CHAPTER 112

THE BREAKING AND RUN-UP OF SOLITARY WAVES ON BEACHES

Ashwini Otta[1], Ib A. Svendsen[2], and S. T. Grilli[3]

ABSTRACT

A high accuracy boundary element method is used to compute the propagation of solitary waves from a constant depth region onto a plane slope. Initial wave heights range from $H/h = 0.06$ to 0.775, slopes between 1:35 and 1:1.73 (30°) have been investigated. The prebreaking shoaling shows very different characteristics on gentle slopes (1:20 and less) and on steeper slopes.

A diagram constructed on the basis of a large number of numerical experiments gives a simple limit between which waves break on which slopes and which not. Typical examples of the range of wave behavior are shown. Waves that do not break at run up often break during run down. The velocity fields for the two types of breaking are compared and found to be very different. A simple explanation for this is offered.

1. INTRODUCTION

The shoaling, run-up and breaking of solitary waves is of interest in connection with the analysis of the behavior of tsunamis in coastal regions.

A significant amount of literature has been published which analyses long wave propagation on slopes using the non-linear shallow water (NSW) equations or Boussinesq approximations. Thus, analytic solutions to periodic problems were obtained by Carrier & Greenspan (1958), Carrier (1966) and to the solitary wave problem by Synolakis (1987). Numerical solutions have been developed by Hibberd and Peregrine (1979), Pedersen & Gjevik (1983), and Zelt (1991). Furthermore, Kim et al. (1983) used a boundary integral formulation to analyze the problem. Experimental results for run-up were obtained by Ippen & Kulin (1954), Camfield & Street (1969) and recently, by Synolakis (1987). In particular

[1]Delft Hydraulics Laboratory, De Voorst, Postbus 152, 8300 AD, Emmeloord, The Netherlands

[2]University of Delaware, Department of Civil Engineering, Newark, DE 19716

[3]The University of Rhode Island, Department of Ocean Engineering, Kingston, RI 02881

Synolakis, comparing results from the NSW equations with experimental run-up, finds some deviations in the surface profiles in the final stages when the waves get steep. Papanicolaou & Raichlen (1987) give a few results for breaker heights on very gentle slopes (< 1:50). Finally, Svendsen & Grilli (1990) found using a high accuracy boundary element method (BEM) that when solitary waves of steepness up to 0.50 run-up on a relatively steep slope, the velocity profiles would differ quite significantly from the depth uniform velocity which is intimately linked with the NSW-equations. Similarly, the run-up of non-breaking waves, while in accordance with experimental results, would differ from the NSW predictions by as much as 75%.

Synolakis also developed a criterion for whether the waves would eventually break during the run-up and concluded that as the slope angle becomes small, the theory is only valid for small H/h whereas for steep beaches, he estimated the theory would be valid for relatively large H/h.

In the present paper we apply the version of the BEM method developed by Otta, Svendsen & Grilli (1992) to analyze the development of solitary waves on beaches. The wave heights initially (on the constant depth region in front of the slope) have height to depth ratios H/h from a moderate 0.10 to 0.775, the latter being almost equal to the steepest stable solitary wave on a horizontal bottom. The beach slopes are between 1.35 and 1/1.73 (30°). Both the shoaling behavior and the question of whether the waves break (and how) are addressed.

2. PROBLEM FORMULATION

The situation considered is described in Fig. 1. Solitary waves generated with initial height H_0 in a region of constant depth h_0 propagate towards a plane beach with slope angle s. Hence the only independent parameters of the problem are H_0/h_o and s. The motion is assumed irrotational and hence can be described by a velocity potential ϕ. Hence the velocity field $\vec{v}$ is given as

