

CHAPTER 76

Reflection Analysis with Separation of Cross Modes

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Abstract

During physical model tests in a wave flume an accurate determination of the incident waves is crucial. Normally, the determination of the incident waves is conducted by a traditional reflection analysis and in this case the influence of any cross modes is ignored. The present paper concerns the advantage of including the possible presence of cross modal activity in the reflection analysis when determining the incident waves. It is shown both analytically and numerically how the cross modes affect the results of a traditional reflection analysis, and that ignoring cross modal activity can lead to an inaccurate determination of the incident waves. A new method is introduced where the cross modes are separated in the reflection analysis. This method requires at least three wave probes while a traditional reflection analyses only requires two wave probes. The applicability of the method is verified both numerically and by physical flume tests at the Hydraulic Laboratory of Aalborg University, Denmark. In both cases the new method seems to give very good results.

1 Introduction

One of the most important tasks in physical model studies is the determination of incident waves. The determination of the incident waves is often carried out using some kind of reflection analysis capable of separating the incident and reflected wave components mathematically.

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Goda and Suzuki (1976) presented a method capable of separating the incident and reflected waves on the basis of wave measurements at two known positions on a line parallel to the direction of wave propagation. Mansard and Funke (1980) improved this method by applying three wave gauges instead of only two. This enabled an optimization of the determination of the incident and reflected waves by the use of a least squares technique. Zelt and Skjelbreia (1992) improved this optimization procedure further by applying an arbitrary number of wave probes.

However, all these commonly used methods of reflection analysis are based on the assumption that the waves measured in the flume are composed of only two wave components; The incident and reflected components. If the measured waves contain other components, for instance cross modes, which is a well-known phenomenon in flume tests, the reflection analysis can produce inaccurate results.

In this paper, a new method of reflection analysis is presented. This method, which also uses an arbitrary number of wave probes, assumes that the measured waves are composed of three components; The usual incident and reflected wave components *and* a component representing any cross modal activity present in the wave flume. Before presenting the new method of reflection analysis, the influence of cross modes on the results of a traditional reflection analysis is addressed. This is done using the method of Zelt and Skjelbreia (1992).

2 Influence of Cross Modal Activity

The presence of cross modal activity affects the composite wave field measured in the flume. Depending on the phase differences between the cross modes and the longitudinal waves, the ordinary wave heights measured at each wave probe either increase or decrease. In case of regular waves, the influence of the cross modes is verified analytically by decomposing the measured Fourier coefficients into a primary term representing the longitudinal waves and secondary term representing the cross modes. The analytical approach is supported by numerical simulations using the method of Zelt and Skjelbreia (1992) with 5 wave probes on simulated time series with and without cross modes. In case of irregular waves, the influence of cross modes is investigated by comparing the incident and reflected wave spectra with and without cross modes contained in the simulated time series.

2.1 Analytical Verification of the Influence of Cross Modes

To investigate the influence of cross modes in traditional methods of reflection analysis analytically, the method of Zelt and Skjelbreia (1992) has been used. The method is explained very briefly in the following.

Using standard Fourier analysis techniques the surface elevation (η_p) measured at wave probe p (see Figure 1) can be expressed as:

$$\eta_p(t) = \sum_{n=0}^{N-1} A_{n,p} e^{i\omega_n t} \quad ; \quad A_{n,p} = a_{n,p} - ib_{n,p} \tag{1}$$

where $a_{n,p}$ and $b_{n,p}$ are the Fourier coefficients corresponding to frequency component n and wave probe p . ω_n is the n th cyclic frequency. Assuming linear wave theory, no frequency modulation due to the reflection process, and a composite wave field composed of incident waves and reflected waves only, the theoretical surface elevation at wave probe p (see Figure 1) can be written as:

$$\eta(x_p, t) = \sum_{n=0}^{N-1} (a_{I,n} e^{-ik_n x_p} + a_{R,n} e^{ik_n x_p}) e^{i\omega_n t} \tag{2}$$

where $a_{I,n}$ and $a_{R,n}$ are complex numbers containing the Fourier coefficients of the n th frequency component of the incident waves and reflected waves. k_n is the wave number of the n th frequency component and x_p is the position of wave probe p .

