Time-Dependent Depth-Integrated Turbulence Modeling of Breaking Waves

Kazuo Nadaoka1 and Osafumi Ono2

Abstract

A breaking wave model is presented to simulate non-equilibrium evolution of turbulence with a time-dependent depth-integrated equation of turbulence intensity, which is to be coupled with nonlinear wave. As the crucial part of the modeling, a method to estimate the production rate of turbulent kinetic energy is developed. This method is based on the experimental observation of the pseudo-periodic generation of large-scale eddies near the wave crest of turbulent bore. As a demonstration to show the performance of the present model, numerical examples are given for regular and irregular waves on a slope and regular waves on a double step, indicating that this model can give favorable features of evolution of regular and irregular breaking waves and the production and diffusion of turbulent kinetic energy.

Introduction

Although the recent progress in the developments of nonlinear-dispersive wave equations enables us to precisely describe even nonlinear evolution of irregular waves up to breaking, the broken wave simulation is still limited mainly because of the lack of proper turbulence model with wide applicability and sound physical background.

In the previous studies, many efforts have been made to incorporate the effect of wave breaking by introducing damping terms in a time-dependent wave model. Among these, e.g., Karambas and Koutitas (1992) presented a breaking wave model based on Boussinesq equations with a diffusion-type term, in which the damping factor is determined by time-averaged equation of the turbulence energy. However their model has a difficulty in further extension to unsteady wave field.

Schäffer et al. (1992) and Madsen et al. (1997) developed a breaking wave

¹ Prof., Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8552, Japan. (Email: nadaoka@mei.titech.ac.jp, Fax: +81-3-5734-2650)

² Graduate student, Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8552, Japan.

model based on a surface roller concept and a time domain formulation of Boussinesq-type equations. As the wave breaking criterion they assumed the presence of a critical value in the local water surface slope according to Deigaard (1989). However Nadaoka et al. (1997a) indicated through a laboratory experiment of irregular breaking waves that the front surface slope at the breaking exhibits appreciable scatter. Besides there is no experimental evidence to support the validity of the surface roller concept as the physical process, although it has been one of useful concepts for the breaking wave modeling. For example Nadaoka et al. (1989) found experimentally that the actual process of turbulence production is governed by a *pseudo-periodic* generation of large-scale eddies near the wave crest.

In the present paper, a new breaking wave model is presented to simulate nonequilibrium evolution of the turbulence with a time-dependent depth-integrated equation of turbulence intensity, which is to be coupled with nonlinear wave equations. As the crucial part of the modeling, a method is developed to estimate the production rate of turbulent kinetic energy, which governs the non-equilibrium development of turbulence and hence the breaking wave evolution. This method is based on the experimental observation of the pseudo-periodic generation of largescale eddies near the wave crest as mentioned above. As the wave breaking criterion the free-fall condition by Nadaoka et al. (1997a), which is based on the vertical pressure gradient near the wave crest, is adopted for the simulation. Numerical examples as a demonstration to show the performance of the present model are given for regular and irregular waves on a slope and regular waves on a double step.

Depth-Integrated Turbulence Modeling

Since the turbulence production of breaking waves is closely related with the instantaneous wave profile, the turbulence model developed in the present study is in a form to be *simultaneously* solved with the wave equations. In general, the existing equations to describe evolution of water waves such as Boussinesq equations are those of the depth-integrated type. Therefore, as a consistent way of formulation, the present turbulence model is to be of a depth-integrated type.

For the subject of suspended sediment transport in the surf zone, as one of important applications of the turbulence model, one needs to know the variation of turbulence intensity within a wave cycle for the evaluation of the sediment pick-up from the sea bottom. For such a purpose a time-dependent formulation is required for the turbulence modeling.

Besides in the turbulence development on a beach with an arbitrary profile like a stepped beach, the non-equilibrium aspect becomes crucial. Therefore both the turbulence production and dissipation must be properly estimated in the turbulence model.