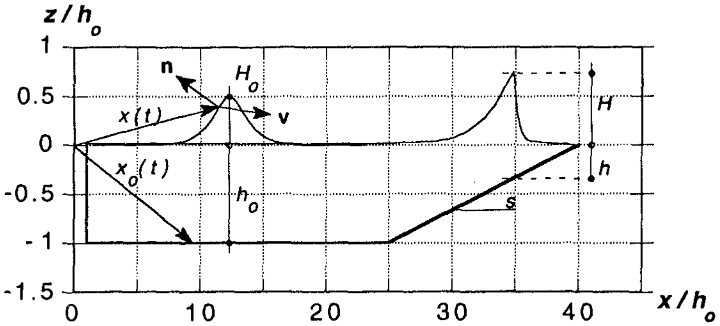

Figure 1: Definition Sketch

$$\vec{v} = \nabla\phi \tag{1}$$

and in the entire flow region ϕ satisfies the Laplace equation

$$\nabla^2\phi = 0 \tag{2}$$

subject to the boundary conditions

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{at the bottom } z = -h_0(x) \tag{3}$$

$$\frac{D\phi}{Dt} = \frac{1}{2}|v|^2 - g\eta \tag{4}$$

at the free surface $z = \eta(x,t)$

$$\frac{D\eta}{Dt} = \vec{v} \tag{5}$$

where $D/Dt = \partial/\partial t + (\vec{v}\cdot\nabla)$ represents the total derivative following a fluid particle. The geometrical quantities are shown in Fig. 1 and we have assumed a constant pressure along the surface.

To achieve the necessary accuracy in the computations, the solitary waves are generated initially by a method described by Tanaka (1986). At $t = 0$, we assume such a wave present in the computational wave tank with crest sufficiently far from the toe of the beach to be essentially undisturbed by the beach. Similarly, the seaward boundary is placed sufficiently far away not to disturb the computations with its reflection.

3. METHOD OF SOLUTION

The exact equations described in section 2 are solved by transforming the Laplace equation (2) into a Boundary Integral Equation which reads

$$\alpha(\vec{x},t)\phi(\vec{x},t) = \int_\Gamma \frac{\partial\phi}{\partial n}G(\vec{x},\vec{x}_0) - \phi(\vec{x}_0)\frac{\partial G(\vec{x},\ \vec{x}_0)}{\partial n}d\Gamma(\vec{x}_0) \tag{6}$$

Γ represents the (closed) boundary of the computational domain given by the bottom and slope, the entire free surface and the seaward boundary (see Fig. 1). $\vec{x}$ and $\vec{x}_0$ indicate points on the boundary curve, $\vec{x}_0$ being the point with respect to which the integral is performed. A free space Greens function is used for $G(\vec{x},\ \vec{x}_0)$. (6) is an exact representation of the original equation (2) which was exact too.

The solution of (6) at each time step is found numerically at modal points on the boundary. These nodes divide the boundary Γ into M segments Γ_j each spanning over an element. Thus, (6) can be written

$$\alpha(\vec{x},\ t)\phi(\vec{x},\ t) = \sum_{j=1}^{M}\int_{\Gamma_j} G(\vec{x},\ \vec{x}_0)\frac{\partial\phi}{\partial n}d\Gamma - \sum_{j=1}^{M}\int_{\Gamma_j}\phi(\vec{x}_0)\frac{\partial G(\vec{x},\ \vec{x}_0)}{\partial n}d\Gamma \tag{7}$$

which is still exact. Hence, the solution is reduced to the question of expressing the two types of integrals in (7) in terms of ϕ and $\frac{\partial \phi}{\partial n}$ at the nodes.

The details of this technique, which also includes upgrading to the next time step the boundary values of ϕ and $\partial\phi/\partial n$ at each point using the boundary conditions (3) and (4) are described by Grilli et al. (1989), Grilli & Svendsen (1990a,b).

The approach used for the computations reported here differ from previous versions of the method in the way the derivatives along the boundary and integrals in (7) have been computed. A third order polynomial based on four nodes has been used, centrally positioned around each interval (termed midinterval approximation).

This turns out to yield a substantial improvement in accuracy over the spline approximations used earlier. Fig. 2 shows a comparison of results for H/h and the maximum error in the normal velocity $\partial\phi/\partial n$ for the propagation of a very steep solitary wave ($H/h = 0.775$) on a horizontal bottom. This is close to highest wave of $H/h = 0.78$ that remains stable according to Tanaka (1986).

