CHAPTER 83

BOUSSINESQ EQUATIONS WITH IMPROVED DOPPLER SHIFT AND DISPERSION FOR WAVE/CURRENT INTERACTION

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Abstract

Boussinesq-type equations with improved dispersion characteristics for the combined motion of waves and currents are introduced. The ambient current is assumed to be uniform over depth and to have a magnitude as large as the shallow water wave celerity, allowing for the consideration of wave blocking of fairly long waves. The temporal variation of the current is ignored, while the spatial variation is assumed to vary on a larger scale than the wave-length scale. Boussinesq-type equations are derived by explicit use of four scales v, δ, ϵ and μ representing the particle velocity and the surface elevation of the total wave-current motion, as well as the wave-nonlinearity and the wave-dispersion, respectively. Firstly, equations are derived in terms of the depth-averaged velocity to obtain a generalization of the equations of Yoon & Liu (1989) to allow for stronger currents. Secondly, these equations are formulated in terms of the velocity variable at an arbitrary z-location resulting in an improved dispersion relation which corresponds to a Padé [2,2] expansion in the wave number of the squared intrinsic celerity for the fully dispersive linear theory. For vanishing currents, these equations reduce to the equations of Nwogu (1993). Finally, this formulation is enhanced to achieve Padé [4,4] dispersion characteristics. Model results for monochromatic and bichromatic waves being fully or partly blocked by opposing currents are given and the results are shown to be in reasonable agreement with theoretical calculations based on the wave-action principle.

1. Introduction

Various forms of lower-order Boussinesq equations are reported in the literature and they may be classified into three groups as follows: (1) the classical Boussinesq equations for wave motion (e.g. Peregrine, 1967); (2) the Boussinesq-type equations with improved linear dispersion properties (e.g. Madsen et al., 1991; Nwogu, 1993; Schäffer and Madsen, 1995); (3) the Boussinesq equations derived for the combined motion of waves and ambient currents (e.g. Yoon & Liu, 1989 and Prüzer & Zielke, 1990). As shown by Chen et al. (1996), only the equations in the third group incorporate a correct form of Doppler shift in connection with wave-current interaction. Their dispersion relation, however, suffers the same inaccuracy as the classical Boussinesq equations for higher wave numbers. In case of opposing

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currents this limitation quickly becomes critical for the applicability of the equations as wave numbers increase with the current speed. It is therefore desirable to improve the equations to achieve Padé-type expansions of the dispersion relation similar to what has been achieved by e.g. Madsen et al. (1991) and Schäffer & Madsen (1995) for the case of pure wave motion. First attempts in this direction were made by Kristensen (1995).

This paper focuses on the derivation and analysis of Boussinesq-type equations with Padè [4,4] dispersion characteristics for coupled wave-current motion. A one-dimensional version of the new equations is solved by the finite-difference method and simulation results of waves blocked by a strong, opposing current are presented.

2. Scaling Assumptions and Governing Equations

As a starting point we consider the combined motion of waves and ambient currents and split the velocity variable into two parts, a wave orbital velocity, u^w and a current velocity, u^c, which is assumed to be uniform over depth. This splitting is only made for scaling purposes and it does not appear in the final equations. A Cartesian coordinate system with the x'-axis and y'-axis located at the still water level (SWL) and the z'-axis pointing vertically upwards is employed. The free surface is defined by $z'=\eta'(x',y',t')$ while the sea bed is defined by $z'=-h'(x',y')$. Non-dimensional variables are used as follows:

$$x=x'/l'_0, \quad y=y'/l'_0, \quad z=z'/h'_0, \quad t=t'\sqrt{(gh'_0)}/l'_0 \tag{2.1}$$

where prime denotes dimensional variables and h'_0 and l'_0 denote a characteristic water depth and wave length, respectively. In the following discussion of scales of waves and currents in shallow water we respectively utilize the linear and nonlinear version of the shallow water equations (SWE). The discussion is kept brief and detailed analyses can be found in the work by Chen et al. (1996).

In connection with pure wave motion in shallow water we introduce the classical measures of nonlinearity and frequency dispersion by

$$\epsilon=a'_0/h'_0 \quad , \quad \mu=h'_0/l'_0 \tag{2.2}$$

where a'_0 denotes a characteristic wave amplitude. As in conventional Boussinesq theory we shall assume that $\epsilon=O(\mu^2)$ and $\epsilon<<1$. Linear long-wave theory can be employed to estimate the order of magnitude of the wave particle velocities (u'^w, w'^w) the free surface elevation (η'^w) and the pressure(p'^w) as well as their temporal and spatial variation.

For pure current motion, the temporal variation of the current is ignored as it is assumed to be several orders of magnitude slower than that of the wind waves. The spatial variation of the current is closely related to the variation of the bottom bathymetry and we generally assume this to vary on a larger spatial scale than the wave-length scale. This can be expressed by $u'^c=u'^c(\sigma x,\sigma y)$, $\eta'^c=\eta'^c(\sigma x,\sigma y)$ and $h'=h'(\sigma x,\sigma y)$, where σ denotes the slow scale, which is yet to be chosen. Coastal currents are typically stronger than the wave particle velocity and weaker than the wave celerity, but in the derivation we use the shallow water wave celerity as the scale of the current speed in order to be able to consider wave-blocking in shallow water. Consequently, we express the order of magnitude of the current velocity by $u'^c=O(\nu)\sqrt{(gh'_0)}$ in which $\epsilon\leq\nu\leq 1$. The corresponding surface elevation due to the current is expressed by $\eta'^c=O(\delta)\,h'_0$ where $\delta=O(\nu^2)$ as analysed in Chen et al. (1996)'s work. In comparison, Yoon & Liu (1989) used $\nu=\mu$ and $\delta=\mu^2$ so that wave-blocking in shallow water was not allowable under their assumptions. By the use of σ, ν, δ and μ we can also determine the order of magnitude of the temporal and spatial variation of the variables.