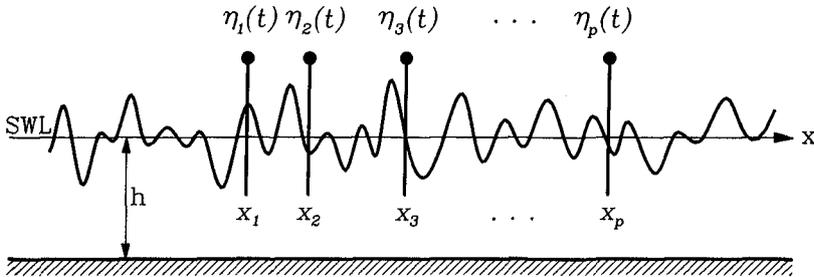


Figure 1 - Measuring composite wave fields at P wave probes.

Combining Equations (1) and (2) a complex equation for each wave probe p and frequency component n is obtained:

$$A_{n,p} = a_{I,n} e^{-ik_n x_p} + a_{R,n} e^{ik_n x_p} \tag{3}$$

The unknowns of this equation ($a_{I,n}$ and $a_{R,n}$) are the same for each wave probe. If only two wave probes are applied, Equation (3) is solved exactly but singularities at certain frequencies will occur (Goda and Suzuki, 1976). If more than two wave probes are applied the overdetermined system of equations is solved by introducing an error $\epsilon_{n,p}$ to the decomposed wave field:

$$\varepsilon_{n,p} = a_{L,n} e^{-ik_n x_p} + a_{R,n} e^{ik_n x_p} - A_{n,p} \tag{4}$$

$a_{L,n}$, and $a_{R,n}$ are then determined by minimizing a weighted sum of the squares of the error $\varepsilon_{n,p}$ (for details please refer to Zelt and Skjelbreia, 1992). The explicit solution is given by:

$$a_{L,n} = \left[S_n \sum_{p=1}^P W_{n,p} A_{n,p} e^{i\Delta\phi_{n,p}} - \sum_{p=1}^P W_{n,p} A_{n,p} e^{-i\Delta\phi_{n,p}} \sum_{p=1}^P W_{n,p} e^{2i\Delta\phi_{n,p}} \right] \frac{e^{i\phi_{n,1}}}{D_n} \tag{5}$$

$$a_{R,n} = \left[S_n \sum_{p=1}^P W_{n,p} A_{n,p} e^{-i\Delta\phi_{n,p}} - \sum_{p=1}^P W_{n,p} A_{n,p} e^{i\Delta\phi_{n,p}} \sum_{p=1}^P W_{n,p} e^{-2i\Delta\phi_{n,p}} \right] \frac{e^{-i\phi_{n,1}}}{D_n} \tag{6}$$

where:

$$S_n = \sum_{p=1}^P W_{n,p} \quad ; \quad \Delta\phi_{n,p} = \phi_{n,p} - \phi_{n,1} = k_n (x_p - x_1) \tag{7}$$

$$D_n = S_n^2 - \sum_{p=1}^P W_{n,p} e^{2i\Delta\phi_{n,p}} \sum_{q=1}^P W_{n,q} e^{-2i\Delta\phi_{n,q}} \tag{8}$$

$W_{n,p}$ is a weight coefficient determined for each wave probe and frequency component as introduced by Zelt and Skjelbreia (1992).

If cross modes are present in the flume their influence can be investigated by decomposing the measured Fourier coefficients ($A_{n,p}$) into one part representing the longitudinal waves ($A_{L,n,p}$) and one part representing the cross modes ($A_{C,n}$), which is independent of the probe number.

$$A_{n,p} = A_{L,n,p} + A_{C,n} \tag{9}$$

Equations (5) and (6) can now be rewritten as:

$$a_{L,n} = a_{L,L,n} + A_{C,n} B_n \tag{10}$$

$$a_{R,n} = a_{L,R,n} + A_{C,n} B_n^* \tag{11}$$

where:

$$B_n = \left[S_n \sum_{p=1}^P W_{n,p} e^{i\Delta\phi_{n,p}} - \sum_{p=1}^P W_{n,p} e^{-i\Delta\phi_{n,p}} \sum_{p=1}^P W_{n,p} e^{2i\Delta\phi_{n,p}} \right] \frac{e^{i\phi_{n,1}}}{D_n} \tag{12}$$

and B_n^* is the complex conjugate of B_n , $a_{L,L,n}$ and $a_{L,R,n}$ in Equation (10) and (11)

represent the solution without cross modes, whereas the last terms represent the cross mode contribution, consisting of one part representing the Fourier coefficients ($A_{c,n}$) and one pure geometrical part depending on the probe spacings (B_n and B_n^*).