Ideally, we should expect each of the quantities in these computations to stay constant during the propagation. Initially, however, the wave undergoes minor changes before settling down to a largely constant value. These adjustments are caused by the fact that the numerical representation of the wave in our computations is based on different node position and also uses interpolation between nodes that differ (slightly) from the interpolation used in the computation of the initial wave (at $t = 0$) by Tanaka's method.

In all, however, the constancy of the parameters in Fig. 2 shows the substantial improvement over previous versions of the method, which were not quite able to cover the propagation of waves with heights so close to the maximum stable wave height. The figure represents propagation over approximately 30 water depths.

4. SHOALING OF SOLITARY WAVES

The tool thus developed has been used for numerical experiments with the shoaling and breaking of solitary waves on beaches with slopes varying from 1/35 to approximately 1/2.

Fig. 3 shows computations on a slope of 1/35 of waves with steepness between 0.10 and 0.40. The local value of H/h on the slope is plotted against x/h_0, the distance from the initial position of the wave crest.

The variation is essentially the same for all value of H/h and all waves in the figure end up breaking. Noticeable is the fact that the value of H/h at the breaking point almost independent of the initial height of the wave. The smaller waves just travel to smaller depths before they break. This pattern was already

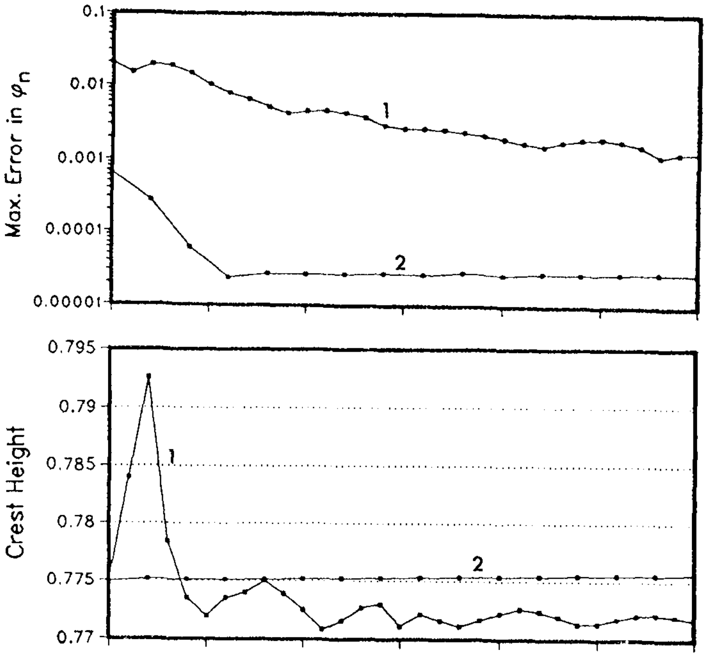

Figure 2: Accuracy of Computations for Very Steep Solitary Wave ($H/H = 0.775$) on a Constant Depth. 1) Quasi-spline interpolation. 2) Cubic mid-interval approximation.

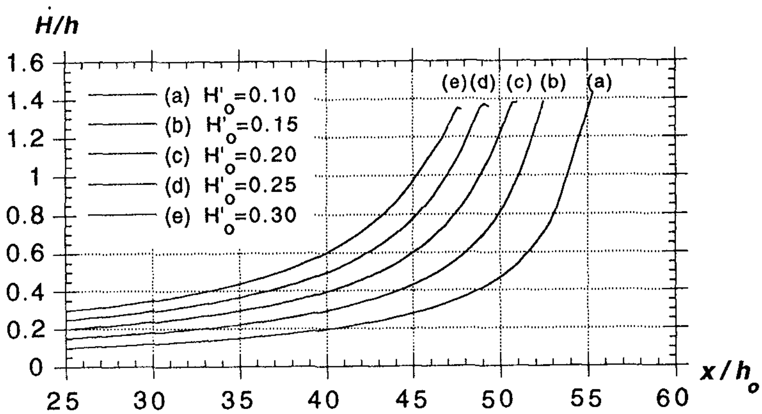

Figure 3: Shoaling of Solitary Waves of Different Initial Height H_o/h_o, on a slope 1/35. H/h on the Slope versus x/h_o. Slope Starts at $x/h_o = 25$.