On the basis of the scaling analyses for pure wave and pure current motion, the scaling of variables in case of coupled wave-current motion becomes

$$u' = O(\epsilon,\nu)\sqrt{(gh'_0)}\ ; \qquad w' = O(\epsilon\mu,\sigma\nu\mu)\sqrt{(gh'_0)} \tag{2.3}$$

$$\eta' = O(\epsilon,\nu^2)\,h'_0\ ; \qquad p' = O(\epsilon,\nu^2)\,(\rho g h'_0) \tag{2.4}$$

The order of magnitude of the leading terms in the continuity equation of the SWE becomes

$$\eta'_{t'} = O(\epsilon\mu)\sqrt{(gh'_0)};\quad h'\,u'_{x'} = O(\epsilon\mu,\sigma\nu\mu)\sqrt{(gh'_0)};\quad u'\,h'_{x'} = O(\epsilon\mu,\sigma\nu\mu)\sqrt{(gh'_0)} \tag{2.5}$$

The order of magnitude of the terms in the momentum equation of the SWE becomes

$$u'_{t'} = O(\epsilon\mu)\,g;\quad g\eta'_{x'} = O(\epsilon\mu,\sigma\nu^2\mu)\,g;\quad u'\,u'_{x'} = O(\epsilon^2\mu,\sigma\nu^2\mu)\,g \tag{2.6}$$

The spatial variation of the current and the bathymetry was defined by σ which is rather arbitrary. We adopt the assumption in Madsen & Schäffer's (1996) work specifying $h'\ u'^{c}_{x'} = O(h'\ u'^{w}_{x'})$. This assumption in combination with the expression in (2.5) yields

$$\sigma = O(\epsilon/\nu) \tag{2.7}$$

This means that strong currents (with $\nu=O(1)$) can be treated only in connection with weakly varying bathymetries, while weak currents (with e.g. $\nu=O(\epsilon)$) do not imply any restrictions on the bathymetric variations. The condition expressed by (2.7) is basically in agreement with the assumptions by Yoon & Liu (1989) and Dingemans (1994), who used $\epsilon=\mu^2$, $\nu=\mu$ and $\sigma=\mu$.

The governing equations serving as our starting point of derivation are the depth-integrated conservation laws for mass and momentum with the dimensionless variables as defined by

$$\boldsymbol{u} = \frac{\boldsymbol{u}'}{\sqrt{gh_0'}},\quad w = \frac{w'}{\mu\sqrt{gh_0'}},\quad p = \frac{p'}{\rho g h_0'},\quad \eta = \frac{\eta'}{h_0'} \tag{2.8}$$

where $\boldsymbol{u} = (u, v)$ is the horizontal velocity vector; w is the vertical velocity; p is the pressure and η is the free surface elevation of the combined wave and current motion. The actual magnitude of each term appearing in the derivation will be explicitly determined by the use of the scaling assumptions (2.3)-(2.7).

In terms of the dimensionless variables defined by (2.1) and (2.8) we express the depth-integrated mass equation as

$$\eta_t + \nabla\cdot\int_{-h}^{\eta}\boldsymbol{u}\,dz = 0 \tag{2.9}$$

where $\nabla = (\partial/\partial x,\ \partial/\partial y)$ is the horizontal gradient operator, and the depth-integrated horizontal momentum equations as

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}u\,dz + \frac{\partial}{\partial x}\int_{-h}^{\eta}u^2 dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}uv\,dz + \frac{\partial}{\partial x}\int_{-h}^{\eta}p\,dz - p\,|_{z=-h}h_x = 0 \tag{2.10a}$$

$$\frac{\partial}{\partial t}\int_{-h}^{\eta}v\,dz + \frac{\partial}{\partial x}\int_{-h}^{\eta}uv\,dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}v^2 dz + \frac{\partial}{\partial y}\int_{-h}^{\eta}p\,dz - p\,|_{z=-h}h_y = 0 \tag{2.10b}$$

while the pressure field reads

$$p(x,y,z,t) = (\eta - z) + \mu^2\left\{\frac{\partial}{\partial t}\int_z^{\eta} w\,dz + \nabla\cdot\int_z^{\eta} \boldsymbol{u}\,w\,dz - w^2\right\} \tag{2.11}$$

The vertical velocity w is evaluated by the vertical integration of the continuity equation from the sea bed to z and the use of the bottom boundary condition, which yields

$$w = -\nabla\cdot\int_{-h}^{z} \boldsymbol{u}\,dz \tag{2.12}$$

The above depth-integrated equations are as exact as their original continuity equation and the Euler equations of motion and the detailed derivation procedure can be found in the literature (e.g. Phillips, 1977). The closure of the equations (2.9)-(2.12) consists of determining the vertical distribution of the horizontal velocity vector $\boldsymbol{u}$ by the use of the vorticity equations of the fluid. As in Yoon & Liu (1989)'s work, the current field is allowed to be horizontally sheared while the vertical shear is limited. Hence, the vorticity equations read

$$\boldsymbol{u}_z - \mu^2\nabla w = O(\mu^4) \tag{2.13}$$

$$u_y - v_x = O(1) \tag{2.14}$$

3. Derivation of a Generalized Version of the Equations by Yoon & Liu (1989)

In the previous section the governing equations were listed in non-dimensional form using μ as the only explicit scaling parameter. In the following derivation of the horizontal and vertical particle velocities, the dynamic pressure and the resulting mass and momentum equations we introduce the parameters ϵ, ν, δ and σ as explicit measures of the order of magnitude of each term in the equations. As defined in Section 2 we take $\sigma=\epsilon/\nu$ and $\delta=O(\epsilon,\nu^2)$. Further specifying $\epsilon\leq\nu\leq1$ ensures that the equations will be also valid in the limit of vanishing currents. Generally, the order of magnitude of the different terms is determined as the maximum of all possible combinations of wave and current components and in this process the difference in horizontal scaling of current and wave components is taken into account. As a key step of the development of Boussinesq-type equations, we must determine the depth-dependence of the horizontal velocity field which can be expanded as a Taylor series with respect to the velocities $\hat{\boldsymbol{u}} = \boldsymbol{u}(x,y,0,t)$ at the still water level.

