CHAPTER 45

Particle Velocity Distribution in Surface Waves

Geir Moe[1] and Øivind A. Arntsen[2]

Abstract

In offshore and ocean engineering it is often of interest to be able to model particle velocities in the so-called splash zone, in which during a wave cycle, a given point is sometimes submerged, sometimes in air. (Emergence effects.) This paper applies the so-called Gerstner wave theory, extends it to narrow banded irregular waves, and presents it in an Eulerian description to second order in wave amplitude, so that the results may be compared with measurements made at fixed positions in the fluid. Mean horizontal velocities have been determined, and shown to compare excellently to laboratory measurements, both for regular and irregular waves. At this point in time, it can not be said whether the fit will be equally good in real ocean waves.

Introduction

The design of structures in an ocean environment is often governed by wave loading, requiring the determination of water particle kinematics. However, real ocean waves are irregular and nonlinear, and no universally accepted theory is available for predictions of such flows, especially not in the splash zone. This is unfortunate, since rather large contributions to the total loading may originate in the splash zone. The most important statistical properties of particle kinematics at fully submerged points for mildly nonlinear, irregular waves have been successfully determined by Longuet-Higgins (1963). The term "particle kinematics" is here used to denote particle velocities as well as accelerations. When emergence effects in the splash zone are to be included, most of the available models are of an approximate, "engineering" type, such as Wheeler stretching or similar, e.g. the models associated with the names of Chakrabarti, Gudmestad or

---

[1] Professor, Department of Structural Eng., Norwegian University of Science and Technology, N-7034 Trondheim, Norway.
[2] Associated Professor, Department of Structural Eng., Norwegian University of Science and Technology, N-7034 Trondheim, Norway.

Heideman, see e.g. Skjelbreia et al. (1991) for full references. A different type of approach was introduced by Tung (1975). He used Airy wave theory, but modified the results by checking whether a splash zone point at a given instant was submerged, or in air, by comparing the vertical coordinate to the instantaneous location of the wave profile, again predicted by Airy theory. Technically this was done by multiplication with a step function, constructed such that the flow velocity became zero when the considered point was in air. Cieslikiewicz & Gudmestad (1993) used a similar approach in conjunction with the previously mentioned higher order wave approach of Longuet-Higgins. However, their results compared only moderately well to Laser Doppler measurements of particle velocities made by Skjelbreia et al. (1989).

In the present paper an alternative theory will be explored, which gives quite good predictions of the mean of the measured horizontal particle velocity. Good agreement were found for several cases tested, in irregular as well as regular waves. The idea underlying this model is quite simple: it is assumed that the particle paths are circles which are traversed at constant velocity. This assumption leads to a deep water wave theory commonly denoted as the Gerstner wave, which will be presented in the next section. A word of caution may be appropriate at this point: The comparison is made between a Gerstner wave and experimental results from a closed wave tank in which waves were generated into quiescent water. The measurements cover only the initial phases after the wave generator had been turned on, the flow field will change if the wave generator is left on for a long time, see Mei (1972). Real ocean waves may have different characteristics from this, since there the waves have been generated in a different manner, viz. over vast areas and during long times, and by different mechanisms of generation. Even so, it is suggested that studies of waves under carefully controlled conditions in test basins may shed considerable light on the general wave kinematics problem.

Basic Equations

The derivation of Airy wave theory is well known and will not be repeated here. It is normally developed in an Eulerian frame of reference, the key point being that the boundary condition at the free surface is applied at the still water level, and that quadratic velocity terms are dropped from the surface boundary condition. Its justification rests on a number of assumptions, notably that the wave amplitude is vanishingly small compared to the wavelength. However in engineering practice linear theory is routinely applied to situations where the amplitude typically is 0.01 to 0.05 of the wavelength, and experience indicates that acceptable accuracy will usually be obtained for most of the physical quantities involved, provided the observation point stays submerged at all times. One major advantage to the Airy theory is that it is linear, and that therefore the principle of superposition applies. The Gerstner wave is also linear, (Kinsman, 1965). It satisfies continuity and the surface conditions exactly, but the flow is rotational. The Gerstner wave will now be described in the usual coordinate system with the x-axis in the direction of wave propagation and z vertically upwards, and the origin at the still water level.

