CHAPTER 81

Irregular Wave Kinematics from a Pressure Record

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Abstract:

A local Fourier approximation method is presented for the prediction of the complete kinematics of irregular waves from a submerged pressure trace. The method seeks a potential function and local water surface elevation that fit the pressure record and the full nonlinear free surface boundary conditions very closely in a small window in time. The result is a complete prediction of the kinematics of the waves throughout the water column that satisfies the complete nonlinear equations for irrotational gravity waves. Comparisons with the predictions of steady wave theory are excellent.

Introduction

A knowledge of wave kinematics is necessary for most aspects of coastal engineering. Fluid velocities and accelerations are necessary for the study of the wave loading of structures through the use of the O'Brien-Morison equation. Knowledge of the kinematics near the sea bed are necessary for studies of sediment transport processes. High order steady wave theories are quite successful at the prediction of the kinematics of steady waves, but are not directly applicable to the irregular waves usually found in the field.

Subsurface pressure transducers are a commonly used method for the measurement of waves in the near-shore zone, as they are relatively easy to deploy. They are frequently used in shallow and transitional depth water. Most methods currently in use for the interpretation of these measurements rely on linear wave

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theory. Linear theory is appealing in that solutions are readily available and can easily be applied to a variety of situations. Unfortunately, linear theory is based on the assumption that wave amplitudes are small. It is also a Stokes theory, with optimum theoretical applicability in deep water. In fact, the cases of most interest in the interpretation of pressure measurements are for large waves in shallow water, exactly the conditions in which linear theory is least adequate.

In this paper, a method is presented for the interpretation of a measured point pressure time history that preserves the full non-linearity of the process, and can be applied to an irregular sea in any depth of water.

Global and Local Approximations

Methods used for the interpretation of irregular wave records fall into two general categories, global and local approximations. Global methods are those that seek a solution that matches an entire measured record, or a single complete measured wave, from trough to following trough, or zero crossing to zero crossing. These methods apply the same frequency and wave number for all z (vertical variation) and t (time).

Local methods seek an approximation to each small local segment of a measured record. In these methods, the frequency and wave number vary, providing a separate solution for each small window in time.

Global Methods

The most commonly used global method for the analysis of irregular waves is spectral analysis, coupled with superposition of linear waves. This method has a number of shortcomings, including the high frequency contamination of the kinematics above the crest (Forristall, 1985; Sobey, 1992). Fundamentally, the difficulties arise from the approximations made by linear wave theory to the free surface boundary conditions. If the full nonlinear free surface boundary conditions are not satisfied, one can expect that the resulting predictions will be inaccurate, particularly near the free surface. Empirical modifications to linear theory have been adopted eg.(Wheeler, 1969), but these no longer conserve mass (Sobey, 1992).

Other global methods rely on zero crossing analysis to identify particular waves that are then analyzed by using steady wave theory for a wave of the same height and period. This approach can provide an order of magnitude estimate for the kinematics, but does not take into account the detail of the record. Dean (Dean, 1965) extended his stream function method to irregular waves, seeking a Fourier expansion for the stream function that fit a water surface record from trough to following trough. While taking into account the detail of the record, the assumption that the wave is globally steady is a major compromise.

Baldock and Swan (Baldock and Swan, 1994) presented a method for the

analysis of a point water surface record that includes unsteady motion. Their method employs a potential function in the form of a double Fourier expansion in time and space. The coefficients of the expansion are found by minimizing the errors in the full non-linear free surface boundary conditions over a grid of nodes in time and space. While comparisons of their results with laboratory data were quite good, the method involves a huge matrix of unknown coefficients and must solve for the water surface far from the actual measurement location. While making no assumptions about the steadiness of the wave field, the method requires an assumed periodicity (usually the length of the record). In order to obtain a good fit to the measured record, a substantial weighting function must be applied to assure that the errors in the boundary conditions are small at the measured location. This need for a weighting function suggests that a local solution may be advantageous.

Local Methods

The Nielsen method (Nielsen, 1986; Nielsen, 1989) uses a local frequency and linear wave theory to find the location of the water surface from a pressure record. Best results were achieved from a stretching method, similar to Wheeler's (Wheeler, 1969), or a semi-empirical transfer function derived from Fourier steady wave theory. In either case, the method does not supply the complete kinematics, and does not satisfy the governing equations.