$$\boldsymbol{u}(x,y,z,t) = \boldsymbol{u}(x,y,0,t) + z\boldsymbol{u}_z(x,y,0,t) + \frac{1}{2}z^2\boldsymbol{u}_{zz}(x,y,0,t) + \cdots \tag{3.1}$$

We make use of the vorticity equations (2.13) and the local continuity equation to evaluate $\boldsymbol{u}_z$ and $\boldsymbol{u}_{zz}$ in (3.1). By the use of the definition of the depth-integrated velocity, $\boldsymbol{U}$ and algebraic manipulation (see Chen et al., 1996 for details) we obtain

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{U} &+ \frac{\epsilon}{\nu}\mu^2\left\{\left(\frac{h^2}{6} - \frac{z^2}{2}\right)\nabla(\nabla\cdot\boldsymbol{U}) - \left(z + \frac{h}{2}\right)\nabla[\nabla\cdot(h\boldsymbol{U})]\right\}\\ &-\delta\frac{\epsilon}{\nu}\mu^2\eta\left\{\frac{1}{6}h\nabla(\nabla\cdot\boldsymbol{U}) - \frac{1}{2}h\nabla[\nabla\cdot(h\boldsymbol{U})]\right\} + \delta^2\frac{\epsilon}{\nu}\mu^2\frac{1}{6}\eta^2\nabla(\nabla\cdot\boldsymbol{U}) + O(\frac{\epsilon}{\nu}\mu^4)\end{aligned} \tag{3.2}$$

Substitution of (3.2) into (2.12) gives the vertical velocity in terms of $\boldsymbol{U}$

$$w = -\nabla\cdot[(z+h)U] + O(\mu^2) \tag{3.3}$$

By inserting (3.2) and (3.3) into (2.11) and determining the ordering, the pressure field may be expressed as

$$\begin{aligned} p = & \left(\eta - \frac{z}{\delta}\right) + \frac{\epsilon}{\delta}\mu^2\left[z\nabla\cdot(hU_t) + \frac{z^2}{2}\nabla\cdot U_t\right] \\ & - \epsilon\mu^2\left\{\frac{1}{2}\delta\eta^2\nabla\cdot U_t + \eta\nabla\cdot(hU_t)\right\} \\ & + \frac{\nu\epsilon}{\delta}\mu^2\left\{zU\cdot\nabla[(\nabla\cdot(hU))] + \frac{z^2}{2}U\cdot\nabla(\nabla\cdot U)\right\} \\ & - \nu\epsilon\mu^2\left\{\frac{1}{2}\delta\eta^2U\cdot\nabla(\nabla\cdot U) + \eta U\cdot\nabla[\nabla\cdot(hU)]\right\} + O(\frac{\epsilon^2}{\delta}\mu^2, \frac{\epsilon}{\delta}\mu^4) \end{aligned} \tag{3.4}$$

Use of the definition of depth-averaged velocity in (2.9) and substitution of (3.2) and (3.4) into the depth-integrated equations (2.10) lead to a modified version of the equations by Yoon & Liu (1989) for wave/current interaction in shallow water as follows,

$$\eta_t + \nabla\cdot(hU) + \delta\eta\nabla\cdot U + \nu U\cdot\nabla\eta = 0 \tag{3.5}$$

and

$$\begin{aligned} & U_t + \nu(U\cdot\nabla)U + \nabla\eta \\ & \quad + \mu^2[\Lambda_0^I + \nu\Lambda_1^I + \delta(\Lambda_2^I + \nu\Lambda_3^I) + \delta^2(\Lambda_4^I + \nu\Lambda_5^I)] = O(\epsilon\mu^2, \mu^4) \end{aligned} \tag{3.6}$$

where

$$\Lambda_0^I = h\Gamma_t^I; \qquad \Lambda_1^I = (U\cdot\nabla)(h\Gamma^I) \tag{3.7a}$$

$$\Lambda_2^I = -\eta\{\Gamma_t^I + \nabla[\nabla\cdot(hU_t)]\}; \qquad \Lambda_3^I = -\eta(U\cdot\nabla)\{\Gamma^I + \nabla[\nabla\cdot(hU)]\} \tag{3.7b}$$

$$\Lambda_4^I = -\frac{1}{3}\eta^2\nabla(\nabla\cdot U_t); \qquad \Lambda_5^I = -\frac{1}{3}\eta^2(U\cdot\nabla)[\nabla(\nabla\cdot U)] \tag{3.7e}$$

where

$$\Gamma^I = \frac{h}{6}\nabla(\nabla\cdot U) - \frac{1}{2}\nabla[\nabla\cdot(hU)] \tag{3.7f}$$

In comparison with the original Yoon & Liu's equations, the new scaling assumptions result in additional terms Λ_2^I, Λ_3^I, Λ_4^I and Λ_5^I. These terms take into account the change in the mean water level due to an ambient current and should be included if we consider $\nu = O(1)$. The Doppler shift properties of both sets of equations however, remain identical and correspond to the Padé [0,2] expansion of the linear dispersion relation of fully dispersive waves. When the speed of an ambient current becomes as weak as the wave particle velocity, all μ^2-terms in the momentum equation except for Λ_0^I will become negligible in the lower-order Boussinesq-type equations. Then this set of equations reduces to the equations by Peregrine (1967).