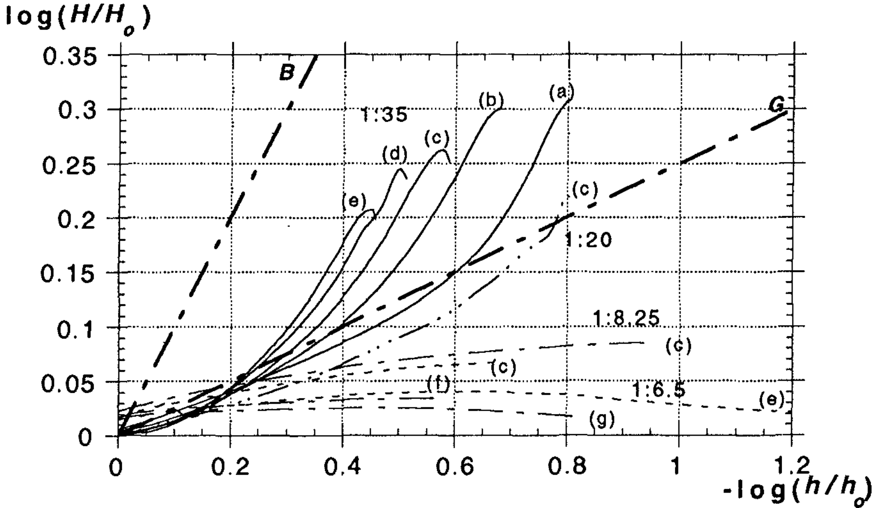

Figure 4: Shoaling of Solitary Waves of Different H_o/h_o on Different Slopes.

observed experimentally by Ippen & Kulin (1954). It is also worth noticing that the H/h value of breaking for all the waves is 1.35–1.45, i.e., dramatically above the height 0.78 or 0.8 of the highest stable wave. The reason for this is of course that the (in)stability of a high symmetrical wave on a constant depth has little to do with unsymmetrical deformation to breaking of a wave on a gradually decreasing water depth.

The absolute change in the height of the waves can not be deduced from Fig. 3. Fig. 4, however, shows the variation of wave height H relative to the original height H_0. This figure also includes a sample of results for a number of steeper slopes (1/20, 1/8.25, 1/6.5). We observe in this figure a substantial difference in the behavior on different slopes of waves initially of the same height. Whereas the waves on a 1/35 slope studied in the previous figure increase in height by generally a factor of 2 ($\log H/H_0 \sim 0.3$), the same waves on the steeper slope of 1/8.25 or 1/6.5 propagate on the slope essentially without change in absolute height. It even turns out that on a slope with steepness 1/6.5, only the initially steepest of those waves reach breaking (see below).

The figure also shows that non of the two theoretical laws proposed in the literature for the variation of the height of shoaling long waves are particularly satisfactory. Green's law $H \propto h^{-\frac{1}{4}}$ (represented by line G in Fig. 4) clearly has some merit during the initial stages of waves on the slope of 1/35 as one should expect from its derivation based on small amplitude waves on gentle slopes. More surprising perhaps is that the relation $H \propto h^{-1}$ predicted by the Boussinesq theory (line B in Fig. 4) can be said to approximate the later part of the wave height development towards breaking as judged from the fact that H/H_0 curves for waves on a 1/35 slope have the same slope in that region. These features were discussed by Synolakis (1991) on the basis of experimental results. On the steeper slopes, however, even 1/20, none on these laws will apparently provide accurate predictions for the wave height variations. In fact, the steepest wave of 0.40 (curve g) on a 1/6.5 slope even decreases slightly in height as it climbs the slope.