The influence of the cross modes is not determined by the amplitude only, also the phase of the cross modes affects the incident and reflected Fourier coefficients as it appears from Equations (10) and (11). Due to the fact that B_n appears as the complex conjugate in Equation (11) the presence of cross modes does not affect the determination of the incident and reflected waves equally. Essentially, this means that no constant ratio of the deviations between the estimated incident and reflected wave heights and their true values exists. Furthermore, it is seen that depending on the amount of reflection, the relative deviation of the reflected amplitudes is larger than the relative deviation of the incident amplitudes.

2.2 Numerical Verification of the Influence of Cross Modes

To investigate the influence of cross modes numerically, the cross modal activity has been introduced to the longitudinal wave train by adding a secondary wave train propagating perpendicular to the ordinary direction of wave propagation. This secondary wave train is assumed to have similar spectral properties as the primary wave train but with a different phase spectrum. For regular waves the amount of cross modes in the simulated wave trains is denoted by the ratio of the amplitudes of the cross modes to the amplitude of the incident waves. For irregular waves a JONSWAP spectrum was synthesized and the ratio of the significant cross mode wave height ($H_{m0,c}$) to the incident significant wave height ($H_{m0,i}$) has been used to describe the amount of cross modal activity.

It is of course possible that natural cross modal activity would possess different spectral distributions with the spectral density concentrated at the natural frequencies of the flume. However, it seems reasonable to investigate the case where the noise in each spectral band is proportional to the energy in this band. This was also used by Mansard and Funke (1987).

Analytically, it was shown that the influence of cross modes is affected by the phase of the cross modes, but the significance of the influence has not been shown. For regular waves the influence is illustrated by varying the phase from 0 to 2π . In Figure 2, the relative deviation between the calculated wave height and the simulated wave height is shown as a function of the phase difference between the cross mode and the incident wave ($\Phi_c - \Phi_i$). A positive deviation corresponds to an overestimation of the incident or reflected wave height (H_i and H_R).

The relative deviations shown in Figure 2 are calculated on the basis of 10% and 20% cross modes. The simulated incident wave height is $H_i=0.5$ m, the period is $T=1.86$ s and the reflection coefficient is $C_r=0.25$. The phase of the incident

waves is zero, essentially meaning that the phases in Figure 2 correspond to the phase of the cross mode. However, from Equations (10) and (11) it is seen that the phase of the reflected waves also affects the results. This is also seen in Figure 2 since a constant ratio of -0.25 between the deviations of the incident and reflected wave heights exists when shifting the phase difference approximately $\pi/10$ between the two curves. Of course, the absolute value of this ratio corresponds to the ratio between the theoretical reflected and the incident wave heights.