4. Equations with Padé [2/2] Characteristics for Wave-Current Interaction

It is desirable to improve the dispersion accuracy of the modified equations we obtained in the last section. For pure wave motion, Nwogu (1993) introduced an alternative to the Madsen & Sørensen (1992)'s equations with improved linear dispersion properties. Nwogu's equations are formulated in terms of the velocity at an arbitrary level instead of depth-integrated or depth-averaged velocities. As shown by Chen et al. (1996), the equations by Nwogu (1993) are not applicable to coupled wave/current motion due to the lack of accuracy in Doppler shift. We shall start our derivation from the generalized version of the equations by Yoon & Liu (1989) (i.e. (3.5) & (3.6)) and consistently replace the depth-averaged velocities by the velocities at an arbitrary elevation by keeping the same scaling assumptions as discussed in Section 2. This also demonstrates an alternative approach to obtaining Nwogu's equations directly from the equations by Peregrine (1967).

As shown in Chen et al. (1996)'s work, the relation between the depth-averaged velocity, $\boldsymbol{U}$ and the velocity at an arbitrary level, $\boldsymbol{u}_\alpha$ may be expressed by

$$\begin{aligned} \boldsymbol{U} = \boldsymbol{u}_\alpha &+ \frac{\epsilon}{\nu}\mu^2\left\{\left(z_\alpha + \frac{h}{2}\right)\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)] + \left(\frac{z_\alpha^2}{2} - \frac{h^2}{6}\right)\nabla(\nabla\cdot\boldsymbol{u}_\alpha)\right\} \\ &- \frac{\delta\epsilon}{\nu}\mu^2\eta\left\{\frac{1}{2}\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)] - \frac{h}{6}\nabla(\nabla\cdot\boldsymbol{u}_\alpha)\right\} - \frac{\delta^2\epsilon}{\nu}\mu^2\frac{1}{6}\eta^2\nabla(\nabla\cdot\boldsymbol{u}_\alpha) + O\left(\frac{\epsilon}{\nu}\mu^4\right) \end{aligned} \tag{4.1}$$

Substitution of (4.1) into the continuity equations (3.5) and the momentum equations (3.6) leads to a set of equations formulated in terms of the velocities at an arbitrary elevation as follows

$$\begin{aligned} &\eta_t + \nabla\cdot(h\boldsymbol{u}_\alpha) + \delta\eta\nabla\cdot\boldsymbol{u}_\alpha + \nu\boldsymbol{u}_\alpha\cdot\nabla\eta \\ &\quad + \mu^2(\Pi^I_{\alpha 0} + \delta\Pi^I_{\alpha 1} + \delta^2\Pi^I_{\alpha 2} + \delta^3\Pi^I_{\alpha 3}) = O(\epsilon\mu^2, \mu^4) \end{aligned} \tag{4.2}$$

where

$$\Pi^I_{\alpha 0} = \nabla\cdot h\Gamma^I_\alpha - \nabla\cdot\left\{\frac{h^3}{6}\nabla(\nabla\cdot\boldsymbol{u}_\alpha) - \frac{h^2}{2}\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)]\right\}; \qquad \Pi^I_{\alpha 1} = \eta\nabla\cdot\Gamma^I_\alpha \tag{4.3a}$$

$$\Pi^I_{\alpha 2} = -\frac{\eta^2}{2}\nabla\cdot\{\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)]\}; \qquad \Pi^I_{\alpha 3} = -\frac{\eta^3}{6}\nabla\cdot[\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] \tag{4.3b}$$

in which

$$\Gamma^I_\alpha = \frac{z_\alpha^2}{2}\nabla(\nabla\cdot\boldsymbol{u}_\alpha) + z_\alpha\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)] \tag{4.3c}$$

and

$$\begin{aligned} &\boldsymbol{u}_{\alpha t} + \nu(\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha + \nabla\eta \\ &\quad + \mu^2[\Lambda^I_{\alpha 0} + \nu\Lambda^I_{\alpha 1} + \delta(\Lambda^I_{\alpha 2} + \nu\Lambda^I_{\alpha 3}) + \delta^2(\Lambda^I_{\alpha 4} + \nu\Lambda^I_{\alpha 5})] = O(\epsilon\mu^2, \mu^4) \end{aligned} \tag{4.4}$$

where

$$\Lambda^{I}_{\alpha 0} = \Gamma^{I}_{\alpha t} \; ; \qquad \Lambda^{I}_{\alpha 1} = (\boldsymbol{u}_\alpha \cdot \nabla)\Gamma^{I}_{\alpha t} \tag{4.5a}$$

$$\Lambda^{I}_{\alpha 2} = -\eta\nabla[\nabla\cdot(h\boldsymbol{u}_{\alpha t})] \; ; \qquad \Lambda^{I}_{\alpha 3} = -\eta(\boldsymbol{u}_\alpha\cdot\nabla)\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)] \tag{4.5b}$$

$$\Lambda^{I}_{\alpha 4} = -\frac{1}{2}\eta^2\nabla(\nabla\cdot\boldsymbol{u}_{\alpha t}) \; ; \qquad \Lambda^{I}_{\alpha 5} = -\frac{1}{2}\eta^2(\boldsymbol{u}_\alpha\cdot\nabla)[\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] \tag{4.5c}$$

equations (4.2) to (4.5) form a new set of equations with the improved Doppler shift corresponding to the Padé [2,2] expansion of the linear dispersion relation given by the first order Stokes theory by choosing the appropriate z_α as suggested by Nwogu (1993). We shall analyse the dispersion properties in Section 6. When the speed of an ambient current becomes as weak as the wave particle velocity, those terms $\Pi^{I}_{\alpha i}$ $(i=1,2,3)$ in (4.3) and $\Lambda^{I}_{\alpha i}$ $(i=1,2,3,4,5)$ in (4.5) will become negligible in the lower-order Boussinesq-type equations. Then this set of equations reduces to the equations by Nwogu (1993) for pure wave propagation in nearshore regions.