5. WAVE BREAKING ON SLOPES

Variation with slope angle and wave height

The second problem addressed in the present paper is the question of which waves break on which slopes.

Fig. 5 shows the propagation of the same wave, a very steep solitary wave with initial height $H_0/h_0 = 0.75$, on three different slopes, 30°, 7.12° (1/8), 3.81° (1/15). The three figures illustrate the range of behavior described in section 4, and we particularly see that even a wave initially close to the maximum stable steepness does not break if the slope is too steep. It turns out that a wave of nearly maximum steepness will only break if the slope angle is smaller 1:4.

We also see the radical difference between the behavior of the same wave on a 30° and a 7.12° slope. Where the crest in the first case essentially rushes up the slope with a total runup of close to three times the initial wave height the wave on a 7.12° slope hardly changes height at all and mainly undergoes a rapid deformation that ends with breaking.

On a 7.12° slope, the front of the wave becomes an almost vertical wall at the point of breaking and the breaking occurs violently as the entire wall tumbles over. On the gentler slope of 3.81° the wave breaking is a classical plunging breaker although the size of the jet is relatively small.

In Fig. 6 is shown the influence (or lack of influence) of wave height on the breaking on a fairly gentle slope (3.81°). Each of the four waves with initial heights of $H_0/h_0 = 0.3$, 0.45, 0.6 and 0.7 have shapes at the point of breaking that virtually are scaled versions of the others. Only the height to depth ratios H/h are slightly different as indicated in section 4 with the smaller wave reaching the largest H/h-value before it breaks.

Breaking criterion

Based on a sufficient number of such calculations as numerical experiments, we are then able to construct the diagram shown in Fig. 7 which shows which waves break, which do not. The axes are $\log 1/s$, ($s = h_x$ being the slope) and $\log(H_0/h_0)$, H_0 the original wave height on the constant depth h_0. The full line in the diagram separates the waves that break before or during run-up on

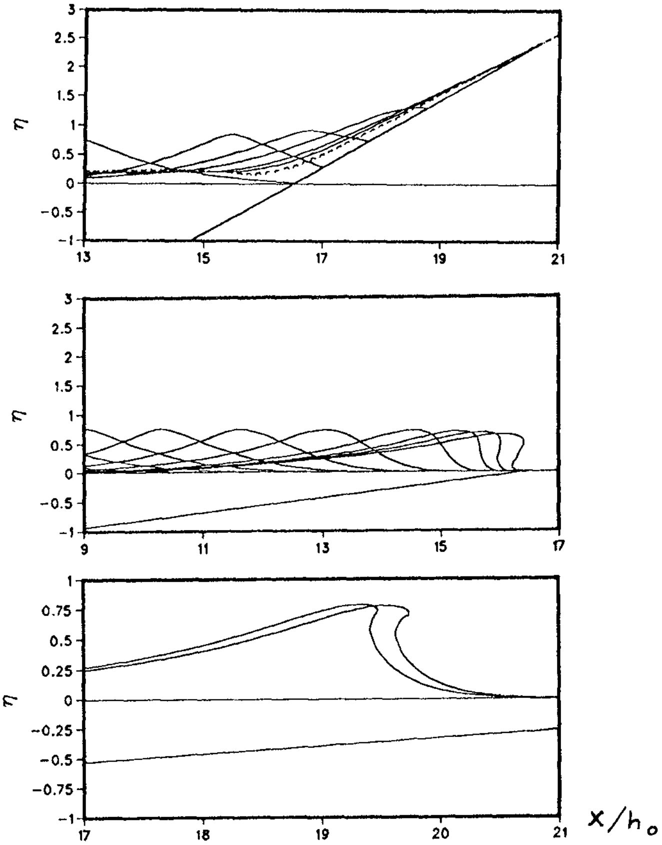

Figure 5: The Breaking of the Same Wave ($H_o/h_o = 0.75$) on Three Different Slopes: a) 30°, b) 7.18° (1:8), c) 3.81° (1:15).