Fenton (Fenton, 1986) employed a local polynomial approximation to the complex potential function. In this method, the potential function is represented by a separate polynomial in each small window in time. Coefficients of the polynomial are sought that fit the measured pressure record, and the full nonlinear free surface boundary conditions. This approach provides the complete kinematics and satisfies the full governing equations. Based on a polynomial variation with depth, it should work well in shallow water, but may have difficulty in transitional or deep water.

Sobey's Locally Steady Fourier Method (LSFI) (Sobey, 1992) employs a potential function represented by a low order Fourier expansion in a small window in time. It is a method derived for the analysis of a point water surface trace. Local frequency, wave number, and Fourier coefficients are sought that fit the measured record and the full free surface boundary conditions. This method provides the complete kinematics, satisfies the full governing equations, and is successful in all depths of water. The method presented in this paper is an adaptation of Sobey's method to the analysis of a measured pressure trace.

Governing Equations

The formulation of the problem of uni-directional irregular waves is similar to that for classical steady wave theory. The flow is taken to be incompressible

and irrotational. The kinematics can therefore be represented by a potential function, ϕ, where

$$u = \frac{\partial \phi}{\partial x} \qquad w = \frac{\partial \phi}{\partial z} \tag{1}$$

where u and w are the horizontal and vertical velocities, respectively. Mass conservation then becomes the Laplace equation:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2}$$

The boundary conditions are the bottom boundary condition (BBC),

$$w = \frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \qquad z = -h \tag{3}$$

the kinematic free surface boundary condition (KFSBC),

$$w - \frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} = 0 \qquad \text{at} \qquad z = \eta \tag{4}$$

and the dynamic free surface boundary condition (DFSBC),

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \frac{1}{2}w^2 + g\eta - \overline{B} = 0 \qquad \text{at} \qquad z = \eta \tag{5}$$

where η is the location of the free surface and $\overline{B}$ is the Bernoulli constant.

In steady wave theory, periodic lateral boundary conditions are imposed, forcing the solution to be periodic in space and time. With irregular waves, there is no periodicity. Rather, a solution is sought that fits a local segment of the record, the Laplace equation, bottom boundary condition, and both nonlinear free surface boundary conditions.

A form for the potential function in each window is motivated by Fourier steady wave theory. This is the same form as that used by Sobey (Sobey, 1992).

$$\phi(x,z,t) = U_E x + \sum_{j=1}^{J} A_j \frac{\cosh jk(h+z)}{\cosh jkh} \sin j(kx - \omega t) \tag{6}$$

U_E and h are the known depth uniform Eulerian current and water depth, J is the truncation order of the Fourier series, A_j are the local Fourier coefficients, and ω and k are the local fundamental frequency and wave number. This form exactly satisfies mass conservation and the BBC. A different set of parameters is found for each segment of the pressure record. While this form for the potential function is periodic, the periodicity is not defined apriori, but found to fit the record, defining a local frequency and wave number.

Fitting to a Pressure Record

While the potential function provides the complete kinematics, the dynamics are found through the unsteady Bernoulli equation:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + w^2) + \frac{P_d}{\rho} - \overline{B} = 0 \tag{7}$$

where P_d is the dynamic pressure ($P_d = P - \rho g z$). In steady wave theory, the Bernoulli constant is (Longuet-Higgins, 1975):

$$\overline{B} = g\overline{\eta} + \frac{1}{2}\overline{u_b^2} \tag{8}$$

where u_b is the velocity at the bed, and the over-bars indicate time averaging. In this case, it will be different in each window. With z defined to be zero at the mean water level, and Eqn. 6 as the potential function, $\overline{B}$ becomes (Sobey, 1992):

$$\overline{B} = \frac{1}{2}U_E^2 + \frac{1}{4}\sum_j \left(\frac{jkA_j}{\cosh jkh}\right)^2 \tag{9}$$

thus all the terms in Eqn. 7 are defined by the potential function except the dynamic pressure, which is given by the measured record.

Eqns. 6 and 7 apply to many periodic flows. There might be any number of these flows that could produce a given pressure record at a single location. It is the free surface boundary conditions (Eqn. 4 and 5) that identify a potential flow as a surface gravity wave. As the solution sought is a gravity wave, these boundary conditions must be included in the formulation. To include the free surface boundary conditions, the location of the water surface, together with the potential function, become the unknowns in each window.