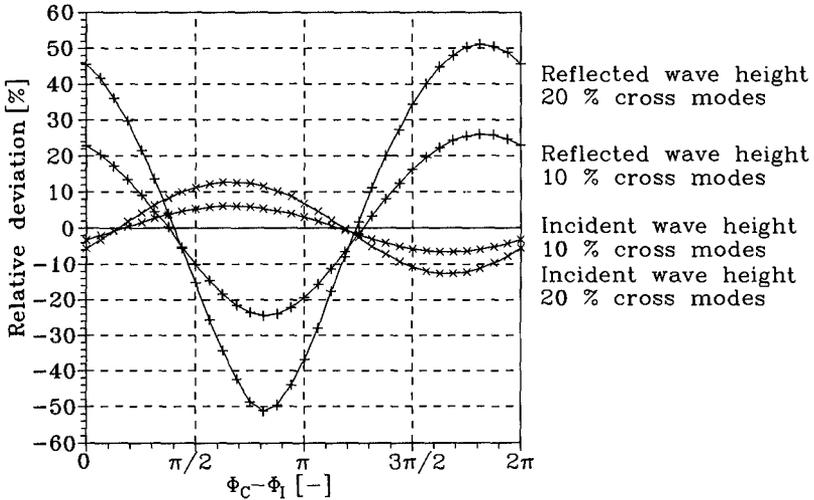


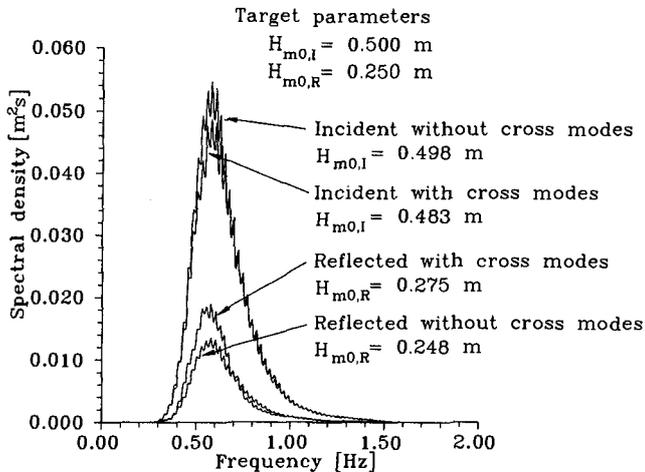
Figure 2 - Dependency of phase difference ($\Phi_C - \Phi_I$). $H_I=0.50$ m, $H_R=0.125$ m.

It appears from Figure 2 that the relative deviations are strongly dependent on the phase of the cross mode (Φ_C), the phase of the incident wave (Φ_I), and the phase of the reflected wave (Φ_R). For 10% cross modes, the relative deviation of the reflected wave height varies from zero to approximately 25%, whereas the relative deviation of the incident wave height does not exceed 7%.

Equations (10) and (11) explain the fact that the deviation in the determination of the incident and reflected waves is in almost opposite phase because of the appearance of B_n as the complex conjugate in Equation (11). If the deviations should be in exactly opposite phases, it can be seen from Equations (10) and (11) that the cross mode contribution must be zero.

As experienced through the analysis with regular waves, the appearance of cross modes do affect the individual frequency components, which is also seen for irregular waves in Figures 3 and 4, where 25 % cross modes have been introduced. Again, the cross modes affect the determination of the reflected spectrum more significantly than the determination of incident spectrum. However, the deviations are small and they can be reduced by further smoothing of the of the wave spectra.

Thus, the sensitivity to cross modes for irregular waves must be evaluated in terms of the applied smoothing of the spectra. Keeping in mind that the wave heights of the incident and reflected waves are determined as H_{m0} -values, the deviations in Figure 3 do not represent the real picture of the deviations, which is seen from the listed parameters in Figure 3. In case of regular waves it was seen from Figure 2 that if the incident wave height was overestimated, the reflected wave heights was underestimated for most phase differences. For irregular waves the deviations will be the mean value of the curves shown in Figure 2.



**Figure 3 - The influence of cross modes on irregular waves. $DOF=64$.
 Bandwidth=0.156 Hz.**

It is seen from Figure 2* that for irregular waves the incident wave height is underestimated in general (mean value < 0), whereas the reflected wave height is overestimated (mean value > 0). This is also seen from the irregular test shown in Figure 3.

As regards the reflection coefficient spectrum (Figure 4), the presence of cross modes causes a relative deviation of approximately 30% in the vicinity of the peak frequency, that is, a 30% overestimation of the reflection coefficient at the peak frequency. However, as it appears from Figure 4, the reflection coefficient is both overestimated and underestimated (depending on the frequency), which obviously can be reduced by further smoothing of the spectra.

* However, this implies that the cross mode phases, the phases of the incident waves and the phases of the reflected waves are independent and uniformly distributed between 0 and 2π . Normally, this assumption is not fulfilled for "real" waves. For instance, the phases of the incident and the reflected waves are correlated.

