CHAPTER 47

A Nonlinear Model for Wave Propagation

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Abstract

Employing the Hamitonian theory, the canonical equations of water waves is used to derive a nonlinear model. In this paper, a unified nonlinear model for water wave propagation is presented. This model can be simplified to the mild-slope equation in the linear case. It is consistent with Stokes wave theory when water depth is deep and reduces to an equation of Boussinesq's type in shallow waters. Results of numerical computations of nonlinear water waves propagating over a submerged bar and a rectangular step are also presented in one-dimensional case. Nonlinear behaviors of water waves are captured, but further works are needed.

Introduction

A unified mathematical model for wave propagation from deep sea into the coastal waters has been long in pursuit. In early seventies, the mild-slope equation was first derived independently by Berkhoff (1972) and Smith & Sprinks (1975). The mild-slope equation reduces to the Helmholtz equation in deep waters and constant water depth. It reduces to shallow water wave equation when water depth becomes shallow. Because of depth integration, the mild-slope equation has simplified the three-dimensional problem into a two-dimensional one on the horizontal plane. Although the original equation is derived for monochromatic waves, it has been used for the entire spectrum of wave frequency. Based on the mild-slope equation, several numerical models have been developed to describe the combined wave refraction and diffraction successfully (Bettess and Zienkiewicz, 1977; Tsay and Liu, 1983).

In order to take other physical mechanisms, such as absorbing boundary, energy dissipation and fast-varying water depth, etc, into accounts, a variety of model equations has been employed to develop numerical mod-

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els for computation of wave scattering (Chen, 1986; Kirby, 1986; Tsay et al., 1989). However, all of the model equations are limited to linear waves. The nonlinear effects can not be ignored for waves of finite amplitude or when linear waves propagate into the shallow water regions. For weakly nonlinear waves over varying topography, mathematical models have been proposed either using Stokes' approach or employing Boussinesq equations (Liu & Tsay, 1984; Witting, 1984; Liu, et al., 1985; Nwogu, 1993; Liu, 1993). These models are limited to different water depth regimes. A unified model to describe wave transformation when it propagates from deep sea into coastal shallow water is strongly desired (Witting, 1984; Nwogu, 1993; Liu, 1993).

The governing equations for weakly nonlinear water waves can be obtained by employing the Hamiltonion theory in variational calculus (Broer, 1974). Radder and Dingmans (1985) had shown that the canonical equations of the Hamiltonian theory can be reduced to Stokes' wave form in deep waters and reduced to a Boussinesq-type equation in shallow waters. However, Radder and Dingemans did not derive an explicit, nonlinear equation for wave propagation. In this paper, employing Taylor's series expansion of free surface displacement and keeping the terms up to the third-order, we derive a unified nonlinear model equation for wave propagation. Validity of present nonlinear model is demonstrated theoretically by comparing model equations for different regimes of applications. Present nonlinear model is applied to calculate waves propagating over a submerged bar and a rectangular step in the third section. Discussions of present nonlinear wave model are followed.

Mathematical Formulation

For self-completeness of this paper, we give a brief derivation following Broer (1974), and Radder and Dingmans (1985).

For water body defined between bottom, $z = -h$, and free surface ζ, the cannonical equations of Hamiltonian theory can be expressed as:

$$\frac{\delta H}{\delta \phi} = \frac{\partial \zeta}{\partial t} \tag{1}$$

$$\frac{\delta H}{\delta \zeta} = -\frac{\partial \phi}{\partial t} \tag{2}$$

where ϕ is the free surface velocity potential, and ζ is the free surface displacement. H is a functional and represents the total energy of water body.

$$H = \int \int H_0 dx dy \tag{3}$$

where H_0 is the energy density function and can be written as:

$$H_0 = \frac{1}{2} g \zeta^2 + \frac{1}{2} \int_{-h}^{\zeta} [(\nabla \Phi)^2 + (\frac{\partial \Phi}{\partial z})^2] dz \tag{4}$$

where the velocity potential at any point, Φ, is related to the velocity potential at the free surface by a distribution function in z–direction.

$$\Phi = f(z)\phi \tag{5}$$

with

$$f(z) = \frac{\cosh k(z+h)}{\cosh k(\zeta+h)} \tag{6}$$

It is noted that $f(\zeta) = 1$ and k is a characteristic value. When ζ is dropped from the equation, the distribution function reduces to the same one as linear cases and the characteristic value k represents wave number.