5. Further Enhancement of the Dispersion Accuracy

It is possible to improve the Doppler shift accuracy of the equations we obtained in the last section even further. Starting from (4.2) to (4.5), we shall formulate another set of Boussinesq-type equations by consistent incorporation of the Padé [4,4] expansion of the Doppler shift relation predicted by the first order Stokes theory for waves on uniform ambient currents. Following Schäffer & Madsen (1995), we introduce four free parameters $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ which are less than or equal to $O(1)$. Use of each of the operators $-\beta_1\mu^2\nabla\cdot(h^2\nabla \quad)$ and $\beta_2\mu^2\nabla\cdot\nabla(h^2 \quad)$ on the continuity equation (4.2) leads to

$$\begin{aligned}-\beta_1\mu^2\{&\nabla\cdot(h^2\nabla\eta_t) + \nabla\cdot\{h^2\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)]\} \\ &+\delta\eta\nabla\cdot[h^2\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] + \nu\boldsymbol{u}_\alpha\cdot\nabla(h^2\nabla^2\eta)\} = O(\epsilon\mu^2,\mu^4)\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\beta_2\mu^2\{&\nabla\cdot[\nabla(h^2\eta_t)] + \nabla\cdot\{\nabla[h^2\nabla\cdot(h\boldsymbol{u}_\alpha)]\} \\ &+ \delta\eta\nabla\cdot[\nabla(h^2\nabla\cdot\boldsymbol{u}_\alpha)] + \nu\boldsymbol{u}_\alpha\cdot\nabla[\nabla\cdot(h^2\nabla\eta)]\} = O(\epsilon\mu^2,\mu^4)\end{aligned} \tag{5.2}$$

Similarly, employing each of the operators $-\gamma_1\mu^2h^2\nabla(\nabla\cdot \quad)$ and $\gamma_2\mu^2h\nabla(\nabla\cdot h \quad)$ on the momentum equations (4.4) yields

$$\begin{aligned}&-\gamma_1\mu^2h^2\,[\nabla(\nabla\cdot\boldsymbol{u}_{\alpha t}) + \nabla(\nabla\cdot\nabla\eta)] \\ &-\gamma_1\nu\mu^2h^2(\boldsymbol{u}_\alpha\cdot\nabla)[\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] = O\,(\epsilon\mu^2,\,\mu^4)\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}&\gamma_2\mu^2h\,\{\nabla[\nabla\cdot(h\boldsymbol{u}_{\alpha t})] + \nabla[\nabla\cdot(h\nabla\eta)]\} \\ &+ \gamma_2\nu\mu^2h(\boldsymbol{u}_\alpha\cdot\nabla)[\nabla(\nabla\cdot h\boldsymbol{u}_\alpha)] = O(\epsilon\mu^2,\,\mu^4)\end{aligned} \tag{5.4}$$

Adding (5.1) and (5.2) to (4.2), we obtain a new continuity equation

$$\eta_t + \nabla\cdot(h\boldsymbol{u}_\alpha) + \delta\eta\nabla\cdot\boldsymbol{u}_\alpha + \nu\boldsymbol{u}_\alpha\cdot\nabla\eta + \mu^2(\Pi^{II}_{\alpha 0} + \nu\Pi^{II}_{\alpha 1} + \delta\Pi^{II}_{\alpha 2} + \delta^2\Pi^{II}_{\alpha 3} + \delta^3\Pi^{II}_{\alpha 4}) = O(\epsilon\mu^2,\ \mu^4) \tag{5.5}$$

where

$$\Pi^{II}_{\alpha 0} = \Pi^{I}_{\alpha 0} + \nabla\cdot\left\{\beta_2\nabla[h^2\nabla\cdot(h\boldsymbol{u}_\alpha)] - \beta_1 h^2\nabla[\nabla\cdot(h\boldsymbol{u}_\alpha)]\right\} + \nabla\cdot\left\{\beta_2\nabla(h^2\eta_t) - \beta_1 h^2\nabla\eta_t\right\} \tag{5.6a}$$

$$\Pi^{II}_{\alpha 1} = \boldsymbol{u}_\alpha\cdot\nabla[\beta_2\nabla\cdot(h^2\nabla\eta) - \beta_1\nabla(h^2\nabla^2\eta)] \tag{5.6b}$$

$$\Pi^{II}_{\alpha 2} = \Pi^{I}_{\alpha 1} + \eta\nabla\cdot[\beta_2\nabla(h^2\nabla\cdot\boldsymbol{u}_\alpha) - \beta_1 h^2\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] \tag{5.6c}$$

while $\Pi^{II}_{\alpha 3} = \Pi^{I}_{\alpha 2}$, $\Pi^{II}_{\alpha 4} = \Pi^{I}_{\alpha 3}$, $\Gamma^{II}_\alpha = \Gamma^{I}_\alpha$ as defined by (4.3b-c). Similarly, adding (5.3) and (5.4) to (4.4) leads to new momentum equations

$$\boldsymbol{u}_{\alpha t} + \nu(\boldsymbol{u}_\alpha\cdot\nabla)\boldsymbol{u}_\alpha + \nabla\eta + \mu^2[\Lambda^{II}_{\alpha 0} + \nu\Lambda^{II}_{\alpha 1} + \delta(\Lambda^{II}_{\alpha 2} + \nu\Lambda^{II}_{\alpha 3}) + \delta^2(\Lambda^{II}_{\alpha 4} + \nu\Lambda^{II}_{\alpha 5})] = O(\epsilon\mu^2,\mu^4) \tag{5.7}$$