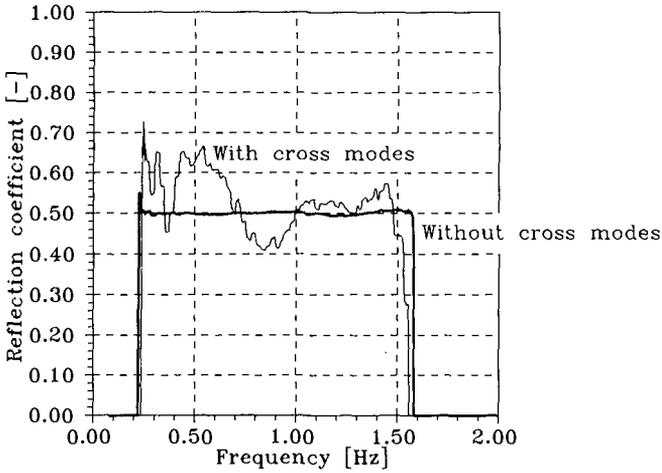


Figure 4 - The influence of cross modes on irregular waves. $DOF=64$.
Bandwidth=0.156 Hz.

3 Reflection Analysis with Cross Mode Separation

It has been shown above that the presence of cross modes can cause inaccurate estimations of the incident and reflected waves when using traditional reflection analysis methods. In this section a new method of reflection analysis will be presented, capable of taking the cross modal activity into account and thereby improving the estimation of the incident and reflected waves. First, the mathematical formulation of the reflection analysis with separation of cross modes is developed. Secondly, results from reflection analysis with separation of cross modes based on simulated data are presented. Finally, the result of a flume test carried out at the Hydraulics Laboratory of Aalborg University, Denmark is presented.

3.1 Mathematical Formulation

The cross modes are characterized by an oscillating signal which is the same at all probe locations. Obviously, this implies that the probes are placed on a straight line in the direction of wave propagation. Expanding Equation (2) the new composite wave field at wave probe p can be written as:

$$\eta(x_p, t) = \sum_{n=0}^{N-1} (a_{I,n} e^{-i\phi_{n,p}} + a_{R,n} e^{i\phi_{n,p}} + a_{C,n}) e^{i\omega_n t} \quad (13)$$

where $a_{C,n}$ is the Fourier coefficients of the cross modes.

The measured wave field is still given by Equation (1) and equating the coefficients in Equations (1) and (13) yields:

$$A_{n,p} = a_{I,n} e^{-i\phi_{n,p}} + a_{R,n} e^{i\phi_{n,p}} + a_{C,n} \quad ; \quad p = 1, 2, \dots, P \quad (14)$$

A complex equation for each wave probe with three unknowns (as opposed to two unknowns in Equation (3)), which are the same for all wave probes for each frequency has now been obtained. Therefore, an exact solution to Equation (14) requires at least three wave probes. However, using only three wave probes will cause singularities at certain frequencies, similar to those for two wave probes in the traditional methods (Andersen et al., 1995). For more than three wave probes, Equation (14) is solved by a least squares minimization procedure similar to the one described in Section 2.1 (please refer to Andersen et al. (1995) for further details). The solution is given implicitly by Equation (15).

$$\begin{aligned} a_{I,n} \sum_{p=1}^P W_{n,p} e^{-2i\phi_{n,p}} + a_{R,n} S_n + a_{C,n} \sum_{p=1}^P W_{n,p} e^{-i\phi_{n,p}} &= \sum_{p=1}^P W_{n,p} A_{n,p} e^{-i\phi_{n,p}} \\ a_{I,n} S_n + a_{R,n} \sum_{p=1}^P W_{n,p} e^{2i\phi_{n,p}} + a_{C,n} \sum_{p=1}^P W_{n,p} e^{i\phi_{n,p}} &= \sum_{p=1}^P W_{n,p} A_{n,p} e^{i\phi_{n,p}} \\ a_{I,n} \sum_{p=1}^P W_{n,p} e^{-i\phi_{n,p}} + a_{R,n} \sum_{p=1}^P W_{n,p} e^{i\phi_{n,p}} + a_{C,n} S_n &= \sum_{p=1}^P W_{n,p} A_{n,p} \end{aligned} \quad (15)$$

The explicit solution for the three complex unknowns has not been derived. Because the unknowns are complex numbers, Equation (15) is solved as six linear equations with six unknowns using standard procedures. In the present study the weight function ($W_{n,p}$) introduced by Zelt and Skjelbreia (1992) has been set equal to one (see Andersen et al., 1995).

