CHAPTER 87

A Fourth Order Boussinesq-Type Wave Model

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Abstract

A fully nonlinear Boussinesq-type model with dispersion accurate to $O((kh)^4)$ is derived. As an extension to the second order extended model proposed by Nwogu (1993), a new dependent variable is defined as a weighted average between the velocity potential at two distinct water depths to force the model to have a (4,4) Padé approximation of the exact dispersion relationship. The present model is similar to the fully nonlinear extension of Nwogu's model proposed by Wei et al (1995), except that the dependent variable is expanded in a fourth (rather than second) order polynomial in the vertical coordinate.

Introduction

Important progress has been made in variable-depth Boussinesq-type models since the development of the more-or-less standard model of Peregrine (1967). Madsen et al (1991) introduced higher order dispersive terms into the governing equations to improve linear dispersion properties. By redefining the dependent variable, Nwogu (1993) achieved the same improvement without the need to add such terms to the equations. Wei et al (1995, referred to as WKGS) used the approach of Nwogu to derive a Boussinesq-type model which retains full nonlinearity. Numerical computations show that the WKGS model agrees well with solutions of the full potential problem over the range of relevent water depths, except for some discrepancies in the vertical profile of horizontal velocity in nearly-breaking waves. These inaccuracies in the prediction of vertical profiles in existing Boussinesq-type models are due to the fact that they assume the velocity profiles to be second order polynomials in the vertical coordinate z. In this paper, we derive a fourth order Boussinesq model in which the velocity potential is approximated by a fourth order polynomial in z. A new dependent variable is defined to be the weighted average of the velocity potential at 2 different elevations in the water column, and the weight and positions are chosen to give a (4,4) Padé approximant of the exact linear dispersion relationship.

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Derivation of New Model

The full boundary value problem for potential flow is given in terms of nondimensional variables by

$$\phi_{zz} + \mu^2 \nabla^2 \phi = 0; \quad -h \le z \le \delta\eta \tag{1}$$

$$\phi_z + \mu^2 \nabla h \cdot \nabla \phi = 0; \quad z = -h \tag{2}$$

$$\eta + \phi_t + \frac{\delta}{2}\left[(\nabla\phi)^2 + \frac{1}{\mu^2}(\phi_z)^2\right] = 0; \quad z = \delta\eta \tag{3}$$

$$\eta_t + \delta \nabla\phi \cdot \nabla\eta - \frac{1}{\mu^2}\phi_z = 0; \quad z = \delta\eta \tag{4}$$

Here, x and y are the horizontal coordinates scaled by a representative wave number $k_0 = 2\pi/L_0$, z is the vertical coordinate starting at the still water level and pointing upwards and h is the water depth, both scaled by a typical depth h_0. η is the water surface displacement and scaled by a representative amplitude a. Two dimensionless parameters are apparent; $\delta = a/h_0$ and $\mu^2 = (k_0 h_0)^2$. Time t is scaled by $(k_0(gh_0)^{1/2})^{-1}$, and ϕ, the velocity potential, is scaled by $\delta h_0 (gh_0)^{1/2}$. We integrate (1) over the water column and use (2) and (4) to obtain a mass conservation equation

$$\eta_t + \nabla \cdot \mathbf{M} = 0; \qquad \mathbf{M} = \int_{-h}^{\delta\eta} \nabla\phi dz. \tag{5}$$

For simplicity, we assume a constant depth h_0; the variable depth model can be derived in straightforward manner and is presented in Gobbi et al (1996). We assume a fourth order polynomial approximation for ϕ and choose the coefficients to satisfy the bottom boundary condition (2) and Laplace's equation (1), retaining terms up $O(\mu^4)$. The approximate potential is given by (Mei, 1989)

$$\phi = \phi_0 - \frac{\mu^2(1+z)^2}{2}\nabla^2\phi_0 + \frac{\mu^4(1+z)^4}{24}\nabla^2\nabla^2\phi_0 + O(\mu^6) \tag{6}$$

where ϕ_0 is the velocity potential at the bottom. Commensurate with the extension of the velocity potential to $O(\mu^4)$, we seek to derive a set of model equations having a corresponding dispersion relation in the form of a (4,4) Padé approximant, given by

$$\frac{\tanh\mu}{\mu} = \frac{1 + (1/9)\mu^2 + (1/945)\mu^4}{1 + (4/9)\mu^2 + (1/63)\mu^4} + O(\mu^6) \tag{7}$$