Locating the water surface

The water surface is defined at N surface nodes in each window ($\eta(t_n), n = 1 \ldots N$). The elevation of these nodes is unknown, and will be sought as part of the solution. Eqn. 5 is directly applied at each node. In order to apply Eqn. 4 at the surface, the time gradient is estimated by cubic spline interpolation among these nodes. This provides a smooth and consistent estimate at all locations within the window. The spatial gradient can be computed from the time gradient by assuming that the water surface is locally steady. This assumption follows from the steady form of the potential function, and is the same assumption used by Sobey.

$$\frac{\partial \eta}{\partial x} = -\frac{k}{\omega}\frac{\partial \eta}{\partial t} \tag{10}$$

Including the water surface nodes as part of the sought solution introduces N additional unknowns for a total of $3 + J + N$ unknowns in each window $(k, kx, \omega, A_j, \eta_n)$. Eqn. 4 is applied at the locations between the water surface nodes. Eqn. 5 is applied at each of the nodes, yielding $2N - 1$ additional equations. Eqn. 7 is applied at I nodes on the pressure record $(P_d(t_i), i = 1 \ldots I)$ within the local window (see Fig. 1).

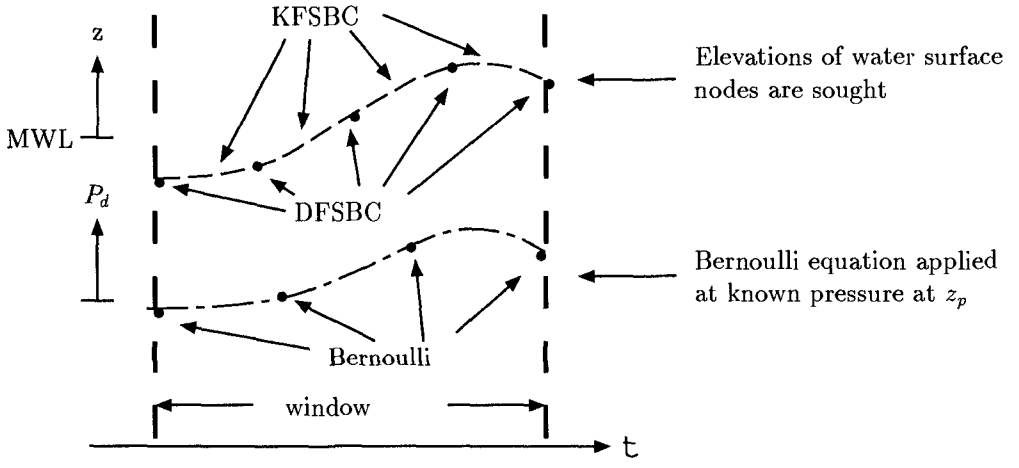

Figure 1: Schematic of system of equations in a window

The problem is uniquely specified for $I + N = 4 + J$ and overspecified for $I + N > 4 + J$. If overspecified, the solution is that which results in the minimum error in all equations, in the least squared sense. Overspecification, with additional nodes on the pressure record, is likely to be advantageous for an actual record to minimize the effect of unavoidable measurement noise.

Computation Methods

Non-dimensionalization

The comparisons of errors of different dimensional quantities would be meaningless. All parameters and variables are scaled by factors computed from physically identifiable parameters. These are the mass density of water, acceleration of gravity, and the mean zero crossing frequency. The characteristic length scale

is g/ω_z^2, time scale is $1/\omega_z$ and the mass scale is $\rho g^3/\omega_z^6$ where ω_z is the mean zero crossing frequency of the record. The familiar dimensional forms of the equations have been presented in this paper to aid in readability. The unknown parameter, x, appears in the equations only when coupled with the parameter, k. It is simpler to solve for the non-dimensional parameter, kx, essentially a phase parameter in the potential function.

Initial Estimate

The primary process in this LSFI-P method is nonlinear optimization to find the minimum error in a system of nonlinear algeraic equations typically involving 14 equations in 12 unknown parameters. A system as complex as this is likely to have a number of local minima that result in spurious physical solutions. The best way to avoid these solutions, as well as to allow for efficient optimization, is to start with a good estimate for the unknowns in the system, and to have a basic set of criteria for identifying spurious solutions.

The first step in each window is to establish a initial estimate for the optimization procedure. Linear wave theory can be used to produce estimates for the parameters of approximately correct magnitudes.