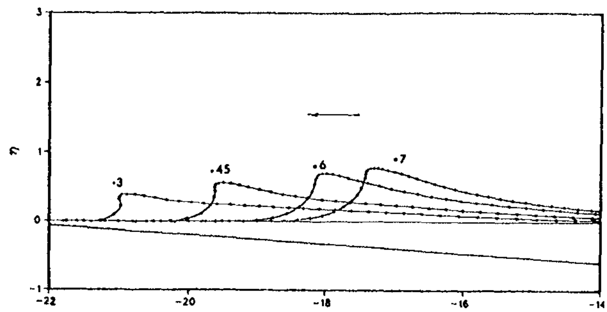

Figure 6: Breaking of Four Different Waves ($H_o/h_o = 0.3$, 0.45, 0.6 and 0.7) on the Same (relatively gentle) Slope 3.81° (1:15).

a given slope. We see that the line indicates that waves with

$$\frac{H_0}{h_0} > 8.4\ s^{15/9} \tag{8}$$

will break sometime during run-up on a slope.

The question of which waves break on a slope was also addressed by Synolakis (1987). Using the non-linear shallow water equations (NSW), he found that waves would eventually break on the slope if $H_0/h_0 > 0.825^{10/9}$. This criterion is indicated by the dashed line in Fig. 7.

We see that only for waves initially of noticeable steepness the NSW equations predict, that much smaller waves will break, than is found by the present more accurate method. Since run-up is greatly reduced if the waves break, this will also result in the NSW equations predicting much less run-up for the class of waves that actually do not break.

It could be argued that for waves initially of very small height (such as Tsunamis), the extrapolation of the two lines in the figure will meet. This happens for $H_0/h_0 = 0.0078$ (corresponding to $s = 0.015$), so that for that set of parameter values, the two methods give the same limit for which waves break and which do not.

However, in the first place it is not clear in advance that the formula (8) can be extrapolated to such small values of H_0/h_0. We have not at the present time performed any numerical experiments with so small wave heights on such gentle slopes.

Secondly, even such small waves will become quite steep when they approach breaking. The fact that the NSW equations fail to correctly predict breaking of the initially steeper waves may indicate that their prediction of the behavior of waves close to breaking is generally deficient, no matter how those waves start out. This would be in accordance with the general knowledge that near breaking the deviations from the hydrostatic pressure imbedded in the NSW equations is important.

Surging breakers

One of the interesting phenomena to look for in the numerical experiments is what happens to the waves that almost break, or only just break. Those are the waves around the full line in Fig. 7. If they break, this would presumably correspond to a so-called surging breaker. Fig. 8 shows computations for very steep waves on a slope of 1/4. The initial wave heights are 0.65, 0.68 and 0.75, and according to Fig. 7 these waves should not break but, particularly the 0.75 case should be very close to breaking. The figure is undistorted in scales so we can judge the development.