Assuming deep water waves of wave period $T = 2\pi/\omega$ and wave length $\lambda = 2\pi/k$, the location of the particle $x(x_0, z_0, t)$, $z(x_0, z_0, t)$ at time t is given by:

$$x = x_0 - ae^{kz_0}\cos(\omega t - kx_0), \tag{1}$$

$$z = z_0 + ae^{kz_0}\sin(\omega t - kx_0). \tag{2}$$

This is in fact a Lagrangian description of particle motions. It easily seen that for fixed values of x_0, z_0, and varying t, the coordinates (x, z) describe a circle about the point (x_0, z_0), whose radius is a at the surface (i.e. at $z_0 = 0$), and decays exponentially with the distance below the surface. The shape of the free surface $\eta(x,t)$ follows by setting $z_0 = 0$:

$$\eta(x,t) = a\sin(\omega t - kx_0(x,t)) \tag{3}$$

When t is kept constant, the equation defines the surface profile, as a function of x. This represents a trochoid[3] as depicted in Fig. 1 for fixed time $t = 0$.

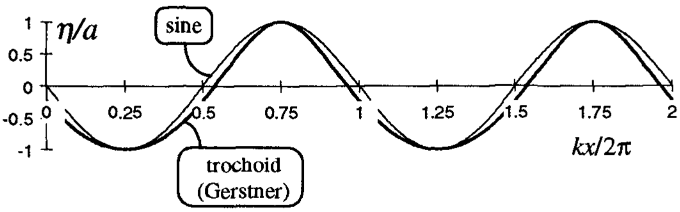

Figure 1. Free surface variation of a Gerstner wave.
Data as for wave case R15b. (Skjelbreia et al., 1989)
$H = 0.13$ m, $T = 1.5$ s, $\lambda = 3.5$ m, d = 1.3 m.

The phase velocity $c = \lambda/T$ is as for the Airy wave, and a close inspection of the free surface condition would reveal that it will be satisfied exactly, provided the deep water dispersion relation is satisfied, see e.g. Kinsman (1965) or Lamb (1932):

$$c = \omega / k \tag{3B}$$

$$k = \omega^2 / g \tag{3C}$$

It is now necessary to convert to a description in terms of fixed coordinates (x, z), in order to describe the particle velocities at a given point, i.e. an Eulerian description is required. Without loss of generality one may consider the point ($x = 0$, z).

---

[3] trochoid: the curve generated by a point somewhere on the radius line (centre distance a) of a circle (radius $r = \lambda/2\pi$) as the circle circumference rolls on a fixed straight line.

This is convenient because then (kx_0) represents a small parameter, of order (ka), so that a low order Taylor expansion may be used for functions of (kx_0). In the final expressions the equations will be expanded to second order in (ka). From (1) with $x = 0$ initially to first order one has

$$x_0^{(1)} = ae^{kz}\cos(\omega t)\,, \tag{4}$$

$$\zeta^{(1)} = z - z_0 = ae^{kz}\sin\omega t\,. \tag{5}$$

The second order expression for x_0 may be found by substitution of the above first order approx. into (1),

$$\begin{aligned} x_0 &= a\exp(kz - k\zeta^{(1)})\cos(\omega t - kx_0^{(1)}) \\ &= ae^{kz}(1-k\zeta^{(1)})(\cos\omega t + kx_0^{(1)}\sin\omega t) \\ &= ae^{kz}\cos\omega t\,. \end{aligned} \tag{6}$$

Similarly for z one has

$$\begin{aligned} \zeta &= z - z_0 \\ &= ae^{kz}(1-k\zeta^{(1)})(\sin\omega t - kx_0^{(1)}\cos\omega t) \\ &= ae^{kz}\sin\omega t - ka^2e^{2kz}\,. \end{aligned} \tag{7}$$

Thus for a point at the surface $z_0 = 0$, and $z(0,0,t)$ represents the surface elevation $\eta(0, t)$ as shown in Fig. 2. At a crest $z = a$ and $\sin\omega t = 1$ and from (7) follows that $\zeta = a$, while at a trough $\zeta = -a$, both accurate to second order in a. Zero crossing occurs at $\sin\omega t = ka$, i.e. at $\omega t = ka$ or $\omega t = \pi - ka$. Thus the crests are sharper and the troughs more rounded than for a pure sine wave. This was to be expected, since the formulae represents the trochoid.