For the case of approximations retaining terms to $O(\mu^2)$, the goal of obtaining the corresponding (2,2) Padé approximant is achieved by redefining the velocity potential in terms of the value of the potential at an elevation $z_\alpha = h[(1+2\alpha)^{1/2} - 1]$; $\alpha = -2/5$ and using the resulting reference value ϕ_α as the dependent variable; see Nwogu (1993), Chen and Liu (1995) and Kirby (1996). This procedure is not adequate in the present context. Instead, we define a new dependent variable

$$\tilde{\phi} = \beta\phi_a + (1-\beta)\phi_b \tag{8}$$

where ϕ_a and ϕ_b are the velocity potentials at elevations $z = z_a$ and $z = z_b$, and β is a weight parameter. $\tilde{\phi}$ may be written in terms of ϕ_0 using (6) to obtain

$$\tilde{\phi} = \phi_0 - \frac{\mu^2}{2}B\nabla^2\phi_0 + \frac{\mu^4}{2}D\nabla^2\nabla^2\phi_0 + O(\mu^6) \tag{9}$$

where

$$\begin{aligned} B &= \beta(1+z_a)^2 + (1-\beta)(1+z_b)^2 \\ D &= \beta(1+z_a)^4 + (1-\beta)(1+z_b)^4 \end{aligned} \tag{10}$$

Inverting (9) gives a formula for ϕ_0 in terms of $\tilde{\phi}$ which is substituted into (6) to get an approximation to the full velocity potential in terms of $\tilde{\phi}$:

$$\begin{aligned} \phi = \tilde{\phi} &+ \frac{\mu^2}{2}\left\{B - (1+z)^2\right\}\nabla^2\tilde{\phi} \\ &+ \frac{\mu^4}{4}\left\{B^2 - B(1+z)^2 - \frac{D}{6} + \frac{(1+z)^4}{6}\right\}\nabla^2\nabla^2\tilde{\phi} + O(\mu^6). \end{aligned} \tag{11}$$

Defining the total depth $H = 1 + \delta\eta$, and substituting (11) into (5) gives a mass flux conservation equation for $\tilde{\phi}$ and η:

$$\begin{aligned} \eta_t &+ \nabla\cdot\left\{H\left[\nabla\tilde{\phi} + \frac{\mu^2}{2}\left(B - \frac{H^2}{3}\right)\nabla\left(\nabla^2\tilde{\phi}\right)\right.\right. \\ &+ \left.\left.\frac{\mu^4}{4}\left(B^2 - B\frac{H^2}{3} - \frac{B}{6} + \frac{H^4}{30}\right)\nabla\left(\nabla^2\nabla^2\tilde{\phi}\right)\right]\right\} = O(\mu^6). \end{aligned} \tag{12}$$

Next we substitute (11) into (3) to obtain an approximate Bernoulli equation, given by

$$\begin{aligned} &\eta + \tilde{\phi}_t + \frac{\mu^2}{2}\left\{B - H^2\right\}\nabla^2\tilde{\phi}_t + \frac{\mu^4}{4}\left\{B^2 - BH^2 - \frac{B}{6} + \frac{H^4}{6}\right\}\nabla^2\nabla^2\tilde{\phi}_t \\ &+ \frac{\delta}{2}\left[(\nabla\tilde{\phi})^2 + \mu^2\left\{B - H^2\right\}\nabla\tilde{\phi}\cdot\nabla(\nabla^2\tilde{\phi}) + \mu^2 H^2\left(\nabla^2\tilde{\phi}\right)^2\right. \\ &+ \frac{\mu^4}{2}\left\{B^2 - BH^2 - \frac{B}{6} + \frac{H^4}{6}\right\}\nabla\tilde{\phi}\cdot\nabla\left(\nabla^2\nabla^2\tilde{\phi}\right) \\ &+ \frac{\mu^4}{4}\left\{B^2 - 2BH^2 + H^4\right\}\left\{\nabla(\nabla^2\tilde{\phi})\right\}^2 \\ &+ \left.\mu^4\left\{BH^2 - \frac{H^4}{3}\right\}(\nabla^2\tilde{\phi})(\nabla^2\nabla^2\tilde{\phi})\right] = O(\mu^6). \end{aligned} \tag{13}$$

If we neglect μ^4 terms from (12) and (13) and set $\beta = 1$, we recover the WKGS model with Nwogu's α being related to B by

$$B = 2\alpha + 1 \tag{14}$$

If, in addition, we neglect products $\delta\mu^2$ or higher, we recover Nwogu's model in the velocity potential form given by Chen and Liu (1995).

Linear Dispersion Properties

Neglecting all terms containing δ in (12) and (13), we obtain the following linear equations for mass flux conservation,

$$\begin{aligned}\eta_t &+ \nabla^2\tilde{\phi} + \frac{\mu^2}{2}\left(B - \frac{1}{3}\right)\nabla^2\nabla^2\tilde{\phi} \\ &+ \frac{\mu^4}{4}\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right)\nabla^2\nabla^2\nabla^2\tilde{\phi} = 0 \end{aligned} \qquad (15)$$

and an approximate Bernoulli equation,

$$\begin{aligned}\eta &+ \tilde{\phi}_t + \frac{\mu^2}{2}(B-1)\nabla^2\tilde{\phi}_t \\ &+ \frac{\mu^4}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\nabla^2\nabla^2\tilde{\phi}_t = 0 \end{aligned} \qquad (16)$$