Nielsen (Nielsen, 1986; Nielsen, 1989) established a method for determining the parameters of a local linear approximation to waves from a pressure record. A similar method is used here. Frequency of a sinusoidal signal of the form $P_d = a\cos(kx - \omega t)$ is available from the second derivative:

$$\omega^2 = -\frac{\partial^2 P_d/\partial t^2}{P_d} \qquad \text{or} \qquad \omega = \sqrt{-\frac{\partial^2 P_d/\partial t^2}{P_d}} \tag{11}$$

Once the frequency is known, the amplitude and phase of a particular segment of record can be found by rearranging the equation as a linear least squares problem by separating the cosine and sine components:

$$P_d = a\cos(kx - \omega t) = b_1\cos\omega t + b_2\sin\omega t \tag{12}$$

The linear terms (a and kx) are a function of ω, reducing the nonlinear problem to one variable (Lawton and Sylvestre, 1971). The estimates for ω, a, and kx are refined by optimizing for the frequency that results in the least error throughout the current window, using Eqn. 11 as a first estimate.

Once the optimum frequency is found, the wave number is estimated from the linear dispersion relation, and the first estimate for the Fourier amplitudes are assigned as follows:

$$A_1 = \frac{a}{\rho\omega\cosh k(h+z)/\cosh kh} \qquad A_2 = \alpha A_1 \qquad A_j = \alpha A_{j-1} \tag{13}$$

$\alpha = 0.1$ was found to be satisfactory. The location of the water surface is estimated from the linear pressure response function with stretching (Nielsen, 1986).

$$\eta = \frac{P_d}{\rho g} \frac{\cosh k(h + (P_d/\rho g))}{\cosh k(h + z)} \tag{14}$$

An estimate for the values of the second time derivative of the record in each window is required. This is accomplished by a least squares fit of a third order polynomial to the record in each window. The derivatives can then be computed from this polynomial.This approach was very successful in the analysis of artificially generated noisy records, resulting in reasonable estimates for the value at the middle of the window, as well as both derivatives at that point.

Optimization

Once there is a reasonable first guess for all the parameters, nonlinear optimization routines can be applied to this system. For the results in this paper, the Leavenburg-Marquart algorithm was used as implemented by the Matlab Optimization Toolbox. If the optimization routine successfully finds a minimum, the solution is checked to see if a clearly spurious solution is found. Spurious solutions can be identified by the following criteria: very large or highly variable errors, first order amplitude smaller than higher order amplitudes, unrealistically large or small frequency or wave number. It is unusual for the routine to converge to a spurious solution. It is far more common for the routine not to converge at all.

If no solution or a spurious solution is found, it is necessary to revise the parameters of the solution to make another attempt. For the next attempt, the window width is increased by a factor of 1.5, and the procedure is repeated. If this is not successful, the window width is increased once more to twice the standard width. When increasing the window width is not successful, the order of the potential function is decreased until a solution is found. If none of these adjustments result in a reasonable solution, the window is skipped, and future analysis must be interpolated through that point. These adjustments are most likely to be needed in the long, flat trough of a shallow water wave, where the window needs to be expanded to include some curvature to indicate the frequency. There can also be difficulties near zero crossings, where there is little curvature in the record, and the effects of amplitude and frequency are not independent. Widening the window to include more of the surrounding record is generally successful in this situation as well.

Another complication can be a record that is symmetric about the crest of a wave. In this case, the equations on either side of the crest are not independent. This situation is unlikely to arise in a field record, and can easily be accommodated by using an asymmetric distribution of points in that window.

Results

In order to remove complications from measurement error in the initial testing of the method, pressure records generated by Fourier Steady wave theory (Sobey, 1989) were used. This also has the advantage of providing a solution with the complete kinematics, to compare with results from the LSFI-P method. High order Fourier wave theory is essentially an exact solution for irrotational steady waves that can be applied at any depth (Rienecker and Fenton, 1981; Sobey, 1989). Fig. 2 presents the results after the first guess, before optimization. This is a window near the crest of a steep, shallow water wave generated by 18th order Fourier theory. The predictions for the dynamic pressure are approximately correct, and the water surface estimate is in the vicinity of the actual water surface. Note that the location of the actual surface is given in the plot, but it is not available to help determine the solution. These points were all generated by the method outlined in the previous section, with only the pressure record as a guide. The third plot shows the non-dimensional errors in the Bernoulli equation and the free surface boundary conditions. The errors are of order .03 and show a systematic pattern, particularly in the Bernoulli equation. It is clear from these plots that a better solution can be found.