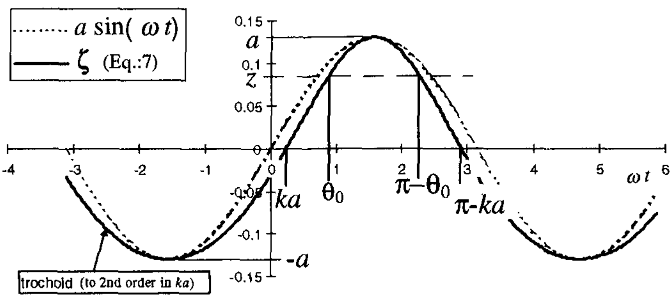

Figure 2. The surface variation of the trochoid correct to second order in *ka*.
Data as in Fig. 1.

In the same manner one may find the horizontal and vertical velocity components at $(x = 0, z)$ as

$$u = a\omega e^{kz_0}\sin(\omega t - kx_0) = a\omega e^{kz}\sin\omega t - a^2 k\omega e^{2kz}\,, \tag{8}$$

$$w = a\omega e^{kz}\cos\omega t\,. \tag{9}$$

Mean particle velocity, regular waves

The above are expressions for regular deep water (Gerstner) waves of amplitude a and period $T = 2\pi/\omega$ and wave length $\lambda = 2\pi/k$. The particles moves in closed circles, hence the mean velocity when following a particle is zero. (Laplacian description). For a stationary observer (Eulerian description) the picture is different, however, and that is the relevant viewpoint when comparing to measurements taken at a fixed point. From (9) is seen that the vertical mean velocity is zero,

$$\overline{w} = 0\,. \tag{10}$$

For points that are always submerged, the mean horizontal velocity can similarly be determined from (8)

$$\overline{u} = -a^2\omega k e^{2kz}\,. \tag{11}$$

For points in the splash zone, $-a < z < a$, emergence effects must be considered. The considered point will be in air for $z_0 > 0$ or, using $z_0 = z - \zeta$ and (7), the point will be in water, provided

$$z_0 = z + ka^2(1 + 2kz) - a(1 + kz)\sin\omega t < 0\,. \tag{12}$$

Solving and retaining terms to second order in amplitude a in the numerator and denumerator, (and remembering that in the splash zone z is of order a,) one obtains

$$\Rightarrow \quad \sin\omega t \geq \sin\theta_0 = \frac{z + ka^2}{a(1 + ka)}\,. \tag{13}$$

Thus the limiting phase angle is $(\omega t)_0 = \theta_0$ (cf. Fig. 2), and the mean value of horizontal particle velocity in the splash zone becomes

$$\begin{aligned}\overline{u(0,z)} &= \frac{1}{\pi}\int_{\theta_0}^{\pi/2} u(0,z,t(\theta))\,d\theta \\ &= \frac{a\omega}{\pi}\left[e^{kz}\cos\theta_0 - ake^{2kz}\left(\frac{\pi}{2} - \theta_0\right)\right].\end{aligned} \tag{14}$$

The considered contribution represents the value of the integral in the phase angle range ($-\pi/2 < \theta < \pi/2$), which in view of symmetry yields the average over a full cycle. The result is shown in Fig. 3, which also includes the experimental points from a project conducted at SINTEF NHL, Trondheim, under the supervision of Dr. Skjelbreia (Skjelbreia et al., 1989). The experimental data points have been read off from a figure presented in Cieslikiewicz & Gudmestad (1992).

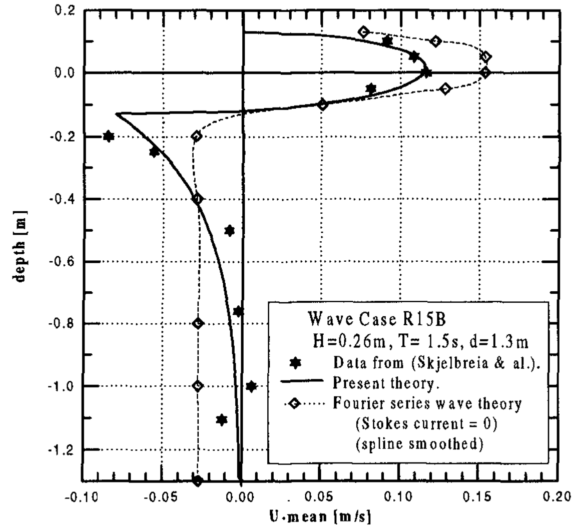

Figure 3. $E[u]$ vs depth. Regular wave case.