To analyse the dispersion properties of these equations, we assume the following general solution to the equations:

$$\eta = ae^{i(\mathbf{x}-\omega t)} \qquad \tilde{\phi} = be^{i(\mathbf{x}-\omega t)} \qquad (17)$$

where ω in the angular frequency nondimensionalized by $k_0(gh_0)^{1/2}$, a and b are amplitudes, and $i = \sqrt{-1}$. Substituting (17) into (15) and (16) we obtain the linear dispersion relationship for the model:

$$\omega^2 = \frac{1 - \frac{1}{2}\left(B - \frac{1}{3}\right)\mu^2 + \frac{1}{4}\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right)\mu^4}{1 - \frac{1}{2}(B-1)\mu^2 + \frac{1}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\mu^4} \qquad (18)$$

The expression (18) is the (4,4) Padé approximant to the exact linear dispersion relationship $\omega^2 = \tanh\mu/\mu$ if we set $B = 1/9$ and $D = 5/189$. The parameters β, z_a, and z_b are chosen in order to obtain these values. Since we have 3 unknowns and 2 equations, there are an infinite number of solutions that give the desired values of B and D. However, an arbitrary choice of β can give imaginary values of z_a or z_b or values lying outside of the fluid domain, making these parameters lack physical significance. It can easily be shown that values of β between 0.018 and 0.467 will give both z_a and z_b to be real values lying inside the water column. In the present paper we arbitrarily choose $\beta = 0.2$, and solve for z_a and z_b to give us the (4,4) Padé approximant to the exact linear dispersion relationship.

Figure (1) shows the ratio of modelled phase speed with Airy's exact linear solution for the standard Boussinesq theory based on depth-averaged velocity, the (2,2) Padé approximant formulation (referred to as Nwogu's formulation for simplicity), and the (4,4) Padé approximant dispersion relationship (referred to as the present formulation). It is clear that the present model has improved linear dispersion properties over Nwogu's already accurate Nwogu's model, and closely reproduces the exact solution through intermediate to deep water. Similarly, the linear group velocity, defined as $C_g = \partial\omega/\partial k$ is shown in figure (2) and the improvement in the present model over Nwogu's model is even more evident.

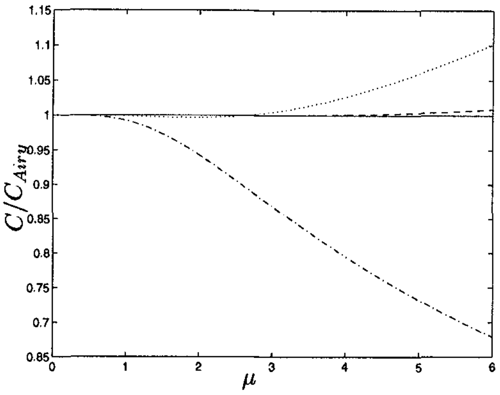

Figure 1: Ratio of model phase speed and Airy's exact linear solution. Standard Boussinesq (dash-dotted), Nwogu's (2,2) Padé approximant(dotted), Present (4,4) Padé approximant(dashed).

Nwogu (1993) found that the range and accuracy of the (2,2) Padé formulation could be extended by adjusting the model coefficients using an error minimization procedure. In the present case, the authors found that the error surface in the neighborhood of the (4,4) Padé approximant is sufficiently flat so that further adjustment of the model parameters is unwarrented.

Internal Kinematics

The internal kinematics of the present model can be obtained from (11). We define a function $f_1(z)$ as the the velocity potential normalized by its value at position $z = 0$:

$$f_1(z) = \frac{1 - \frac{\mu^2}{2}\left[B - (1+z)^2\right] + \frac{\mu^4}{4}\left[B^2 - B(1+z)^2 - \frac{D}{6} + \frac{(1+z)^4}{6}\right]}{1 - \frac{\mu^2}{2}\left[B - 1\right] + \frac{\mu^4}{4}\left[B^2 - B - \frac{D}{6} + \frac{1}{6}\right]} \tag{19}$$

The vertical velocity component w can be obtained by differentiating (11) with respect to z. Similarly to f_1, a vertical velocity profile function can be obtained by defining $f_2(z) = w(z)/w(0)$:

$$f_2(z) = \frac{\mu^2\left[(1+z)\right] + \frac{\mu^4}{2}\left[-B(1+z) + \frac{(1+z)^3}{3}\right]}{\mu^2 + \frac{\mu^4}{2}\left[-B + \frac{1}{3}\right]} \tag{20}$$

The corresponding f_2 from the exact linear theory is $\sinh[\mu(1+z)]/\sinh[\mu]$.

Figure (3) shows comparisons of $f_1(z)$ between the exact linear solution $\cosh[\mu(1+z)]/\cosh[\mu]$, Nwogu's model and the present model, for various values of μ. For moderately shallow water, the two models reproduce the exact solution quite well. As μ increases, Nwogu's model starts to deviate strongly, while the present model remains very accurate until quite deep water.