where

$$\Lambda^{II}_{\alpha 0} = \Gamma^{II}_{\alpha t} - \gamma_1 h^2\nabla(\nabla\cdot\boldsymbol{u}_{\alpha t}) + \gamma_2 h\nabla[\nabla\cdot(h\boldsymbol{u}_{\alpha t})] - \gamma_1 h^2\nabla(\nabla\cdot\nabla\eta) + \gamma_2 h\nabla[\nabla\cdot(h\nabla\eta)] \tag{5.8a}$$

$$\Lambda^{II}_{\alpha 1} = (\boldsymbol{u}_\alpha\cdot\nabla)\Gamma^{II}_\alpha - \gamma_1 h^2(\boldsymbol{u}_\alpha\cdot\nabla)[\nabla(\nabla\cdot\boldsymbol{u}_\alpha)] + \gamma_2 h(\boldsymbol{u}_\alpha\cdot\nabla)[\nabla(\nabla\cdot h\boldsymbol{u}_\alpha)] \tag{5.8b}$$

while $\Lambda^{II}_{\alpha 2} = \Lambda^{I}_{\alpha 2}$, $\Lambda^{II}_{\alpha 3} = \Lambda^{I}_{\alpha 3}$, $\Lambda^{II}_{\alpha 4} = \Lambda^{I}_{\alpha 4}$, $\Lambda^{II}_{\alpha 5} = \Lambda^{I}_{\alpha 5}$, $\Gamma^{II}_\alpha = \Gamma^{I}_\alpha$ as defined by (4.5b-c) and (4.3c). Equations (5.5)-(5.8) form a new set of Boussinesq-type equations for wave/current interaction applicable up to even shorter waves for a suitable choice of the free parameters $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ and z_α as analysed by Schäffer and Madsen (1995) for pure wave motion. In the following section, we shall analyse the Doppler shift behaviour of this new set of equations in comparison with the Stokes theory.

6. Analysis of Linear Dispersion Characteristics

We shall use dimensional form in this chapter and drop primes for convenience. The one-dimensional version of (5.5) and (5.7) with constant water depth can be expressed as

$$\eta_t + hu_{\alpha x} + (u_\alpha\eta)_x - \beta h^2(\eta_{xxt} + u_\alpha\eta_{xxx}) + \left[\left(\alpha - \beta + \frac{1}{3}\right)h^3 + (\alpha - \beta)h^2\eta - \frac{1}{2}h\eta^2 - \frac{1}{6}\eta^3\right]u_{\alpha xxx} \tag{6.1a}$$

and

$$u_{\alpha t} + g\eta_x + u_\alpha u_{\alpha x} - \gamma g h^2 \eta_{xxx} + \left[(\alpha - \gamma)h^2 - \eta h - \frac{1}{2}\eta^2\right](u_{\alpha xxt} - u_\alpha u_{\alpha xxx}) = 0 \tag{6.1b}$$

where $\alpha \equiv z_\alpha/h + 0.5(z_\alpha/h)^2$, $\beta \equiv \beta_1 - \beta_2$, $\gamma \equiv \gamma_1 - \gamma_2$. By the use of Fourier analysis we obtain the linear dispersion relation of the new equations (5.5) to (5.8) as

$$(\omega - u^c k)^2 = \frac{\left[1 - \left(\alpha - \beta + \frac{1}{3}\right)(kh)^2\right][1 + \gamma(kh)^2]k^2 gh}{[1 + \beta(kh)^2]\,[1 - (\alpha - \gamma)(kh)^2]} \tag{6.2}$$

Different choices of the parameters in (6.2) result in different Doppler shift accuracy. For example, $(\alpha, \beta, \gamma) = (-2/5, 0, 0)$ gives

$$(\omega - u^c k)^2 = \frac{\left(1 + \frac{1}{15}k^2h^2\right) k^2 gh}{1 + \frac{2}{5}k^2h^2} \tag{6.3}$$

This is the Padé [2,2] approximation of the linear dispersion relation given by the first-order Stokes theory. It turns out that (6.3) is also the linear dispersion relation of (4.2) to (4.5). Choosing $(\alpha, \beta, \gamma) = (-1/3, 0, 0)$ yields

$$(\omega - u^c k)^2 = \frac{k^2 gh}{1 + \frac{1}{3}k^2h^2} \tag{6.4}$$

which corresponds to the linear dispersion relation of the original and generalized versions of the equations by Yoon & Liu (1989) using the depth-averaged velocities as variables.

Schäffer and Madsen (1995) obtained four sets of coefficients. Each of them leads to the highly accurate linear dispersion relation

$$(\omega - u^c k)^2 = \frac{\left(1 + \frac{1}{9}k^2h^2 + \frac{1}{945}k^4h^4\right) k^2 gh}{1 + \frac{4}{9}k^2h^2 + \frac{1}{63}k^4h^4} \tag{6.5}$$

which is correct to fourth-order in $(kh)^2$ in comparison with the first-order Stokes' solution. Linear shoaling analyses by Schäffer & Madsen (1995) show that these four sets of parameters all give accurate linear shoaling behaviour. The influence of these four sets of parameters on the nonlinearity of the new equations can be analysed by examination of the transfer functions for sub-harmonics and super-harmonics as discussed in Madsen & Schäffer's (1996) work. The best set of parameters can therefore be chosen based on accuracy in the nonlinear properties. We adopt the one recommended by Madsen & Schäffer (1996) as follows.

$$\left.\begin{array}{l}\left(\alpha,\ \dfrac{z_\alpha}{h}\right) = (-0.39476,\ -0.54122\ ) \\ (\beta,\ \beta_1,\ \beta_2) = (0.03917,\ -0.12919,\ -0.16836) \\ (\gamma,\ \gamma_1,\ \gamma_2) = (0.01052,\ -0.07327,\ -0.08379)\end{array}\right\} \tag{6.6}$$