For these requirements, the depth-integrated equation of the turbulent kinetic energy transport may be used, which is

$$\frac{\partial k}{\partial t} + \bar{u} \frac{\partial k}{\partial x} = P_k - \varepsilon + D_k, \qquad (1)$$

where k denotes the depth-averaged turbulent kinetic energy, \overline{u} the depth-averaged velocity, ε the energy dissipation rate, D_k the diffusion term and P_k the production rate of the turbulent kinetic energy. The energy dissipation rate, ε , may be evaluated with k and the turbulence length scale l according to the usual k-equation model,

$$\varepsilon = c^* \frac{k^{3/2}}{l},\tag{2}$$

The length scale *l* may be estimated according to Svendsen (1987),

$$l = 0.2h, \tag{3}$$

where h is the local water depth. The energy diffusion term is simply evaluated with the turbulent eddy viscosity, v_{ν} , as

$$D_k = \frac{\partial}{\partial x} \left(\frac{\nu_l}{\sigma_k} \frac{\partial k}{\partial x} \right). \tag{4}$$

For the model parameters, c^* and σ_k , in eqs.(2) and (4), the standard values, $c^*=0.17$, $\sigma_k=1.0$ are adopted. The eddy viscosity, v_t , is estimated as

$$v_t = k^{1/2} l$$
. (5)

Although eqs.(1) to (5) constitute a depth-integrated one-equation model^{*}, one may employ a two-equation model such as $k - \varepsilon$ model as the turbulence equations to be vertically integrated.

One of the crucial points for the present turbulence modeling is the estimation of the turbulence production near the bore crest, which governs the non-equilibrium development of the turbulence. The local production rate of the turbulent kinetic energy is usually estimated with the product of the local shear stress, τ , and the shear rate, $\partial u/\partial z$. Hnece the depth-averaged form of the turbulent energy production rate, P_k , may be obtained by the vertical integration of $\tau \partial u/\partial z$. However this idea, as described below in detail, may not be applicable directly to the modeling of the surf zone turbulence, in which the the large-scale eddy generation near the wave crest is the dominant turbulence source.

^{*} Recently Nwogu (1996) proposed a time-dependent breaking wave model based on Boussinesq-type equations, in which a one-equation type model is adopted to estimate turbulence development. The evaluation of the turbulence production is made by the assumption that it is proportional to the vertical gradient of the horizontal velocity at the wave crest and that the breaking process is of spilling-type.

Turbulence-Energy Production Modeling

In the initial development of turbulence at the breaking point, the entire process is governed by the generation of a large-scale eddy due to the water masss impingement at the front surface. This process, as illustrated in Fig.1, is a conversion from an irrotational motion in a simply connected region to the rotational flow in the doubly connected region with a circulation, Γ , at the moment when the water jet reaches the front surface. This circulating flow motion, in other words, the horizontal large-scale vortex involves ample vorticity and air bubbles in the water, providing subsequently smaller scale turbulence. In this process the initial turbulence generation may not be evaluated simply by $\tau \partial u/\partial z$. Our question is then what about in the inner region of the surf zone?

In the conventional modeling of the breaking waves, a concept of steady vortex attached at the bore front such as "surface roller" (Svendsen, 1984) as illustrated in Fig. 2 has been employed. However the actual process of the eddy formation at the bore front exhibits *pseudo-periodic* generation of large-scale eddies, which are advected behind the bore crest with appreciable diffusion (Fig. 3; Nadaoka et al., 1989). The essential feature of the development of each eddy is the same with that for the initial vortex development in terms of the generation of the circulation and associated supply of the vorticity and air bubbles.

Based on this observation, in the present model, the turbulent energy production rate, P_k , is estimated with the kinetic energy of a large-scale eddy, e_L , and its generation frequency per unit time, n; both of which may be evaluated with the circulation of the large-scale eddy, Γ_0 as described below.

The kinetic energy of a large-scale eddy, e_L , is estimated here with the Rankine eddy model,



(a) a simply connected region (b) a doubly connected region with Γ

Fig.1 Illustration of initial development of a large-scale eddy with the circulation Γ , which involves ample vorticity and air bubbles into the water.



Fig. 2 Illustration of the surface roller model.



Fig.3 *Pseudo-periodic* generation of large-scale eddies at the bore front (Nadaoka et al., 1989).

$$e_L \simeq \frac{\rho}{8} \Gamma_0^2, \tag{6}$$

where r_0 denotes the eddy radius. On the other hand, the number of eddies generated per unit time, n, may be evaluated simply as

$$n \simeq \frac{\Gamma_0}{2(\pi r_0)^2}.$$
(7)

Therefore, with the assumption that all the kinetic energy of the large-scale eddies is supplied to the turbulence, the production of kinetic turbulent enegy within a wave length per unit time, E_{L} , is

$$E_L = ne_L \cong \frac{\rho |\Gamma_0|^3}{16(\pi_0)^2}.$$
 (8)