The results after optimization are given in Fig. 3. At this point the prediction for the dynamic pressure is essentially exact. This is virtually always the case, as the pressure record is available, and the parameters are found to fit that record. The predictions for the water surface are also extremely close. This is an impressive achievement, as location of the water surface was found only by minimizing the errors in the free surface boundary conditions. In this case, the LSFI-P method was able to accurately capture the crest of a steep shallow water wave.

Fig. 4 shows the results of the method for the complete wave. The parameters of the wave are: 5m water depth, 3m wave height, and 10s period, with the pressure record measured on the bottom. The LSFI-P method finds the water surface and the kinematics on the surface essentially exactly. While these results show the complete wave, it is important to keep in mind that each of the indicated points is in the center of a separate window, and was computed completely independently of the other windows. In this case, the standard window width was 2s, with a sixth order potential function and seven water surface nodes. The 2s window width is one fifth of the period of the wave, and is a reasonable length of time to extend the locally steady approximation. It is not expected that the standard window width will exceed about one fifth of the zero crossing period of a record, nor the solution to be of order higher than six.

The dotted lines on the plot are the water surface and horizontal velocity at the surface as predicted by the linear wave theory pressure response function. It is clear that this method completely misses the high, sharp crest, and the large velocities at the crest.

Fig. 5 shows the results of the LSFI-P method for an entire deep water wave.

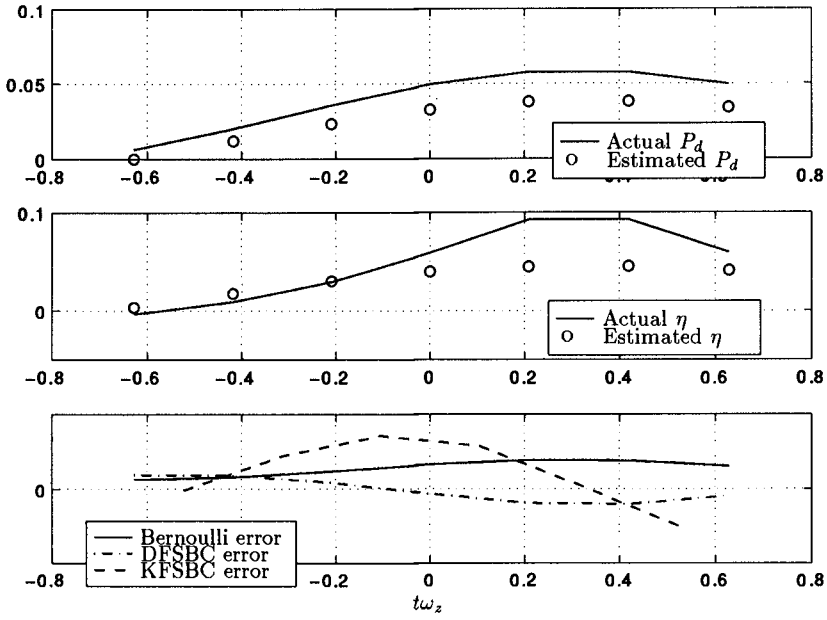

Figure 2: Results in a window after first estimate

The wave was generated by 12th order Fourier wave Theory, with parameters: 100m water depth, 10m wave height, and 10s period, with the pressure record measured 10m below the mean water level. Standard window width is 1s, with a fourth order potential function and five water surface nodes. Once again the LSFI-P solution matches the actual solution exactly. In the case of deep water, linear wave theory performs fairly well on steady waves, but is not applicable to irregular records, as there is no clearly defined single frequency or wave number.

Conclusions

While the given results are on artificially generated steady wave records, they show the potential for the method for a variety of conditions. In the case of steady waves, the LSFI-P method accurately computed the detail of the wave, using only data from a small window in time. In particular, the method was able to capture the pronounced sharp crest of a steep, shallow water wave. It is expected that it will perform well on segments of an irregular record.