We notice that these waves show all the characteristics found from experi-

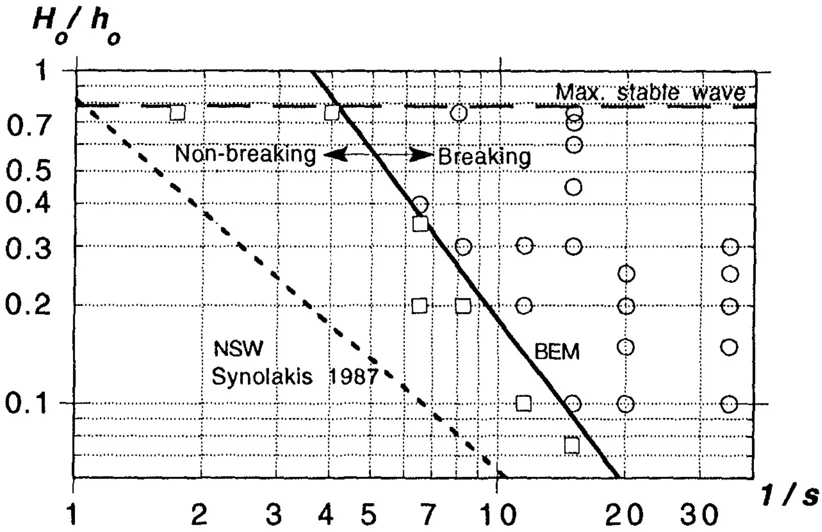

Figure 7: Wave Breaking Criterion

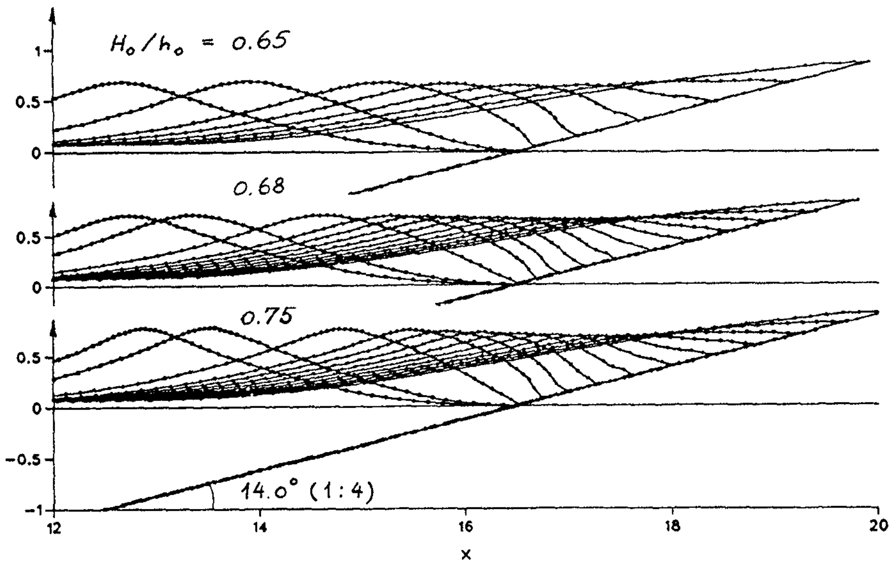

Figure 8: Waves Close to a Surging Breaking

ments for surging breakers. The front, with no water ahead of it, steepens to about 75° and the steepest surface slope is at very bottom of the front. At a certain point, the steepness stops, however, and the toe of the front shoots forward into a run- up while the wave crest behind collapses. Clearly, a very significant part of the potential energy plus the kinetic energy built into the wave crest at the instant of maximum slope is transferred into kinetic energy causing the run-up.

Though we have not tested this through more extensive experiments, we would expect this pattern to apply for a wide range of wave/slope combinations along the line given by (8). The question remains open however, how gentle the slope can be and how nearly breaking waves behave on really gentle slopes such as 1/50.

Wave breaking in the backwash

It turns out that waves that do not break during run-up may still do so during run-down. This was already seen in the computations on steep slopes by Svendsen & Grilli (1990) and Grilli & Svendsen (1991) who also described the velocity fields.

Fig. 9 shows an example of the breaking of a wave during run-down. At the period following maximum run-up (approximately position 1), the water starts rushing down the slope at rapidly increasing speed. The water level nearly parallel to the bottom indicates the acceleration is close to $g \sin \alpha$ where α is the slope angle. Close to the slope, the water level drops well below undisturbed SWL while staying well above SWL a little further out. This creates a front, as of a new incoming wave, which rapidly breaks shoreward. The whole sequence, in particular the breaking, takes place so swiftly that under laboratory conditions it can hardly be followed with the naked eye. After the breaking, the SWL is restored at the shoreline while smaller reflected wave propagates away. This phase can only be reproduced by the present method when the wave does not reach breaking in the downrush.

