beaches, but is not surprising, since the regular wave theory is least appropriate for short isolated wave groups. Many of the features of the groups observed on the horizontal beach are also seen on the sloping beach. The group travels with the linear group velocity in deeper water, but not in shallower water where we may expect the group velocity concept to fail.

Three wave groups propagated on a plane sloping beach

Here the initial conditions were formed of 3 adjacent envelope soliton groups, again with no higher harmonic corrections added, spaced by either 10 wavelengths ($ak =$ 0.05) or 6 wavelengths (other steepnesses). We choose the same combinations of beach slopes and wave steepnesses as for the single groups, except that for the case of beach slope 1:10, initial wave steepness $ak = 0.05$, the beach corner is now located at $x = 15.0$.

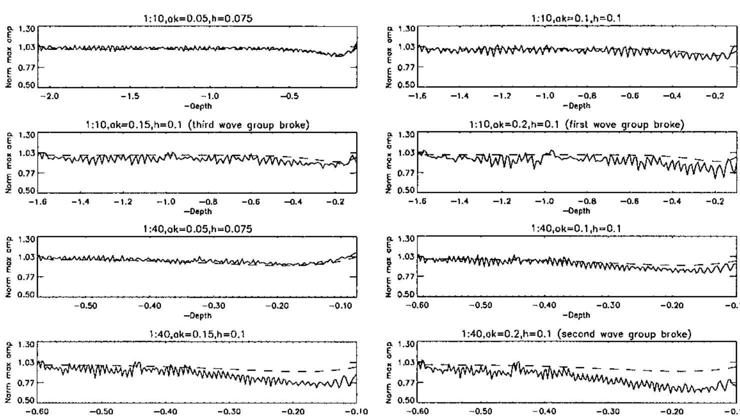

Figure 9: Normalised maximum amplitude of three wave groups progressing on a plane beach (solid line), shoaling coefficient (dashed line).

A comparison between the shoaling coefficient and the maximum height of the computed surface is shown in figure 9. The results are very similar to those for the isolated groups (see figure 6); the computed maximum amplitudes are generally slightly higher for 3 groups than for one, but the differences are quite small.

Examining the surface profiles shown in figure 10, it is not difficult to see why the changes are small: there is not enough time for any significant interactions between the waves to occur, unlike the horizontal bed case. Also there are no significant long waves to raise the individual waves of the group, as there was on the horizontal bed. The tendency of the very short steep groups to spread is very strong, with the maximum amplitude below the shoaling coefficient, as before.

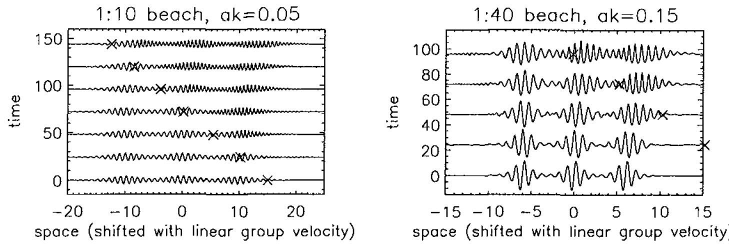

Figure 10: Selected surface profiles for three wave groups progressing on a plane beach. 'x' marks the beach corner position.

Conclusions

We have examined the behaviours of wave groups in water of intermediate depth. Features of both the deep water and Boussinesq type behaviours were evident; however the primary area of concern was the differences between the linear shoaling coefficient and the shoaling of the computed groups. It was noted that basic linear theory is the least appropriate for short, steep isolated groups and that on a horizontal bed, both linear theory including higher order dispersion and the weakly nonlinear theory give better results. The differences with the linear shoaling theory were explained by the tendency of the groups to spread and the lack of interaction with other waves; the differences are smaller when the initial conditions consisted of three wave groups instead of just one. An important result is that the increase in wave amplitude due to wave shoaling is likely to be much less than the linear theory result for short steep wave groups because of the tendency for the group to spread.

Acknowledgements

We are grateful for support from an EPSRC studentship, from the U.S Office of Naval Research, NICOP Grant N00014-97-1-0791 and European Commission research grant MAS3-CT97-0081 (SASME).

References

T.C.D. BARNES and D.H. PEREGRINE. Wave groups approaching a beach: Full irrotational flow computations. *Proc. of Coastal Dynamics, 116-127,* (1995).

C.C. BIRD and D.H. PEREGRINE Wave groups from Deep to Shallow water. *Proc. of Coastal Dynamics, 1013-1022,* (1997).

M.J. COOKER. A boundary integral method for water wave motion over irregular beds. *Engineering Analysis with Boundary Elements,* **7(4):** *178-195,* (1990).

C.C. MEI. The Applied Dynamics of Ocean Surface Waves. John Wiley & Sons, (1983).

D.H. PEREGRINE. Water-wave interaction in the surf zone. *Proc. 14th Coastal Eng. Conf.,* **1:** *500-517,* (1974).

D.H. PEREGRINE. Water waves, nonlinear Schrödinger equations and their solutions. *J. Austral. Math. Soc. Ser. B,* **25:** *16-43,* (1983).

D.H. PEREGRINE. Surf Zone Currents. *Theoretical and Computational Fluid Dynamics,* **10:** *295-309,* (1998).

T.R. TAHA and M.J. ABLOWITZ. Analytical and numerical aspects of certain nonlinear evolution equations II: Numerical, nonlinear Schrödinger equation. *J. Comp. Phys.,* **55:** *203-230,* (1984).

M. TANAKA, J.W. DOLD, M. LEWY and D.H. PEREGRINE. Instability and breaking of a solitary wave. *J. Fluid Mech.,* **185:** *235-248,* (1987).