With this set of parameters, equations (5.5) to (5.8) are applicable to deeper water for waves on ambient currents. For waves on following currents, the currents increase the wave length, thus the application range of the new equations exceeds that of the corresponding equations for pure wave motion. For waves on opposing currents, the currents reduce the wave length so that the applicable area depends on the ambient current speed. We use the linear dispersion properties of the first-order Stokes theory to estimate the applicable range of various forms of the Boussinesq-type equations for wave/current interaction. Solving the linear dispersion relations (6.4) & (6.5) gives two sets of curves corresponding to the dimensionless wave number (*kh*), relative water depth (*h/Lo*) and Froude number $F_r = U^c/\sqrt{gh})$ Figs. 7.1a-b illustrate the applicable areas of the new equations (5.5) & (5.7) and the equations by Yoon & Liu (1989) in case of opposing currents, respectively. The 5% error contour for *kh* as compared with the first-order Stokes' solution is also shown. Obviously, the new form of the equations gives a much larger applicable range than those of Yoon & Liu's equations.

It deserves to be mentioned that Madsen & Schäffer (1996) recently derived equations with equivalent properties for wave-current interaction by following a different line of derivation: In their work Boussinesq-type equations were derived on the basis of the two wave scales μ and ϵ, while the ambient current was not explicitly considered during the derivation procedure. In contrast to the present work, however, Madsen & Schäffer allowed $\epsilon = O(1)$ rather than $\epsilon = O(\mu^2)$ and retained all nonlinear terms to the particular order of dispersion. In retrospect this is the reason why their equations could account also for the case of ambient currents, as it turns out that the equations derived in this paper appear as a subset of the former equations by Madsen & Schäffer (1996). Since the extra nonlinear dispersive terms included by Madsen & Schäffer (1996) are expected to be minor in the present applications, the code developed for their equations is adapted for the following numerical experiments.

7. Numerical Solutions for Wave-Current Interaction

A one-dimensional version of the equations (5.5) to (5.8) is solved by the finite-difference method. The equations are discretized on a space staggered grid by means of fourth-order central differencing for first derivative terms in space and second-order central differencing for second and third spacial derivatives. The time-integration of the governing equations consists of the third-order Adams-Bashforth predictor and fourth-order Adams-Moulton corrector schemes. This numerical method was utilized in the work by Wei et al. (1995) and Banijamali (1997) and essentially designed to eliminate the truncation errors which mathematically have the same form as the Boussinesq-type terms due to the use of conventional second-order schemes for pure wave motion. It can be adapted for modelling fully coupled wave/current motion. For the case of strong ambient currents with significant nonlinear advection a smaller convergence criterion for the iterating corrector step

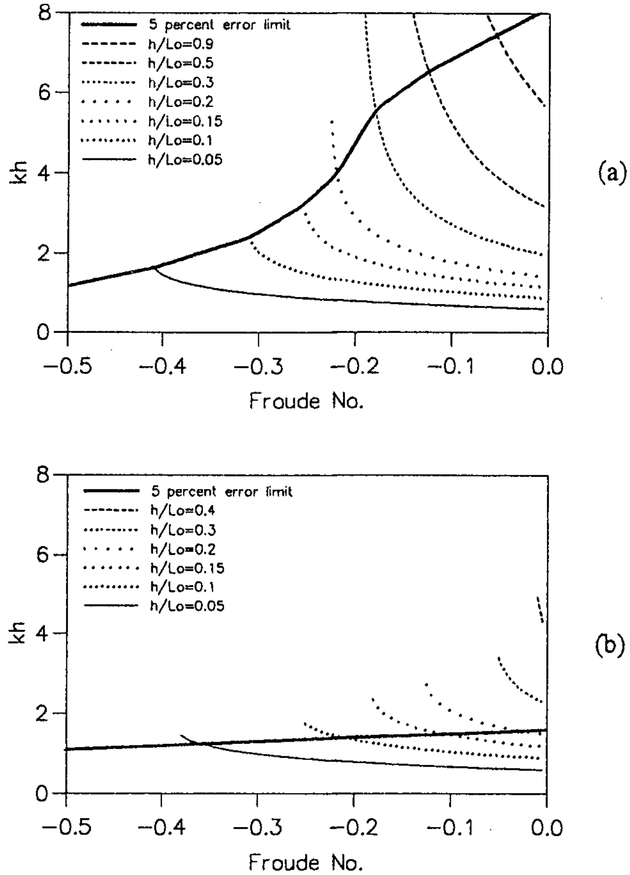
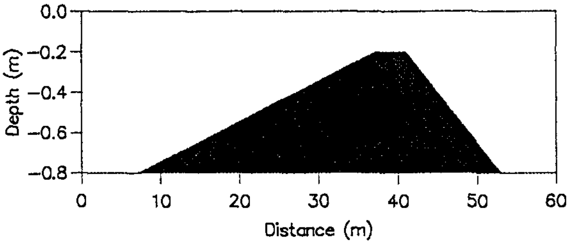


Fig. 7.1 Illustration of the applicable ranges of the equations with (a) the Padè [4,4] expansion (i.e.(5.5) & (5.7)) and (b) the Padè [0,2] expansion (i.e.(3.5) & (3.6)). The area below the 5% kh-error limit predicts the applicable regime.

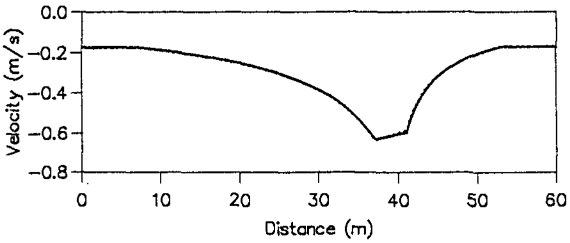
is required, which leads to more iterations. A prototype of the numerical model for pure wave motion developed by Banijamali (1997) is adopted in the present work. We incorporate the model with non-reflective boundary conditions for fully coupled wave/current motion. The sponge layer technique (Larsen & Dancy, 1983) applicable to absorption of short waves is combined with the Sommerfeld radiation condition for radiating long waves or currents. We shall present some model results in connection with waves blocked by strong opposing currents.