Assuming that change rates of water depth and characteristic value are negligible when slope of water bottom is mild, eqs.(1) and (2) after integrating from the bottom to the free surface can be expressed as:

$$\begin{aligned}\frac{\partial \zeta}{\partial t} =& (D+E\zeta)\nabla\phi\cdot\nabla\zeta + (F+G\zeta+H\zeta^2)\phi + I\phi(\nabla\zeta)^2 \\ & - \nabla\cdot[(A+B\zeta+C\zeta^2)\nabla\phi + (D+E\zeta)\phi\nabla\phi] + O(\zeta^4) \qquad (7)\\ \frac{\partial \phi}{\partial t} =& -[g\zeta + \frac{1}{2}(\nabla\phi)^2(B+2C\zeta) + E\phi\nabla\phi\cdot\nabla\zeta + \frac{1}{2}\phi^2(G+2H\zeta)] \\ & + \nabla\cdot[(D+E\zeta)\phi\nabla\phi + I\phi^2\nabla\zeta] + O(\zeta^4) \qquad (8)\end{aligned}$$

where

$$\begin{aligned} A &= k\tanh kh \\ B &= k(k - k\tanh^2 kh) \\ C &= \frac{2kh + \sinh 2kh}{4k\cosh^2 kh} \\ D &= \frac{\cosh kh - kh\sinh kh}{\cosh^3 kh} \\ E &= \frac{-4kh + 2kh\cosh 2kh - 3\sinh 2kh}{4k\cosh^4 kh} \\ F &= \frac{k(-2kh + \sinh 2kh)}{4\cosh^2 kh} \\ G &= \frac{k^3 h\tanh kh}{\cosh^2 kh} \\ H &= \frac{k^3(4kh - 2kh\cosh 2kh + \sinh 2kh)}{4\cosh^4 kh} \end{aligned} \tag{9}$$

The free surface, ζ, is related to the velocity potential, ϕ,

$$\begin{aligned}\zeta =& -\frac{1}{g}\frac{\partial\phi}{\partial t} - \frac{1}{2g}[(B - \frac{2C}{g}\frac{\partial\phi}{\partial t})(\nabla\phi)^2 - \frac{2E}{g}\phi\nabla(\frac{\partial\phi}{\partial t})\cdot\nabla\phi + (G - \frac{2H}{g}\frac{\partial\phi}{\partial t})\phi^2] \\ & - \frac{1}{g}\nabla\cdot[(-D + \frac{E}{g}\frac{\partial\phi}{\partial t})\phi\nabla\phi + \frac{I}{g}\phi^2\nabla(\frac{\partial\phi}{\partial t})] + O(\zeta^4)\end{aligned} \tag{10}$$

Combining eqs.(7) and (8), a nonlinear equation of velocity potential can be obtained as:

$$\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}+\frac{1}{2g}\frac{\partial}{\partial t}[(B-\frac{2C}{g}\frac{\partial\phi}{\partial t})(\nabla\phi)^2-\frac{2E}{g}\phi\nabla(\frac{\partial\phi}{\partial t})\cdot\nabla\phi+(G-\frac{2H}{g}\frac{\partial\phi}{\partial t})\phi^2]$$
$$+\frac{1}{g}\frac{\partial}{\partial t}\nabla\cdot\{[-D+\frac{E}{g}\frac{\partial\phi}{\partial t}]\phi\nabla\phi+\frac{I}{g}\phi^2\nabla(\frac{\partial\phi}{\partial t})\}$$
$$=\frac{-1}{g}[-D-\frac{E}{g}\frac{\partial\phi}{\partial t}]\nabla(\frac{\partial\phi}{\partial t})\cdot\nabla\phi-\{\frac{I}{g^2}[\nabla(\frac{\partial\phi}{\partial t})]^2+F-\frac{G}{g}\frac{\partial\phi}{\partial t}-\frac{GB}{2g}(\nabla\phi)^2$$
$$-\frac{G^2}{2g}\phi^2-\frac{G}{g}\nabla\cdot(D\phi\nabla\phi)+\frac{H}{g^2}(\frac{\partial\phi}{\partial t})^2\}\phi+\nabla\cdot\{[A-\frac{B}{g}\frac{\partial\phi}{\partial t}-\frac{B^2}{2g}(\nabla\phi)^2-\frac{GB}{2g}\phi^2$$
$$+\frac{B}{g}\nabla\cdot(D\phi\nabla\phi)+\frac{C}{g^2}(\frac{\partial\phi}{\partial t})^2]\nabla\phi-\frac{1}{g}[-D+\frac{E}{g}\frac{\partial\phi}{\partial t}]\phi\nabla(\frac{\partial\phi}{\partial t})\}+O(\zeta^4) \tag{11}$$