The analysis of regular waves provides guidelines for the parameters to be

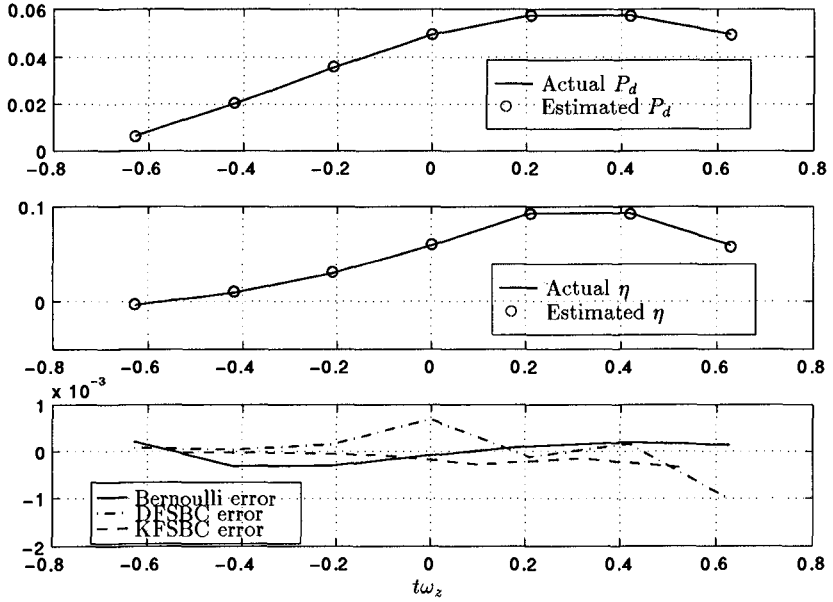

Figure 3: Final results in a window

used in the analysis of irregular waves. Higher order solutions and wider windows must be used in shallow water than in deep. Window widths of one fifth of the zero crossing period and a sixth order potential function are adequate for the shallowest waves, and window widths as small as one tenth of the zero crossing period and a third order potential function are adequate for deep water.

The Locally Steady Fourier approximation for irregular waves is an effective method for the computation of the kinematics of irregular waves from a point pressure record. The method results in a complete description of the water surface and kinematics of the waves that fit the given pressure record and the full free surface boundary conditions very closely in a small window in time. Comparisons with the predictions of steady wave theory are excellent.