3.2 Results

In the following the results of reflection analysis with cross mode separation are shown. These results are presented in terms of spectra and wave heights for a simulated irregular composite wave field. The composite wave field is based on the same geometry and spectral characteristics as used in Section 2. Again $H_{m0,I} = 0.50$ m, and $C_r = 0.50$ for all frequencies and finally 25 % cross modes have been added to the signal i.e., $H_{m0,C} = 0.125$ m. The peak period is 1.86 s.

In Figure 5 the incident, the reflected and the cross mode spectra are shown. It is seen that the calculated wave heights correspond to the target parameters within acceptable accuracy.

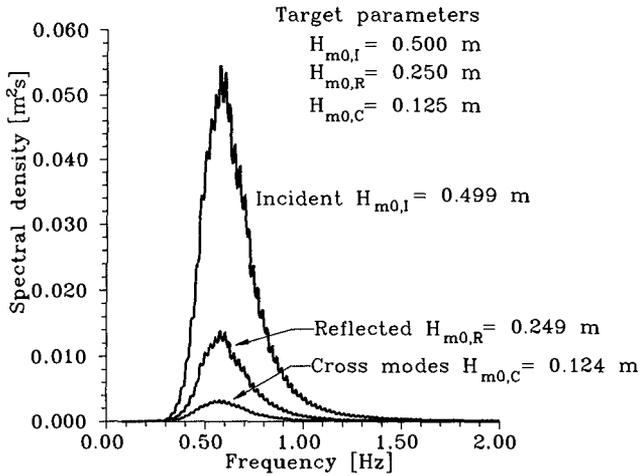


Figure 5 - Incident, reflected and cross mode spectra. Irregular waves. $DOF=64$, $bandwidth=0.156$ Hz.

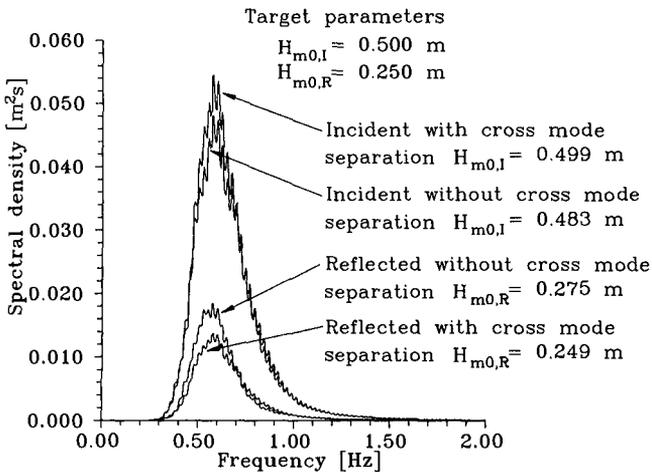


Figure 6 - Spectra with and without cross mode separation. $DOF=64$, $bandwidth=0.156$ Hz.

In Figure 6 the spectra from the reflection analysis with cross mode separation and the spectra from the analysis without cross mode separation are shown. It is seen that the incident spectrum determined with cross mode separation has larger spectral densities than the spectrum determined without cross mode separation. Particularly near the peak this is significant. This is also seen from the wave heights where the former corresponds to the simulated while the latter is 3.5 % too small. Regarding the reflected spectra the opposite is seen and the deviations are more significant, since the wave height from the analysis without cross mode separation is 8.8 % too large. These contrasts were explained in connection to Figures 2 and 3. The difference between the two spectra would be even larger if less smoothing of the wave spectra was applied. Therefore, in the case of the presence of cross modal activity the separation of cross modes is important when determining the incident wave train by inverse Fourier transformation because too much smoothing should be avoided. The deviations between the two incident spectra and the two reflected spectra are of course also seen on the reflection coefficient spectra in Figure 7.