Irregular waves

Typically the variance spectrum, $S_\eta(\omega)$, of the sea surface is known. Let us for simplicity assume that the particle kinematics represent ergodic and narrow band processes, and further that h, u and w are Gaussian, so that their peaks are Rayleigh distributed. The choice of a frequency, that can be considered to be the most representative for the narrow process $u(x, z, t)$, is not straight forward. In the splash zone one might think that the zero crossing frequency of u (which is ω_{42}) would be the best choice, however the surface shape is dictated by the zero crossing frequency of η (ω_{20}), and this would also govern the frequency of the horizontal particle velocity. The shape of the variance spectrum of the horizontal particle velocity will change with depth,since the wave components of higher frequencies attenuate more rapidly with depth, than those of the lower frequencies. The frequency of the depth averaged horizontal particle velocity turns out to be ω_{20} which therefore is chosen as the representative frequency for the narrow band process used herein.

$$\omega_{20} = (M_2 / M_0)^{1/2}, \qquad (15)$$

$$\omega_{42} = (M_4 / M_2)^{1/2} , \qquad (16)$$

$$M_j = \int_0^\infty \omega^j S_\eta(\omega) d\omega . \qquad (17)$$

The expected value of u in an irregular seastate depends on the wave amplitude, which is a slowly varying envelope for η, and is as such Rayleigh distributed with parameter σ_η, viz.

$$f_A(a) = \frac{a}{\sigma_\eta^2} \exp(-\frac{a^2}{2\sigma_\eta^2}) \, ; \, a \geq 0 , \qquad (18)$$

in which $f_A(a)$ is the probability density of a. Then the expectation of u is given by

$$\mathrm{E}[u] = \int_0^\infty f_A(a) \mathrm{E}[u|a] da . \qquad (19)$$

Equation *(19)* states that for a given amplitude, the result depends on the expectations of u, summed over all amplitudes, and weighting according to their frequency of occurrence. The ergodicity theorem implies that the calculation of (19) may be done as a time average. Since the wave spectrum is narrow-banded, the components in (19) represent almost harmonic waves occurring sequentially rather than superposition of components that occur simultaneously. Hence the decision whether the point is in air can be made on a wave by wave basis for each value of the amplitude. Then the expectations in (11) and (14) represent $\mathrm{E}[u|a]$ for points that are submerged or in the splash zone respectively. For a given depth, z, the decision whether the point is submerged or not depends on the amplitude a. Hence (19) must be split in two integrals, one for $(-z)>a$ which can be evaluated analytically, and another in which the lower integration limit must first be determined according to (13), and the integral in (19) then must be evaluated numerically. The results are shown in Fig. 4 together with points representing the results of the measurements made by Skjelbreia and his team. The experimental values have been read off from a figure, this time the figure is taken from Cieslikiewicz & Gudmestad (1993). It is seen that the fit between experiments and theory is very good.

Discussion

The Gerstner theory used herein shows results that correspond very well to the Skjelbreia experiments. In contrast, application of linear (Airy) wave theory resulted in considerable discrepancies between experiments and theory. In the results presented in the two papers by Cieslikiewicz and Gudmestad, the discrepancy relative to maximum of the mean velocity as calculated from the measurements, amounted to about 25% through much of the splash zone and up to 40% in the zone of total submergence. This was the case both for the regular and the irregular wave case. The regular wave case R15b was also modeled using an eighteen Fourier component method (ACES107). As recommended in the guide the integrated Stokes flow in a wave flume should be equal to zero and set accordingly in the Fourier-series model.. The results are presented in Fig. 3. We see that this model fit the observations much less than the Gerstner approach.