In shallow water, where the parameter of $kh=\mu$ is small, we expand the coefficients of A, B, C, D, E, F, G, H and I in eq.(9) up to the accuracy of $O(\mu^3)$ of Taylor series. For one-dimensional cases, when normalized variables, $\varepsilon=a/h(a$:amplitude$), x'=kx, t'=k\sqrt{gh}\,t, \zeta'=\zeta/a, \phi'=k\phi/\varepsilon\sqrt{gh}$ are introduced, and ε and μ^2 are assumed in the same order of magnitude, eq.(11) can be reduced to:

$$\phi'_{t't'}+\varepsilon(2\phi'_{x'}\phi'_{x't'}+\phi'_{t'}\phi'_{x'x'})-\phi'_{x'x'}=O(\varepsilon^2,\mu^3) \tag{12}$$

This is a nonlinear equation of Boussinnesq type for waves propagating over shallow water depths.

Using Stokes expansion,

$$\zeta=a_1\cos\theta+(a_{20}+a_{21})\cos 2\theta+a_3\cos 3\theta$$
$$\phi=b_1\sin\theta+b_2\sin 2\theta+b_3\sin 3\theta+O(a_1^4) \tag{13}$$

where $\theta=kx-\omega t$, ω is the radian frequency, we obtain the following coefficients in terms of amplitude, a_1, up to the accuracy of $O(a_1^4)$:

$$a_{20}=-\frac{ka_1^2}{2\sinh 2kh}$$
$$a_{21}=\frac{8kh-4kh\cosh 2kh+12\sinh 2kh+\sinh 4kh}{4(-2kh+\sinh 2kh)\sinh 2kh}ka_1^2$$
$$b_1=\frac{g}{\omega}[a_1+\frac{k^2(66\cosh kh-3\cosh 3kh+\cosh 5kh-52kh\sinh kh-4kh\sinh 3kh)}{16(-2kh+\sinh 2kh)\sinh 2kh\cosh kh}a_1^3]$$
$$b_2=\frac{g}{\omega}\frac{k(7\cosh kh+\cosh 3kh-4kh\sinh kh)}{4(-2kh+\sinh 2kh)\cosh kh}a_1^2$$
$$a_3=k^2a_1^3(932kh\cosh kh-246kh\cosh 3kh-30\cosh 5kh$$
$$+541\sinh kh-264k^2h^2\sinh kh+616\sinh 3kh+72k^2h^2\sinh 3kh$$
$$+78\sinh 5kh+3\sinh 7kh)/[128(-2kh+\sinh 2kh)^2\sinh kh\cosh^2 kh] \tag{14}$$

with the nonlinear dispersion relation:

$$\frac{\omega^2}{g}=k\tanh kh[1$$
$$+\frac{32kh-16kh\cosh 2kh-4kh\cosh 4kh+31\sinh 2kh+4\sinh 4kh+\sinh 6kh}{8(-2kh+\sinh 2kh)\sinh 2kh^2}k^2a_1^2] \tag{15}$$

The quantity of a_{20} represents change of lower mean water level under nonlinear wave action and is the same as that derived by Bowen (1968) for wave set down.

When kh approaches inifity for deep water depth, those coefficients in eqs. (14), and nonlinear dispersion relation, eq.(15) reduces to:

$$\begin{aligned} a_{20} &= 0 \\ a_{21} &= \frac{1}{2}ka_1^2 \\ a_3 &= \frac{3}{8}k^2a_1^3 \\ \frac{\omega^2}{g} &= k(1+k^2a_1^2) \end{aligned} \tag{16}$$

These results are identical to those in the Stokes wave theory.