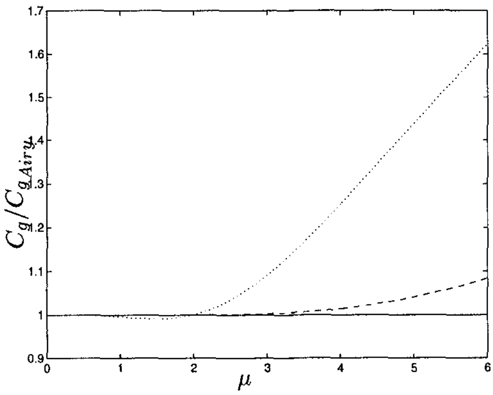

Figure 2: Ratio of model group velocity and Airy's exact linear solution. Nwogu (dotted), Present (dashed)

Figure (4) shows plots similar to figure (3) for f_2. Notice that Nwogu's model has a linear vertical profile for w, a poor representation in intermediate to deep water. The present model stays close to the exact solution for a wide range of μ. Finally, figure (5) shows the ratio to the exact linear solution $\tanh(\mu)$ of the ratio between vertical and horizontal velocities w/u at $z = 0$, $f_3(\mu)$, for the present model and Nwogu's model. The approximate expression for f_3 is:

$$f_3(\mu) = \frac{w(z=0)}{u(z=0)} = \frac{\mu + \frac{\mu^3}{2}\left[-B + \frac{1}{3}\right]}{1 - \frac{\mu^2}{2}[B-1] + \frac{\mu^4}{4}\left[B^2 - B - \frac{D}{6} + \frac{1}{6}\right]} \tag{21}$$

The present model agrees better with the exact linear solution than Nwogu's model for a wide depth range.

Second Order Nonlinear Interactions

In the previous sections we have seen that the proposed model has excellent linear dispersion properties as well as a greatly improved representation of the internal flow kinematics. It is useful to analyse some of the nonlinear properties of the model by using analytical tools such as Stokes' type asymptotic expansions and multiple scales expansions. Since these types of analysis have been extensively applied and studied for the full boundary value potential problem, we can obtain an idea of how well the nonlinear version of the present model would perform by comparing some of its nonlinear properties with those of the full problem, and also with WKGS and Nwogu's model, keeping in mind that a numerical implementation of WKGS model has already been tested and compared to data with success. We will now look at the generation of super- and subharmonics by second order Stokes-type interactions. It is well known that in intermediate and deep water the first nonlinear correction of a linear wave solution is a set of bound waves called the superharmonics (resulting from sum-wave

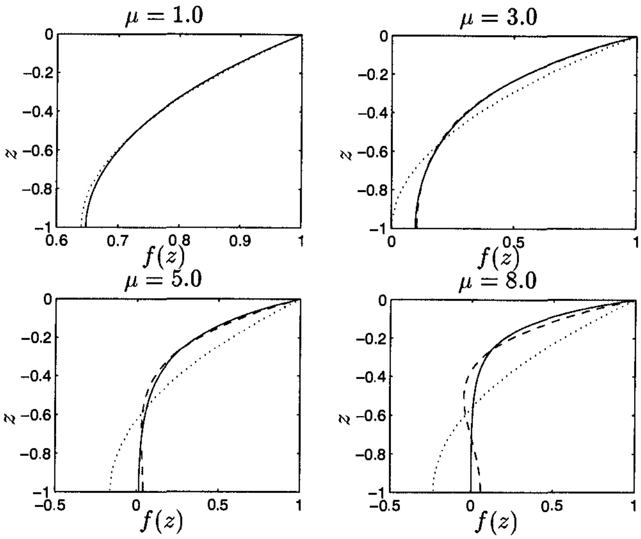

Figure 3: Normalized verical profile of linear horizontal kinematics for (a) $\mu = 1$, (b) $\mu = 3$, (c) $\mu = 5$, (d) $\mu = 8$. Exact (solid), Nwogu (dotted), Present (dash)

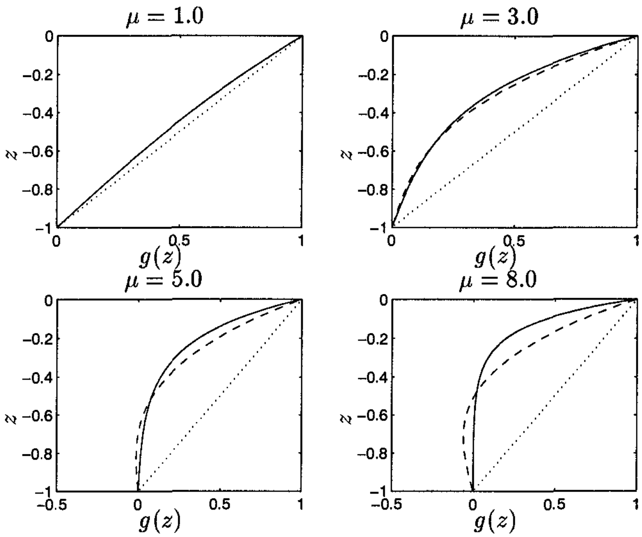

Figure 4: Normalized verical profile of linear verical velocity for (a) $\mu = 1$, (b) $\mu = 3$, (c) $\mu = 5$, (d) $\mu = 8$. Exact (solid), Nwogu (dotted), Present (dash)