The circulation, Γ_0 , itself is estimated by equating the vorticity production rate associated with the eddy generation, $n\Gamma_0$, with the mean production rate of the vorticity at the singular point of the bore front, A, as indicated in Fig. 4. The latter may be evaluated as

$$\Omega = \frac{(u_w - c)^2}{2} \tag{9}$$

where c denotes the bore celerity and u_w represents the velocity of orbital motion at A. Equating this with $\Omega = n\Gamma_0$, we obtain

$$\Gamma_0 = \pi r_0 (u_w - c). \tag{10}$$

Therefore the turbulence production rate, P_k , is estimated as,



Fig.4 Definition sketch of u_w in a turbulent bore.

$$P_k = \frac{\pi r_0 |(u_w - c)|^3}{16(h + \eta)L_1}$$
(11)

where L_1 represents the horizontal length of the vorticity supply region in a wave and is assumed here to be the horizontal length between the zero-up crossing point and the wave crest. Besides the eddy radius, r_0 , and the horizontal velocity, u_w , are simply assumed here to be the half of the crest height and zero, respectively. For further improvement of the evaluation of P_k , one should develop more reasonable way to specify the location of A and the corresponding values of L_1 , r_0 and u_w .

Depth-Integrated Wave Equation

Although the turbulence model described above may be coupled with any depth-integrated time-dependent wave equations including Boussinesq-type equations. In the present study, the new wave model developed by Nadaoka et al.(1994, 1997b) has been used. This wave model can describe nonlinear dispersive waves under general conditions, such as nonlinear random waves with a broad-banded spectrum at an arbitrary depth. Among the various versions of the model, the following multi-component equations are adopted here.

$$\frac{\partial \eta}{\partial t} + \sum_{m=1}^{N} \nabla \cdot \left[\left(\frac{\omega_m^2}{gk_m^2} + \eta \right) U_m \right] = 0, \qquad (12)$$

$$\sum_{m=1}^{N} A_{nm} \frac{\partial U_m}{\partial t} + B_n \nabla \left[g\eta + \eta \frac{\partial w_0}{\partial t} + \frac{1}{2} \left(u_0 . u_0 + w_0^2 \right) \right] = \frac{\partial}{\partial t} \sum_{m=1}^{N} \left[C_{nm} \nabla (\nabla \cdot U_m) + D_{nm} (\nabla \cdot U_m) \right] + \sum_{m=1}^{M} A_{nm} v_t \nabla^2 U_m \qquad (n = 1, 2, \cdots, N) \qquad (13)$$

where,

$$\omega_m^2 = gk_m \tanh k_m h, \quad A_{nm} = \frac{\omega_n^2 - \omega_m^2}{k_n^2 - k_m^2}, \quad A_{nn} = \frac{g\omega_n^2 + h(g^2k_n^2 - \omega_n^4)}{2gk_n^2},$$
$$B_n = \frac{\omega_n^2}{k_n^2}, \quad C_{nm} = \frac{B_n - A_{nm}}{k_m^2}, \quad D_{nn} = \nabla C_{nn},$$
$$D_{nm} = \frac{2}{k_m^2 - k_n^2} \left[\frac{2\nabla k_m}{k_m} \left\{ A_{nm} - \left(k_m^2 - k_n^2 \right) C_{nm} \right\} + \frac{\nabla h \sqrt{\left(g^2k_n^2 - \omega_n^4 \right) \left(g^2k_m^2 - \omega_m^4 \right)}}{gk_n k_m} \right].$$
(14)

 u_0 and w_0 in eq.(12) are the velocities at z=0 and may be evaluated as

$$\boldsymbol{u}_0 = \sum_{m=1}^N \boldsymbol{U}_m, \quad \boldsymbol{w}_0 = -\sum_{m=1}^N \nabla \cdot \left(\frac{\boldsymbol{B}_m}{\boldsymbol{g}} \boldsymbol{U}_m\right). \tag{15}$$

The last term in eq.(13) represents the damping effect due to wave breaking.

Wave Breaking Criterion

By developing a method to experimentally evaluate pressure field of irregular waves with a time sequence of the wave profile data, Nadaoka et al. (1997a) found out that at the wave breaking the vertical pressure gradient near the wave crest becomes nearly zero even for irregular waves. This fact may be used as a general wavebreaking criterion which is applicable to regular or irregular and progressive or standing waves. Nadaoka, et al. (1997a) called this criterion "free-fall condition" and showed that the breaker depths computed by eqns. (12) and (13) with this breaking criterion. Therefore, in the present study, this free-fall condition has been used to detect the breaking point and to impose the production rate of turbulent kinetic energy, P_k , according to the occurrence of the wave breaking; i.e.,

$$\begin{aligned} &\text{if } -\frac{1}{\rho g} \frac{\partial p}{\partial z} \Big|_{crest} \le 0, \quad \text{then } P_k \neq 0, \\ &\text{if } -\frac{1}{\rho g} \frac{\partial p}{\partial z} \Big|_{crest} > 0, \quad \text{then } P_k = 0. \end{aligned}$$
(16)

Numerical Examples and Discussion

To demonstrate the fundamental performance of the present model, numerical simulations for various conditions of regular and irregular waves were made both for uniform slope and stepped bottom.