When all the nonlinear terms are dropped and monochromatic waves are assumed, eq.(11) can be easily simplified to the mild-slope equation (Berkhoff, 1972) with linear dispersion relation of the characteristic value, k, defined as wave number.

The model equation, eq.(11), therefore unifies propagation of nonlinear waves from deep to shallow water depths which is accurate up to the third order of incident wave amplitude. We plausibly use eq.(11) to simulate nonlinear wave propagation.

Numerical Computations

Due to the complexity of the model equation and difficulty of determining the characteristic value for nonlinear waves of third order, we employ Stokes expansion for second order nonlinear monochromatic waves.

$$\zeta = Re\{\zeta_0 + \zeta_1 e^{-i\omega t} + \zeta_2 e^{-2i\omega t}\} + O(|\zeta_1|^3) \tag{17}$$

$$\phi = Re\{\phi_0 + \phi_1 e^{-i\omega t} + \phi_2 e^{-2i\omega t}\} + O(|\phi_1|^3) \tag{18}$$

The model equations for different orders can be written as:

$$(\frac{\omega^2}{g} - F)\phi_1 + \nabla\cdot(A\nabla\phi_1) = 0 \tag{19}$$

$$\zeta_1 = \frac{i\omega}{g}\phi_1 \tag{20}$$

$$\zeta_0 = \frac{-B}{4g}|\phi_1|^2(\nabla S_{\phi_1})^2 - \frac{G}{4g}|\phi_1|^2 \tag{21}$$

$$\begin{aligned} (\frac{4\omega^2}{g} - F)\phi_2 + \nabla\cdot(A\nabla\phi_2) = &\frac{D}{2}\nabla\phi_1\cdot\nabla\zeta_1 + \frac{G}{2}\phi_1\zeta_1 - \nabla\cdot(\frac{B}{2}\zeta_1\nabla\phi_1 + \frac{D}{2}\phi_1\zeta_1) \\ &+ \frac{2i\omega}{g}[-\frac{B}{4}(\nabla\phi_1)^2 - \frac{G}{2}\phi_1^2 + \nabla\cdot(\frac{D}{2}\phi_1\nabla\phi_1)] \end{aligned} \tag{22}$$

$$\zeta_2 = \frac{2i\omega}{g}\Phi_2 - \frac{B}{4g}(\nabla\phi_1)^2 - \frac{G}{4g}\Phi_1^2 + \frac{1}{g}\nabla\cdot(\frac{D}{2}\Phi_1\nabla\phi_1) \tag{23}$$

where S is the phase function and the characteristic value, k, determined by the dispersion relation of

$$\frac{\omega^2}{g} = k\tanh kh \tag{24}$$

Wave field can be calculated by using eqs.(19), (20), (21), (22), and (23) for each component. The free surface displacement is obtained by employing eq.(17). For the case of nonlinear waves propagating over a submerged bar (Beji and Battjes, 1994), Fig. 1, incident waves with wave height $2cm$, period $2sec.$ propagate from left to right. The constant depth before the bar is $0.40m$ and the Ursell number, $(U_r = (a/h)/(kh)^2)$, is 0.054. In the computational domain of $12m$, 481 nodal points are used. For numerical calculations, eqs.(19) and (22) are discretized into finite difference equations. Radiation conditions of outgoing wave components are applied to both ends of the computational domain. The sloping bottom on the right hand side is assumed to absorb all of the wave energy. In experiments (Beji and Battjes, 1993), gage 1 indicated incident waves. We campare the numerical solutions with experimental results for waves over the submerged bar (Figs. 2, 3, 4, 5, 6 and 7). The numerical solutions are obtained by calculating amplitude distribution in space of each component and reconstructing the time history of free surface at each point. It can be observed that there is a significant difference between linear and nonlinear waves. The nonlinear behavior of waves due to change of water depth is captured by present nonlinear model quite nicely. However, discrepancy between numerical solutions and experimental results at wave troughs seems quite persistent.