The first test case considers monochromatic waves propagating against a current in a channel with a submerged bar. A sketch of the bathymetry is shown in Fig 7.1a. The channel is 60m long, 0.8m deep on both sides of the bar and 0.2m deep on top of the bar. The western and eastern slopes of the bar are 1/50 and 1/20, respectively. Bed friction is modelled by the use of the Chezy friction law, using a

Chezy coefficient of 300 $m^{1/2}/s$ in the sections $0 < x < 37m$ and $55m < x < 60m$, and a coefficient of 30 $m^{1/2}/s$ in the section $37m < x < 55m$. The relatively strong friction in the latter section serves as a stabilizing factor for the flow simulation. Initially we impose a constant velocity of -0.17 m/s at the eastern boundary and a radiating condition at the western boundary. This leads to an increase in the surface elevation at the western boundary of approximately 0.05m. Fig 8.1b shows the computed spatial variation of the velocity, which is found to be in fairly good agreement with conventional theory neglecting the vertical accelerations of the flow.



(a)



(b)

Fig. 8.1 Steady open channel flow predicted by the model (dotted line) and the nonlinear shallow water equations (solid line). a) Submerged bar topography; b) Particle velocity.

As the next step we impose a sinusoidal wave train on top of the steady current field. This is done by specifying a velocity condition at the western boundary including the local current obtained in the previous calculation. At the eastern boundary we use a sponge layer which absorbs the short waves while allowing the current to pass through. The incoming wave has a period of 1.2s and an initial height of 0.02m. The grid size and the time step are chosen to be 0.02m and 0.005s, respectively. Fig 8.2 shows the computed surface elevation for the combined wave-current motion. We notice that the oscillatory motion is stopped at the position $x=33.5m$ where wave blocking occurs because the local current velocity exceeds the local group velocity of the wave. The dotted line in Fig 8.2 indicates the theoretical solution obtained by the principle of wave action and we notice a good agreement with the computations with respect to amplitude amplification as well as the position of the blocking point.

The theory of wave action based on linear progressive wave motion obviously fails close to the blocking point as it predicts the wave height to go to infinity, which does not happen in reality (nor in the model). As suggested by Smith (1975), Stiassnie & Dagan (1979) and Shyu & Phillips (1990) the wave action is eventually reflected at the blocking point and energy is transferred to much higher wave numbers. Furthermore, the wave numbers of the reflected waves will decrease rapidly with the distance from the blocking point due to the decreasing current.

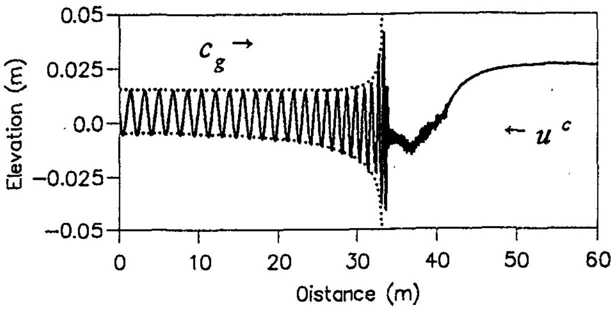


Fig. 8.2 Monochromatic wave propagation on a spatially-varying, opposing current.

In Fig. 8.3, a bichromatic wave train on a spatially-varying current is simulated. The same bathymetry, steady current field, grid size and time step as in the simulation of the monochromatic wave are employed. The bichromatic waves consist of a 1.2s wave and a 3.0s wave. Both of them have the same wave height of 0.02m. The model predicts that the shorter wave of 1.2s is blocked by the opposing current at the position x=33.5m (as before) while the longer wave of 3.0s propagates through the blocking point and reaches the eastern boundary where a sponge layer efficiently absorbs the wave energy. The wave profile in Fig. 8.3 with bichromatic and regular wave forms before and after blocking, respectively, illustrates the blocking of the shorter wave in the bichromatic wave train.

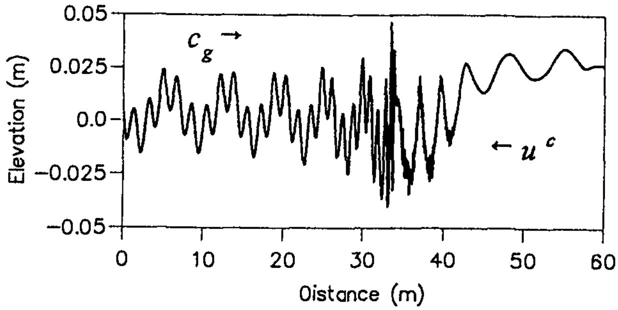


Fig. 8.3 A bichromatic wave train propagating on a spatially-varying, opposing current.

9. Conclusions

This paper deals with the derivation and application of Boussinesq-type equations with Padè [4,4] dispersion characteristics for the combined motion of waves and currents in nearshore areas. The waves are assumed to be weakly nonlinear and the ambient current is assumed to be uniform over depth. In order to allow for the treatment of wave blocking in shallow water we assume the magnitude of the current to be as large as the shallow water celerity. A one-dimensional numerical model has been implemented on the basis of the new equations and as demonstrated it can simulate the complicated phenomenon of monochromatic and bichromatic waves being fully or partly blocked by opposing currents. Further verification of the model against measurements is obviously required, but the results obtained so far are promising and show that the new equations make it possible to simulate a range of complicated phenomena related to the interaction of waves and depth-uniform currents in coastal regions.

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