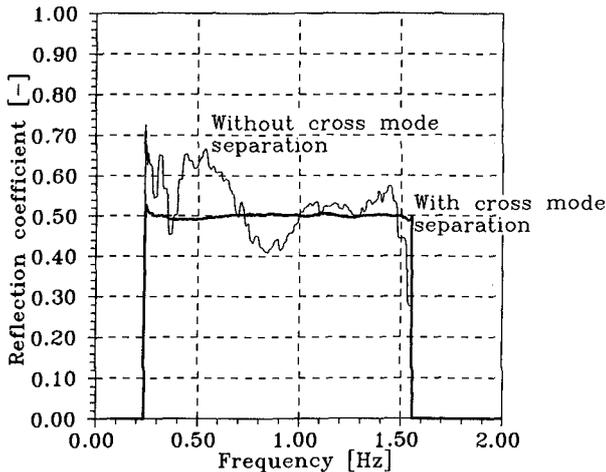


Figure 7 - Reflection coefficient spectra with and without cross mode separation.
DOF=64, bandwidth=0.156 Hz.

In Figure 8, the results of a flume test carried out in the Hydraulics Laboratory at Aalborg University, Denmark are shown. The irregular waves had a peak period of 1.00 s, corresponding to the 2nd natural frequency of the flume. It appears that the cross mode spectrum has a significant peak at the 2nd natural frequency of the flume ($f=1.00$ Hz). The two minor peaks (at $f=1.44$ Hz and $f=2.28$ Hz) correspond to the 4th and the 10th eigenmode, respectively. The deviation between the incident and reflected significant wave heights determined with and without cross mode separation is about 2 % in the present case.

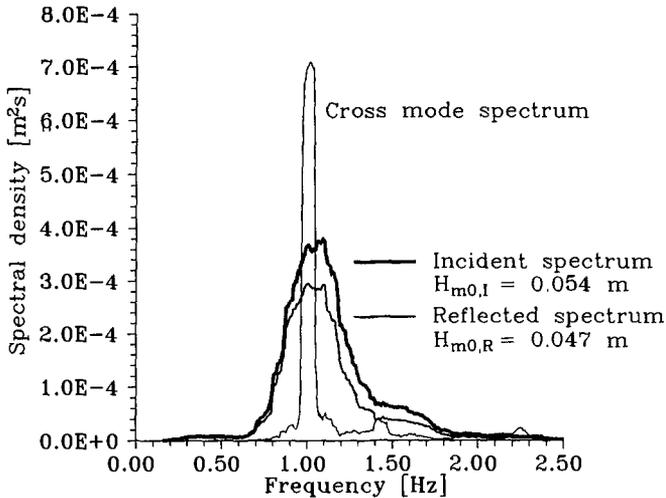


Figure 8 - Incident, reflected and cross mode spectra from flume test with irregular waves. $DOF=128$ and $bandwidth=0.31$ Hz for the incident and reflected spectra. $DOF=32$ and $bandwidth=0.078$ Hz for the cross mode spectrum.

4 Conclusions

It has been shown that if the waves measured during a flume test contain cross modal activity, it can lead to an inaccurate determination of the incident and reflected waves if ordinary methods of reflection analysis are applied. For regular waves it was shown that an error of 7 % can occur in the estimation of the incident wave height for 10 % cross modes. It was also found that the error on the reflected wave height was generally higher than the error on the incident wave height. For irregular described by their H_{m0} -value only, the error is of course smaller due to averaging.

To obtain a better estimation of the incident and reflected waves in case of cross modal activity in the flume, a new method was developed. Instead of only assuming a composite wave field composed of the incident and reflected wave components, the new method assumes that the composite wave field also contains a cross mode component. Whereas the traditional methods of reflection analysis requires at least two wave probes in order to separate the incident and reflected wave components, the new method requires at least three wave probes. To avoid singularities and to obtain more reliable solutions it is however recommended to use a least 5 wave probes.

The performance of the new reflection analysis with cross mode separation was tested both numerically and physically. It was shown that the method is capable of

separating the cross mode spectrum and thereby obtain a more accurate determination of the incident and reflected waves.

Acknowledgements

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