We further extend present nonlinear model to calculate waves over a step, Fig.8 (Kittitansuan et al., 1993). Although the slope at the step violate mild-slope assumption, waves evolve nonlinearly after the step. The depths before and after the step are $0.376cm$ and $0.113cm$, respectively. Incident wave height is $1.63cm$ and period $1.85sec.$ The Ursell number in this case is 0.042. Total nodal points of 601 are used in a $12m$ computational domain. The step is simulated by a sudden change of water depth between two nodes with a slope of 6.575. Present numerical time histories of free surface at points after the step of $1m$ and $2m$ are compared with experimental results, Figs.9 and 10 (Kittitanasuan et al., 1993). Good agreement is observed except profiles at wave troughs. We also calculate free surface profile in space and compare with that of Kittitanasuan et al (1993). The step is located at the origin of x-axis. Waves behaves almost linearly before the step and evolve nonlinearly further away from the step. Present nonlinear model is in good agreement with theory of Kittitanasuan et al (1993).

Discussions and Conclusions

In this paper, we present a derivation of nonlinear model in the accuracy of $O(\zeta^4)$ for water wave propagation. It unifies nonlinear wave models for different depth regimes. Present nonlinear model reduces to the mild-slope equation in the linear case. It is shown that present model can be simplified to a Boussinesq equation for shallow water and Stokes waves when water

depth is deep. However, due the complexity of the governing equation and the difficulty in determining the characteristic value of dispersion relation, we calculate nonlinear waves up to the second order and compare present numerical solutions with experimental results in two one-dimensional cases. Good agreement is obtained. Discrepancy between present numerical solutions and experimental results of free surface profiles at wave troughs remains for future study.

Acknowledgement

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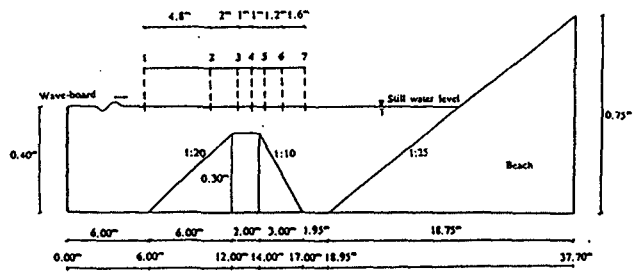

Fig. 1 Experimental set-up for waves over a submerged bar (Beji and Battjes, 1993)