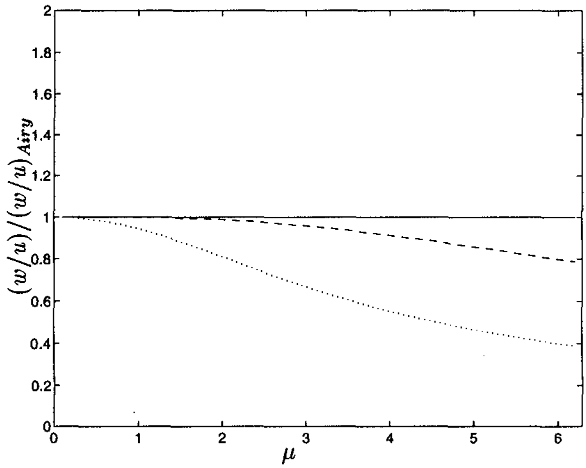

Figure 5: Ratio of approximate results for $w(0)/u(0)$ to the exact linear solution. Nwogu and WKGS (dotted), Present solution (dash).

interactions) and corresponding subharmonics (resulting from difference-wave interactions) (Hasselmann, 1962). These bound waves are proportional to products of the amplitudes of solutions to the linear equations. The constants of proportionality (which are functions of the local depth) will be referred to as transfer coefficients. Nwogu (1993) has investigated the generation of these bound waves in his extended Boussinesq model and found qualitatively reasonable agreement with Stokes' theory. Madsen and Sørensen (1993) have found similar results. Kirby and Wei (1994) extended Nwogu's model to full nonlinearity and found that the retention of terms proportional to $\delta\mu^2$ (which are neglected in Nwogu's model and the standard Boussinesq model by assumption) is essential for a prediction of the transfer coefficients to the level of accuracy implied by the order of retained dispersive terms in the original model equations. Here, we derive the transfer coefficients for the present model and compare to results from previous models.

We investigate nonlinear properties of the present model by introducing the perturbation expansion:

$$\begin{aligned} \eta &= \eta_0 + \delta\eta_1 + \delta^2\eta_2 \\ \tilde{\phi} &= \phi_0 + \delta\phi_1 + \delta^2\phi_2 \end{aligned} \tag{22}$$

into (12) and (13), and order the equations in powers of δ. At each order $O(\delta^n)$ we obtain:

$$\begin{aligned} \eta_{nt} + L_1\phi_n &= F_n \\ \eta_n + L_2\phi_{nt} &= G_n \end{aligned} \tag{23}$$

where L_1 and L_2 are the linear operators:

$$\begin{aligned} L_1 &= \nabla^2 + \frac{\mu^2}{2}\left(B - \frac{1}{3}\right)\nabla^2\nabla^2 \\ &+ \frac{\mu^4}{4}\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right)\nabla^2\nabla^2\nabla^2 \end{aligned} \tag{24}$$

$$\begin{aligned} L_2 &= 1 + \frac{\mu^2}{2}(B-1)\nabla^2 \\ &+ \frac{\mu^4}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\nabla^2\nabla^2 \end{aligned} \tag{25}$$

and the forcing terms for the first 2 orders are given by:

$$\begin{aligned} F_0 &\equiv 0 \\ G_0 &\equiv 0 \\ F_1 &\equiv -\nabla\cdot(\eta_0\nabla\phi_0) - \frac{\mu^2}{2}(B-1)\nabla\cdot\left\{\eta_0\nabla\left(\nabla^2\phi_0\right)\right\} \\ &- \frac{\mu^4}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\nabla\cdot\left\{\eta_0\nabla\left(\nabla^2\nabla^2\phi_0\right)\right\} \end{aligned} \tag{26}$$

$$\begin{aligned} G_1 &\equiv -\frac{1}{2}(\nabla\phi_0)^2 + \frac{\mu^2}{2}\left\{2\eta_0\nabla^2\phi_{0t} - (B-1)\nabla\phi_0\cdot\nabla\left(\nabla^2\phi_0\right) + \left(\nabla^2\phi_0\right)^2\right\} \\ &- \frac{\mu^4}{4}\left\{\left(\frac{2}{3} - 2B\right)\eta_0\nabla^2\nabla^2\eta_{0t} + \left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)\nabla\phi_0\cdot\nabla\left(\nabla^2\nabla^2\eta_0\right)\right. \\ &+ \left.\frac{1}{2}(B-1)^2\nabla\left(\nabla^2\phi_0\right)\cdot\nabla\left(\nabla^2\phi_0\right) + 2\left(B - \frac{1}{3}\right)\left(\nabla^2\phi_0\right)\left(\nabla^2\nabla^2\phi_0\right)\right\} \end{aligned} \tag{27}$$