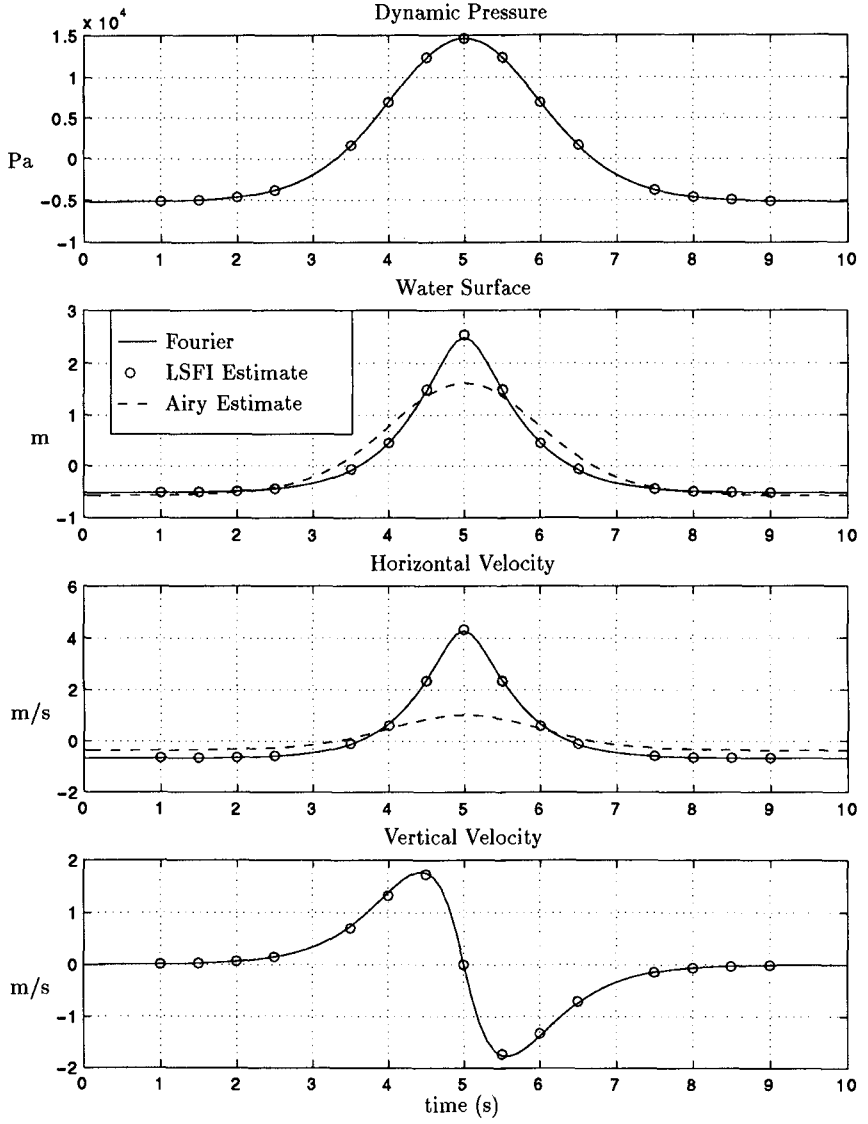

Figure 4: Results for a shallow water wave

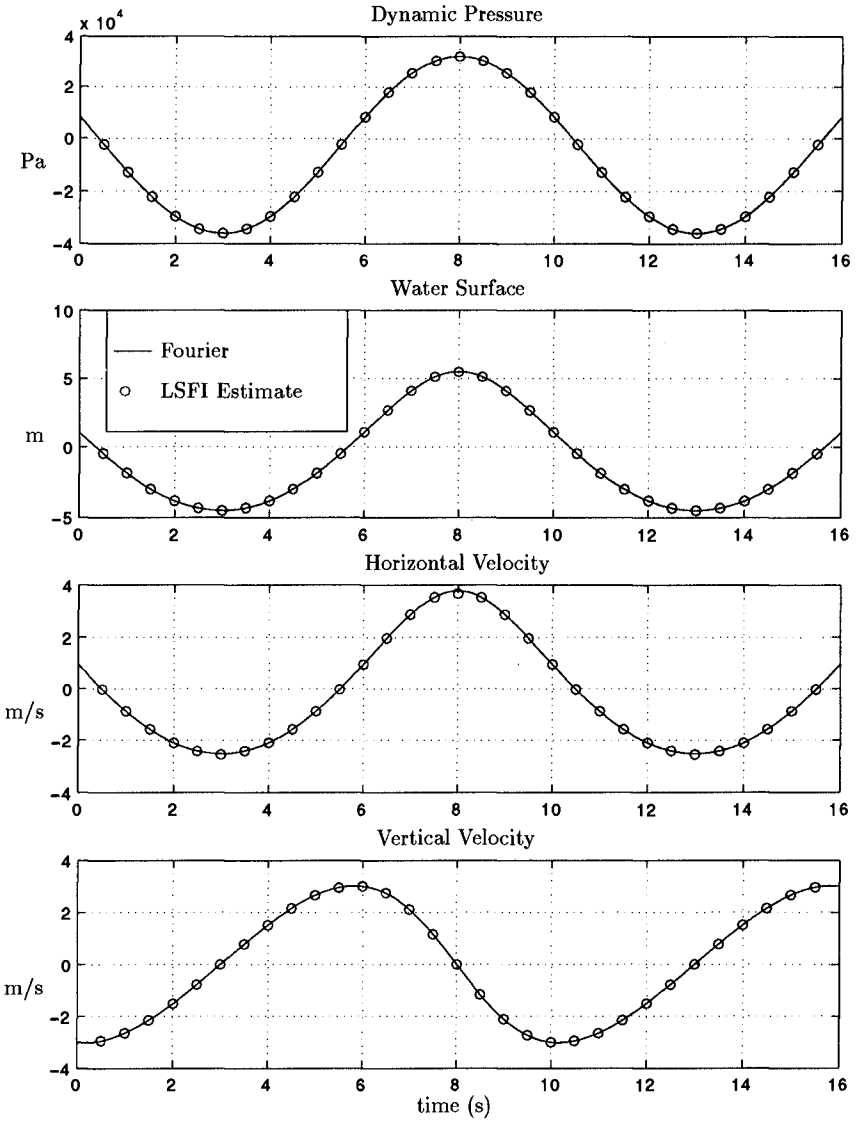

Figure 5: Results for a deep water wave

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