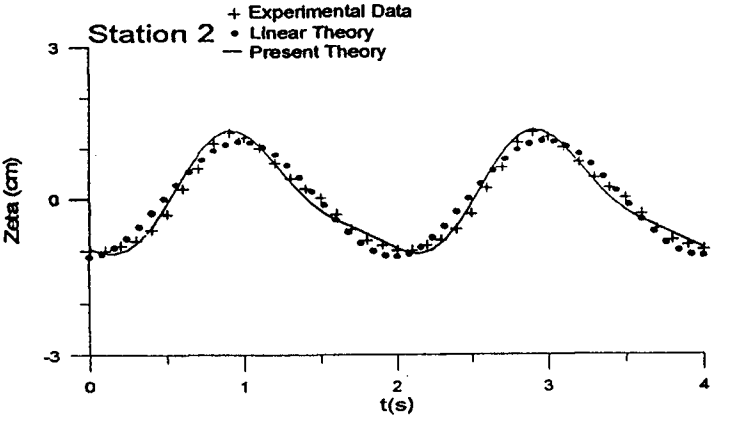

Fig. 2 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 2

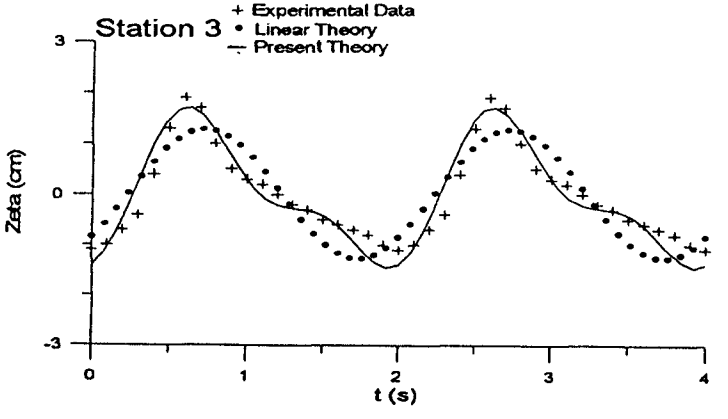

Fig. 3 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 3

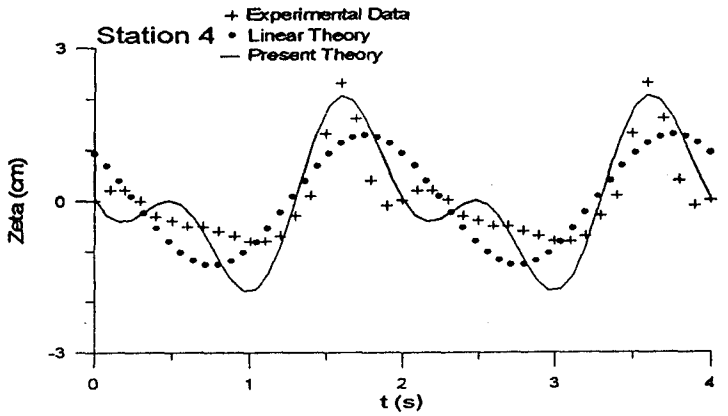

Fig. 4 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 4

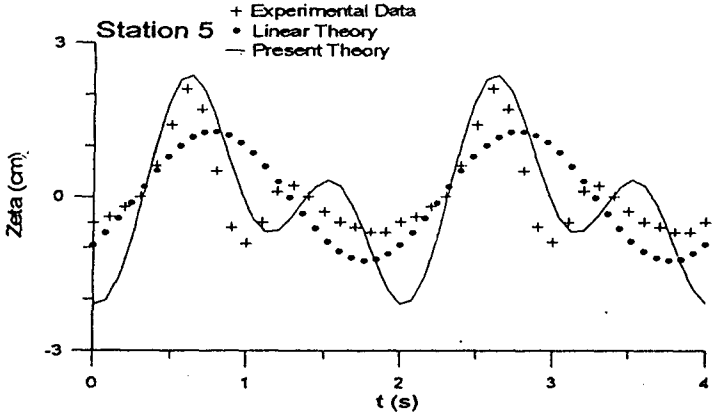

Fig. 5 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 5

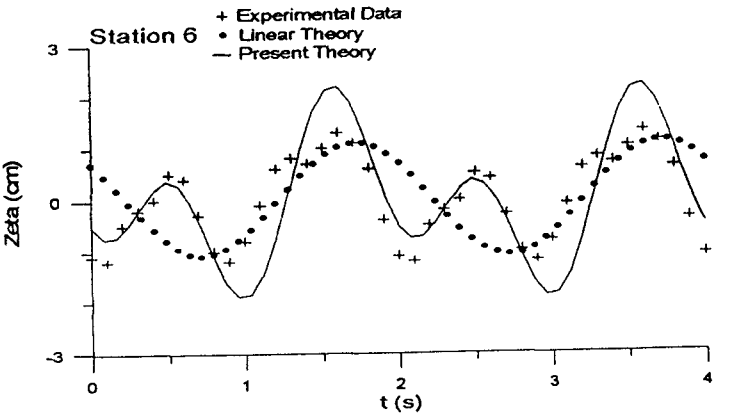

Fig. 6 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 6

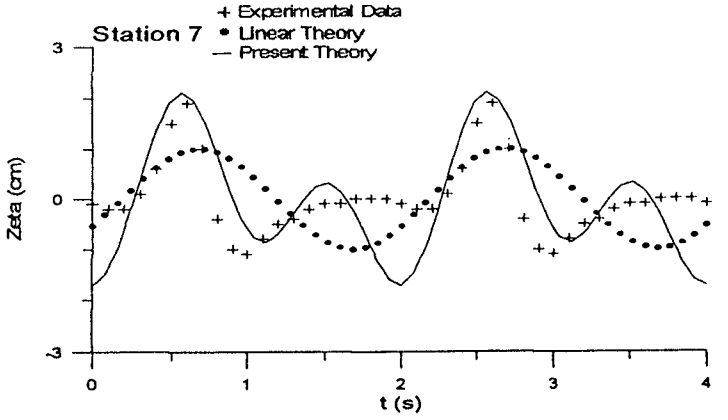

Fig. 7 Comparison of present numerical solutions with experimental results (Beji and Battjes, 1993), station 7