We assume the following random linear sea as the solution to the $O(1)$ problem:

$$\eta_0 = \sum_n a_n \cos\psi_n; \qquad \phi_0 = \sum_n b_n \sin\psi_n; \tag{28}$$

where a_n and b_n are nondimensional amplitudes of the functions η_0 and ϕ_0, $\psi_n = \mathbf{k}_n\cdot\mathbf{x} - \omega_n t - \theta_n$, $\mathbf{k}_n$ is the n-component wavenumber vector nondimensionalized by k_0, $\mathbf{x}$ is the horizontal coordinates vector nondimensionalized by $1/k_0$, ω_n is the n-component angular frequency nondimensionalized by $k_0(gh_0)^{1/2}$. Substitution of (28) into the $O(1)$ set of equations (23) with $n = 0$ gives a set of n relationships between ω_n and $k_n = |\mathbf{k}_n|$; each of them is the same as (18). We also find a relationship between a_n and b_n given by:

$$b_n = \frac{\omega_n}{k_n K_n}a_n; \quad K_n = k_n\left\{1 - \frac{\mu^2}{2}k_n^2\left(B - \frac{1}{3}\right) + \frac{\mu^4}{4}k_n^4\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right)\right\} \tag{29}$$

Following the standard perturbation technique, we substitute the $O(1)$ solution (28) into the right-hand-side of the $O(\delta)$ equations (23) to find the forcing of the $O(\delta)$ problem. The forcings F and G in the mass and dynamic equations (23) respectively are:

$$\begin{aligned} F &= \frac{1}{4}\sum_l\sum_m a_m a_l\left\{\mathcal{F}^+_{ml}\sin(\psi_l+\psi_m) + \mathcal{F}^-_{ml}\sin(\psi_l-\psi_m)\right\} \\ G &= \frac{1}{4}\sum_l\sum_m a_m a_l\left\{\mathcal{G}^+_{ml}\cos(\psi_l+\psi_m) + \mathcal{G}^-_{ml}\cos(\psi_l-\psi_m)\right\} \end{aligned} \tag{30}$$

where

$$\mathcal{F}^{\pm}_{ml} = \frac{\omega_m k_l^2 \pm \omega_l k_m^2 + (\omega_l \pm \omega_m)(\mathbf{k}_l \cdot \mathbf{k}_m)}{\omega_l \omega_m} \tag{31}$$

$$\begin{aligned}
\mathcal{G}^{\pm}_{ml} &= \frac{1}{k_l k_m K_l K_m}\Big[-\omega_l\omega_m(\mathbf{k}_l\cdot\mathbf{k}_m) + \mu^2\Big\{\omega_m^2 k_m^2 k_l K_l \\
&+ \omega_l^2 k_l^2 k_m K_m + \frac{1}{2}(B-1)\omega_l\omega_m(k_l^2+k_m^2)(\mathbf{k}_l\cdot\mathbf{k}_m) \pm \omega_l\omega_m k_l^2 k_m^2\Big\} \\
&+ \mu^4\Big\{-\frac{1}{2}\left(B-\frac{1}{3}\right)\left(\omega_l^2 k_l^4 k_m K_m + \omega_m^2 k_m^4 k_l K_l\right) \\
&- \frac{1}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right)(\mathbf{k}_l\cdot\mathbf{k}_m)\omega_l\omega_m(k_l^4+k_m^4) \\
&- \frac{1}{4}(B-1)^2\omega_l\omega_m k_l^2 k_m^2(\mathbf{k}_l\cdot\mathbf{k}_m) \mp \frac{1}{2}\left(B-\frac{1}{3}\right)\omega_m\omega_l k_m^2 k_l^2(k_m^2+k_l^2)\Big\}\Big]
\end{aligned} \tag{32}$$

Equation (31) is identical to the full Stokes' theory result, except for the approximate dispersion relationship. Equation (32) can be rearranged within the level of approximation of the present model to:

$$\mathcal{G}^{\pm}_{ml} = \frac{-\mathbf{k}_l\cdot\mathbf{k}_m + \mu^2\{\omega_l\omega_m(\omega_l^2+\omega_m^2) \pm \omega_l^2\omega_m^2\}}{\omega_l\omega_m} + O(\mu^6) \tag{33}$$

which is, again, formally the same as the full Stokes' theory result but with an approximate dispersion relationship.