It is worth noticing that although it looks like the down-wash breaking is equivalent to a new incoming wave front, the velocity field in this breaker is radically different from the velocity field in an ordinary shoreward propagating breaker. Fig. 10 shows a comparison between the two situations. It seems that due to the strong downward water flow on the slope meeting the backward breaker, this wave is held in an almost stationary position while it turns over. The strong outward going particle velocities confirms this impression.

6. CONCLUSION

Summarizing the results described, we conclude that

- The wave height variation of solitary waves shoaling on plane slopes depends far more radically on the slope than on the initial wave steepness.

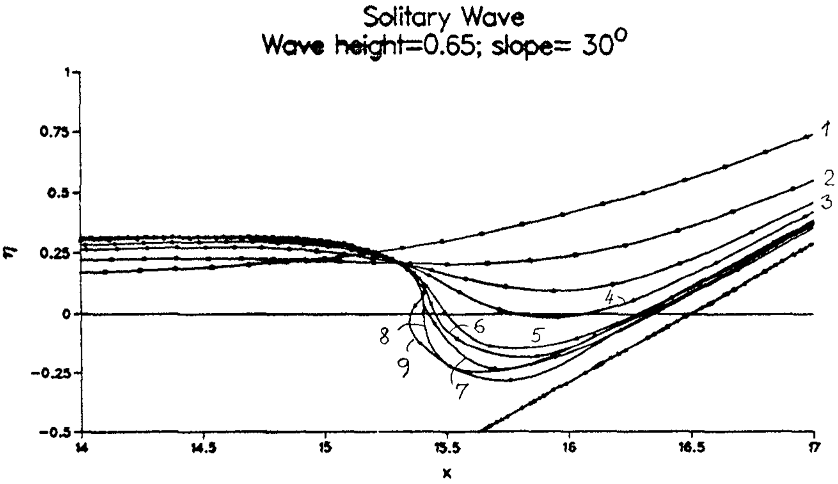

Figure 9: Breaking During Run Down.

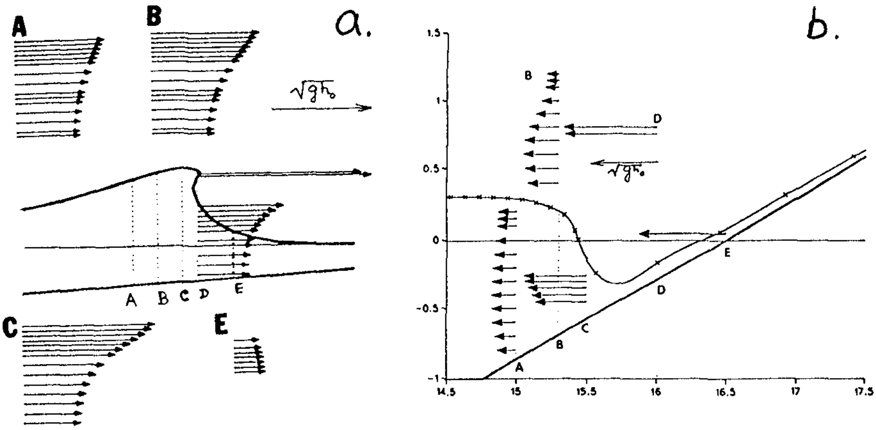

Figure 10: Velocity Profiles Under Forward & Backward Breaking Waves.

On gentle slopes ($< 1{:}20$), the wave height grows monotonously, slowly at first, towards breaking nearly as h^{-1}.

On steeper slopes, the wave height stays almost constant over the slope.

- For all slopes investigated (1:35 being the gentlest), the wave height to water depth ratio at breaking is well above the limit of approximately 0.78 of the steepest stable wave on constant depth. For the moderately steeper slopes, the waves may break even at the shoreline (and the value of H/h then is infinite).
- No waves that can propagate stably on a constant depth break on slopes steeper than 12°.
- Waves that do not break during run up may still generate a (backward) wave breaking during run down. The velocity field in such a breaking is very different from the breaking of a progressive wave. It resembles the velocity field of a (second) shoreward moving wave which is arrested by the downrush. The wave deforms to breaking without propagating.

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