It must be emphasized very strongly that the case considered herein is laboratory measurements, and hence different from real ocean waves in many ways. The Airy wave can be made equal to the Gerstner wave, provided a so-called Stokes drift is added. Stokes drift is a steady current that decays exponentially with depth, and is in fact equal and opposite to the current determined herein for the totally submerged case, as given by (11). Integrated over the whole depth this yields a total flow of water in the direction of wave propagation and per unit width of the wave crest, equal to ωa^2. Under the conditions at which the experiments were conducted, such a flux can not take place, since it violates continuity both at the wavemaker and at the wave front. This can hardly be remedied by superposition of another irrotational flow. Therefore it may not be too surprising that the Gerstner wave is in better agreement with measurements, even though it has the unusual character of being rotational. Another rotational solution to this problem has been presented by Kyozuka (1995). One possibility is that the required rotation is generated from shear along the boundaries of the fluid region. It is not known how well the two theories discussed herein will compare to measurements in real ocean waves.

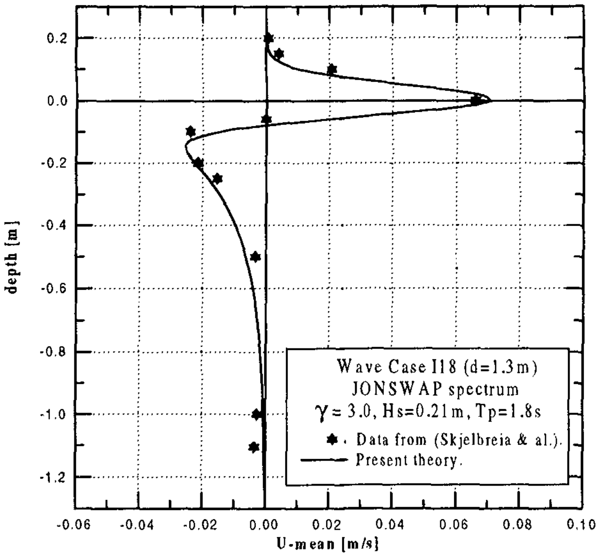

Figure 4. $E[u]$ vs depth. Irregular wave case.

Conclusions

- The Gerstner wave has been presented in an Eulerian description, to second order in the wave amplitude.
- Mean particle velocities have been computed for the regular and irregular case.
- The results have been compared to laboratory measurements, exhibiting an excellent fit, see Fig. 3 and Fig. 4.
- No statements can at this time be made on how the predictions from the present theory would compare to measurements in the ocean.

References

ACES107: "Automated Coastal Engineering System: The Wave Theory Functional Area/Fourier Series Wave Theory", User's guide and technical reference. Version 1.07D, October 1993. Coastal Engineering Research Center, Waterways Experiment Station, 3909 Halls Ferry Road, Vicksburg, Mississippi 39180-6199

Cieslikiewicz, W.,Gudmestad, O.T.: "Stochastic characteristics of orbital velocities of random water waves", J. Fluid Mech.(1993), Vol 255, pp.275-299.

Cieslikiewicz, W.,Gudmestad, O.T.: "Mass transport within the free surface zone of water waves", Wave Motion, Aug 1992

Lamb, H.: *Hydrodynamics,* Sixth edition, Cambridge University Press, 1932.

Kinsman, B.: *Wind Waves*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, USA, 1965.

Kyozuka, Y.: "Mass transport in two-dimensional tanks", preliminary version of paper, private communication, 1995.

Longuet-Higgins, M.S.: "The effects of nonlinearities on statistical distributions in the theory of sea waves", J. Fluid Mech. (1963), Vol 17, pp.459-480.

Mei, C.C., (1972): *Mass transport in Water Waves*, dept. of Civil Engineering, Massachusetts Institute of Technology, Research report R72-15., 287p.

Skjelbreia, J., Tørum, A., Berek, E., Gudmestad, O.T., Heideman, J., Spidsøe, N.: "Laboratory measurements of regular and irregular wave kinematics", Proceedings E & P Forum Workshop, Paris, 25-26 Oct, 1989

Skjelbreia, J., Berek, E., Bolen, Z., Gudmestad, O.T., Heideman, J., Ohmart, R.D., Spidsøe, N., Tørum, A.: "Wave kinematics in irregular waves", OMAE proceedings, Vol 1A, pp 223-228, 1991.

Tung, C.C.: "Statistical properties of the kinematics and dynamics of a random gravity wave field. J. Fluid Mech. (1975), Vol 70, pp. 251-255.