The forced solution for η_1 can be obtained by solving (23) and is given by

$$\eta_1 = \sum_l\sum_m a_m a_l\left\{\mathcal{H}^{+}_{ml}\cos(\psi_l+\psi_m) + \mathcal{H}^{-}_{ml}\cos(\psi_l-\psi_m)\right\} \tag{34}$$

where

$$\mathcal{H}^{\pm}_{ml} = \frac{\omega^{\pm}_{ml}\mathcal{F}^{\pm}_{ml} - k^{\pm}_{ml}\mathcal{G}^{\pm}_{ml}T^{\pm}_{ml}}{4(\omega^{\pm}_{ml})^2 - k^{\pm}_{ml}T^{\pm}_{ml}}, \tag{35}$$

$$T^{\pm}_{ml} \equiv k^{\pm}_{ml}\frac{1-\frac{\mu^2}{2}\left(B-\frac{1}{3}\right)(k^{\pm}_{ml})^2 + \frac{\mu^4}{4}\left(B^2-\frac{B}{3}-\frac{D}{6}+\frac{1}{30}\right)(k^{\pm}_{ml})^4}{1-\frac{\mu^2}{2}(B-1)(k^{\pm}_{ml})^2 + \frac{\mu^4}{4}\left(B^2-B-\frac{D}{6}+\frac{1}{6}\right)(k^{\pm}_{ml})^4},$$

$$k^{\pm}_{ml} = |\mathbf{k}_l \pm \mathbf{k}_m|, \qquad \omega^{\pm}_{ml} = \omega_l \pm \omega_m$$

$\mathcal{H}^{+}_{ml}$, $\mathcal{H}^{-}_{ml}$ are respectively the super- and subharmonic transfer coefficients of the interaction between the (l, m) pair of waves. Figures (6) and (7) show comparisons of the ratio of approximate $\mathcal{H}^{\pm}_{ml}$ to Stokes' solution, for Nwogu's model, WKGS model, and the present model. Note that the poor representation of these coefficients at small μ in Nwogu's model is due to the assumption of weak nonlinearity, as discussed by Kirby and Wei (1994). The present model predicts superharmonic amplitudes very accurately over a wide range of water depths. The asymptotic representation of subharmonic amplitudes is also more accurate than in previous models. However, the new solution deviates more rapidly from the exact solution than do the previous results.

Third Order Nonlinear Interactions

We now extend our analysis to third order interactions. We will focus on obtaining the amplitude dispersion of a simple unidirectional monochromatic wave train. It is well known that at this order, it is necessary to introduce a "slow" time scale into the problem, since resonant interactions take place and the perturbation problem becomes singular. We will concentrate on plane waves traveling in the x direction. The "stretched" time scale is given by:

$$t = t' + \delta t' + \delta^2 t' = t' + T_1 + T_2 \tag{36}$$

We then substitute (36) and (22) into (12) and (13), and order the equations up to $O(\delta^2)$. We assume the solution to each order to be of the form

$$\begin{aligned} \eta_n &= \sum_{m=-(n+1)}^{n+1} \eta_{nm}(T_1, T_2) e^{im(x'-\omega t')} \\ \phi_n &= \sum_{m=-(n+1)}^{n+1} \phi_{nm}(T_1, T_2) e^{im(x'-\omega t')} \end{aligned} \tag{37}$$

We then seek an equation for the $O(1)$ wave amplitude, in T_1 by relating the coefficients (amplitudes) in (37) of each order to the ones of the previous order. After some algebra, the following equation for the wave amplitude $A = \eta_{01}$ is found after we neglect current components (terms involving ϕ_{10}):

$$A_{T_2} + i\sigma_1 |A|^2 A = 0 \tag{38}$$

where, for the present model:

$$\begin{aligned} \sigma_1 \; = \; & \frac{-P_{22}}{\omega Q_1 Q_2}\left[4 + 16C_1\mu^2 - \omega^{-2}\left(1 + 4C_3\mu^2 + 16C_4\mu^4\right)\right] \\ - \; & \frac{\omega}{2Q_1}(E_{20} + E_{22})\left(\mu + C_1\mu^3 - \omega^{-2}\mu^{-1}Q_1\right) \\ + \; & \frac{P_{22}}{4\mu^2\omega^3 Q_1^2 Q_2}\left[1 + (2 + 5C_3)\mu^2 + \left(10C_1 + 17C_4 + 4C_3^2\right)\mu^4\right] \\ - \; & \frac{3}{16Q_1}\left[1 + C_3\mu^2 - \omega^{-2}\left(1 + C_1\mu^2\right)\right] \\ + \; & \frac{1}{\omega Q_1^2}\left[4 + (8C_3 + 1/6)\mu^2\right] \end{aligned} \tag{39}$$

and

$$\begin{aligned} C_1 &= -\frac{1}{2}\left(B - \frac{1}{3}\right); \quad C_2 = \frac{1}{4}\left(B^2 - \frac{B}{3} - \frac{D}{6} + \frac{1}{30}\right) \\ C_3 &= -\frac{1}{2}(B - 1); \quad C_4 = \frac{1}{4}\left(B^2 - B - \frac{D}{6} + \frac{1}{6}\right) \end{aligned} \tag{40}$$

and where $E_{20}, E_{22}, P_{22}, Q_1$ and Q_2 are complicated functions of μ which may be found in Gobbi et al (1996). The corresponding σ_1 for the full boundary value problem is given by:

$$\sigma_{1Full} = \frac{\cosh 4\mu + 8 - 2\tanh^2\mu}{16\sinh^4\mu} \tag{41}$$

Equation (38) can be integrated to give:

$$A = a_0 e^{-i(\sigma_1 a_0^2 T_2)} \tag{42}$$

where $a_0 = |A|$. The leading order solution of η is, then:

$$\eta_1 = a_0 \cos(kx - \omega_1 t) \tag{43}$$

where

$$\omega_1 = \omega + (\delta\mu)^2 \sigma_1 a_0^2 \tag{44}$$

The coefficient σ_1 characterizes the amplitude dispersion occurring at leading order due to third order wave-wave interactions. Figure (8) shows comparison of the ratio σ_1 from the present model and from WKGS model to the Stokes' solution to the full problem. The present model appears to have a better asymptotic approximation to the full problem, with excellent agreement in shallower water and acceptable agreement in intermediate to deep water.