As the most fundamental case, Fig.5 shows the evolution of the spatial distribution of the water surface elevation η and the turbulent kinetic energy k in case of regular waves with T=8s, $H_0/L_0=0.02$ on a uniform slope of 1:50. At the shore-side end of the slope a shallow horizontal section was connected for better computational performance in the treatment of outgoing waves. The generation of turbulence near the wave crest and its subsequent decay behind the wave crest are reasonably simulated. Figure 6 represents the results for the irregular waves with the Bretschneider spectrum having the significant wave height of 8s. The irregular feature of the occurrence of the turbulence energy. Figure 7 shows the results for regular waves with T=8s, $H_0/L_0=0.02$ on a double step. This is a typical case in which the non-equilibrium feature of the wave evolution becomes appreciable due to the imbalance between the turbulence intensity attenuates with the corresponding reform of waves on the step, which is followed by another breaking on the second slope.

Conclusion

With the time-dependent depth-integrated formulation of turbulence, which is to be coupled with nonlinear wave equations, and the development of a method to estimate the turbulence production due to wave breaking, a new model for the breaking wave simulation is presented in this study. As a governing factor to describe non-equilibrium turbulence evolution of breaking waves on a beach with arbitrary profile, the turbulence production is modeled based on *pseudo-periodic* generation of large-scale eddies at the bore front. Through numerical simulations, the present model has been demonstrated to give favorable features of evolution of regular and irregular breaking waves and the production and diffusion of turbulent kinetic energy.

References

Deigaard, R. (1989): Mathematical modelling of waves in the surf zone, *Prog. Rep.* 69, ISVA, Technical Univ., Lyngby, pp.47-59.

Goda, Y. (1970): A synthesis of breaker indices, *Trans. Japan Soc. Civil Eng.*, No.180, pp.39-49. (in Japanese)

Karambas, T.V. and Houtitas, C. (1992): A breaking wave propagation model based on the Boussinesq equations, *Coastal Eng.*, Vol. 18, pp.1-19.

Madsen, P.A., Sørensen, O.R. and Schäffer, H.A. (1997): Surf zone dynamics simulated by a Boussinesq type model. Part I. Model description and cross-shore motion of regular waves, *Coastal Eng.*, Vol. 32, pp.255-287.

Nadaoka, K., Hino, M. and Koyano, Y. (1989): Structure of the turbulent flow field under breaking waves in the surf zone, *J. Fluid Mech.*, vol.204, pp.359-387.

Nadaoka, K., Beji, S. and Nakagawa, Y. (1994): A fully-dispersive nonlinear wave model and its numerical simulations, *Proc. 24th Int. Conf. on Coastal_Eng.*, ASCE, pp.427-441.

Nadaoka, K., Ono, O. and Kurihara, H. (1997a): Near-crest pressure gradient of irregular water waves approaching to break, *Proc. Coastal Dynamics* '97. pp.255-264.

Nadaoka, K., Beji, S. and Nakagawa, Y. (1997b): A fully dispersive weakly nonlinear model for water waves, *Proc. Royal Society of London A*, Vol.453, pp.303-318.

Nwogu, O.K. (1996): Numerical prediction of breaking waves and currents with a Boussinesq model, *Proc. 25th Int. Conf. on Coastal Eng.*, ASCE, pp.4807-4820.

Schäffer, H.A., Deigaard, R. and Madsen, P.A. (1992): A two-dimensional surf zone model based on Boussinesq equations, *Proc. 24th Int. Conf. on Coastal Eng.*, ASCE, pp.576-589.

Svendsen, I.A. (1984): Wave heights and setup in a surf zone, *Coastal Eng.*, Vol.8, pp.303-329.

Svendsen, I.A. (1987): Analysis of surf zone turbulence, J. Geophys. Res., 92, pp.5115-5124.



Fig. 4 Regular waves breaking on a uniform slope.



Fig. 5 Irregular waves breaking on a uniform slope.



Fig. 6 Regular waves breaking on a stepped beach.