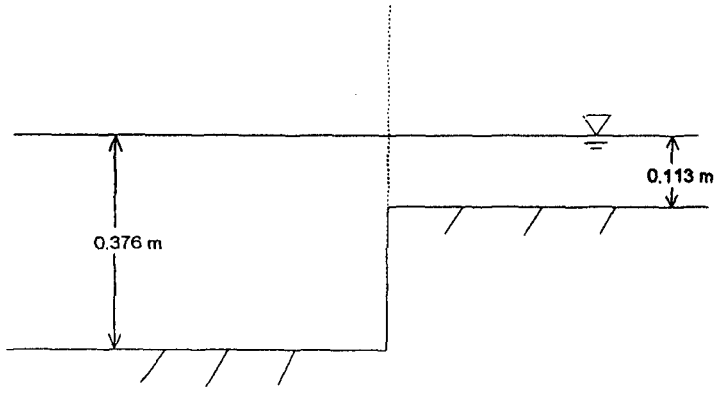

Fig. 8 Experimental set-up for waves over a rectangular step (Kittitanasuau et al.,1993)

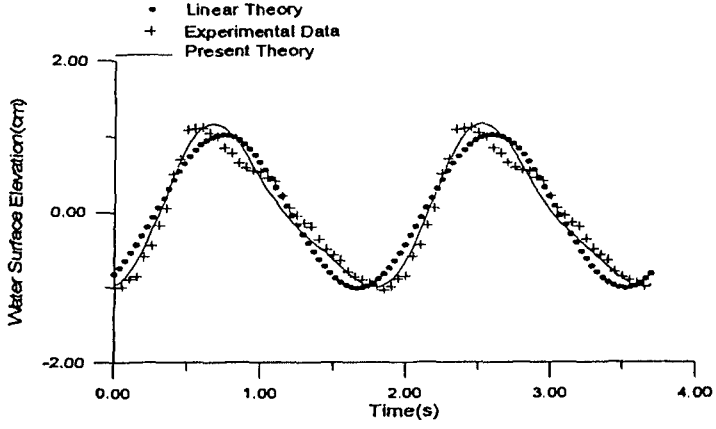

Fig. 9 Comparison between present numerical solutions and experimental results (Kittatanasuan et al., 1993) at 1m after the step

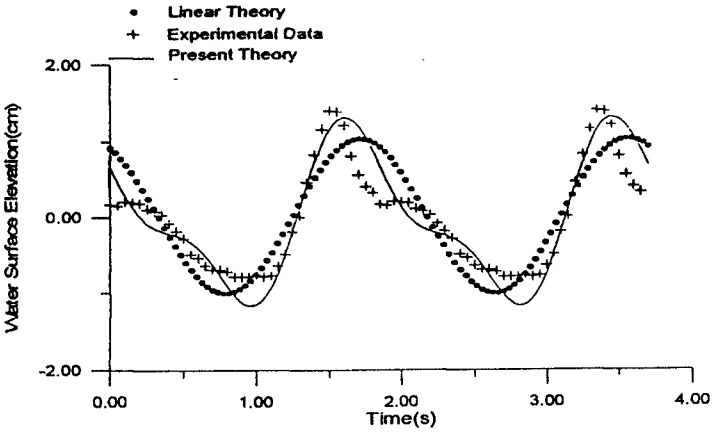

Fig. 10 Comparison between present numerical solutions and experimental results (Kittatanasuan et al., 1993) at 2m after the step

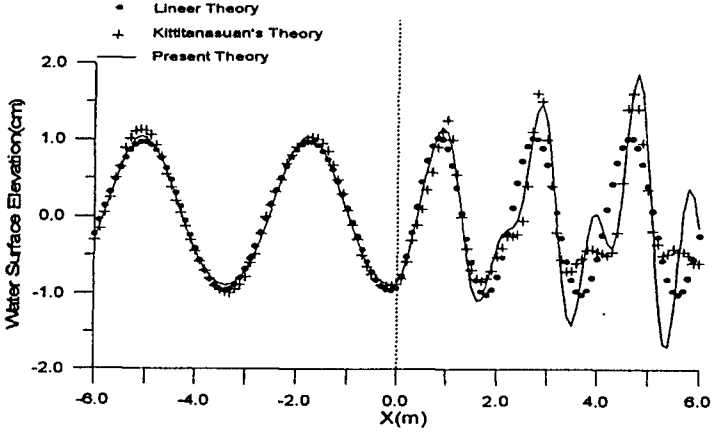

Fig. 11 Comparison of free surface profile in space between present model and Kittatanasuan et al., (1993).