Conclusions

A Boussinesq-type model with $O(1)$ nonlinearity and $O(\mu^4)$ dispersion has been proposed. By defining one the dependent variables as the weighted average of the velocity potential at two distinct water depths, it is possible to achieve an extremely accurate (4,4) Padé approximant for the linear dispersion relationship. A major improvement over the existing second order models has been found in the prediction of the internal flow kinematics. A perturbation approach was carried out to analyse random wave second order nonlinear interactions and it has been shown that the present model predicts the transfer coefficients of super and subharmonics generation very well over a wide range of water depths. Finally, the present model predicts well the amplitude dispersion due to third order nonlinear wave-wave interactions. The authors are now preparing a more thorough paper, and are working on the direct solution of the proposed equations by numerical techniques.

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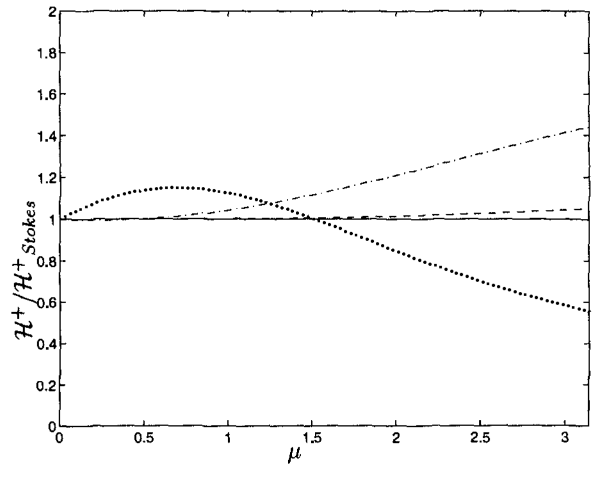

Figure 6: Ratio of approximate superharmonic transfer coefficients to Stokes' solution. Stokes' theory (solid), Nwogu (dotted), WKGS (dash-dot), Present (dash), Present rearranged (thin dot)

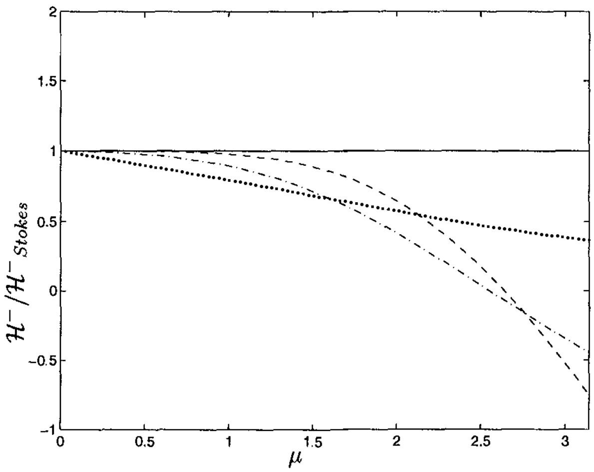

Figure 7: Ratio of approximate subharmonic transfer coefficients to Stokes' solution. Stokes' theory (solid), Nwogu (dotted), WKGS (dash-dot), Present (dash), Present rearranged (thin dot)

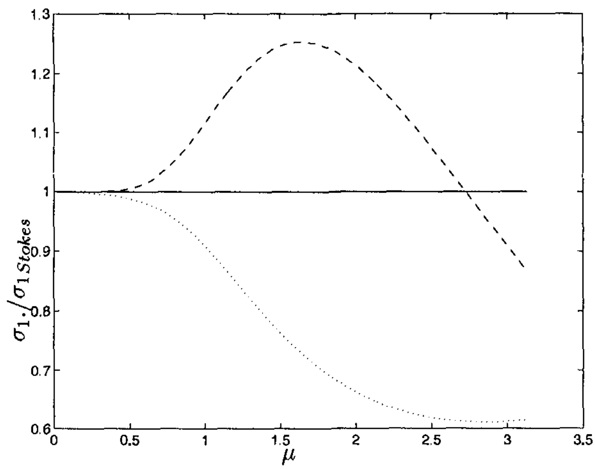

Figure 8: Ratio of Schrodinger equation's cubic term coefficient to full problem's Stokes solution. Wave-wave interaction contribution. Full boundary value problem (solid), WKGS (dotted), Present (dash)