CHAPTER 58

Estimating Incident and Reflected Wave Fields Using an Arbitrary Number of Wave Gauges

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1 Abstract

A method based on linear wave theory is presented to decompose onedimensional wave fields into left and right-travelling components using an arbitrary number of wave gauges. Results are presented to show that an increased accuracy is possible if more than three wave gauges are used. The technique uses a least squares scheme with variable weights. Results will also be presented that indicate a further improvement in accuracy is possible by an appropriate choice of the weighting coefficients.

2 Decomposition Theory

The decomposition of general one-dimensional wave fields into component waves travelling in opposite directions is of fundamental importance in many experimental studies. Breakwater evaluation involves estimating reflection coefficients as a function of wave frequency, and the efficiency of wave-energy extraction devices can be quantified similarly. Reflection coefficients of shorelines are also important quantities since many beach processes are driven by the energy extracted from incident waves through wave breaking. For some studies it is sufficient to obtain the spectra of the incident and reflected waves, but the complete space/time description of these waves can also be important, especially in resonance studies.

Suppose that a one-dimensional wave field is observed by recording the surface elevation $\eta_p(t)$ at a series of locations $\{x_p\}$, $p = 1, 2, \ldots, P$, as shown in Fig. 1. Using standard Fourier analysis techniques, the elevation can be expressed as

$$\eta_p(t) = \sum_{j=-N/2}^{N/2} A_{j,p} \, e^{i\omega_j t} \,, \tag{1}$$

where $\omega_j = 2\pi j/T$, T is the length of the time series, and N is large enough to resolve adequately the frequencies of interest. The time t will be discrete for a

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Figure 1. Measuring incident and reflected waves with *P* wave gauges.

sampled signal $(t \to m\Delta t, \text{ for } m = 0, 1, ..., N-1, \text{ with } \Delta t = T/N)$, but this will not be explicitly indicated here to simplify the notation.

Under the assumption that the waves are one-dimensional, dissipation is negligible, and that linear wave theory is valid, the wave field in Fig. 1 can be approximated by a Fourier sum of left and right travelling waves:

$$\eta(x,t) = \sum_{j=-N/2}^{N/2} a_{Lj} e^{i(k_j x + \omega_j t)} + a_{Rj} e^{i(-k_j x + \omega_j t)} , \qquad (2)$$

where $k_j = 2\pi/\lambda_j$, and k_j is related to ω_j through the linear dispersion relation:

$$\omega_j^2 = gk_j \tanh k_j h . \tag{3}$$

The still water depth is h and g is the acceleration of gravity. Evaluating $\eta(x,t)$ at the location of wave gauge p yields

$$\eta(x_p, t) = \sum_{j=-N/2}^{N/2} \left\{ a_{Lj} e^{i\phi_{j,p}} + a_{Rj} e^{-i\phi_{j,p}} \right\} e^{i\omega_j t} , \qquad (4)$$

where $\phi_{j,p} \equiv k_j x_p$. The ultimate goal is to estimate the $\{a_{Lj}\}\$ and $\{a_{Rj}\}\$ as accurately as possible from the wave records $\{\eta_p(t)\}\$. Equating the coefficients in Eqs. (1) and (4) yields the following equations:

$$A_{j,p} = a_{Lj}e^{i\phi_{j,p}} + a_{Rj}e^{-i\phi_{j,p}} \qquad p = 1, 2, \dots, P$$
(5)

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for each Fourier component j. If there are only two wave gauges, Eq. (5) can be solved exactly for a_{Lj} and a_{Rj} (Goda & Suzuki 1976). However, for P > 2, Eq. (5) is over-determined, and a_{Lj} and a_{Rj} must be estimated by an approximate technique. Mansard & Funke (1980, 1987) treated the case P = 3 using a least squares approach with uniform weighting. Here, a weighted least squares approach will be described that is valid for *arbitrary* P. Instead of forcing strict equality in Eq. (5), the value of an appropriate 'merit' function will be minimized so that Eq. (5) holds *approximately* for each wave gauge p. Let

$$\epsilon_{j,p} = a_{Lj} e^{i\phi_{j,p}} + a_{Rj} e^{-i\phi_{j,p}} - A_{j,p} .$$
(6)

For a given choice of a_{Lj} and a_{Rj} , $\epsilon_{j,p}$ represents the error in matching the j^{th} Fourier coefficient $A_{j,p}$ at wave gauge p. The merit function is chosen to be a weighted sum of the squares of the errors for each wave gauge:

$$E_j \equiv \sum_{p=1}^{P} W_{j,p} \,\epsilon_{j,p} \,\epsilon_{j,p}^* \,, \tag{7}$$

where $W_{j,p} > 0$ is the weighting coefficient for wave gauge p at frequency ω_j , and ()* represents the complex conjugate of the enclosed quantity. At a given frequency specified by j, the reliability of the estimates of a_{Lj} and a_{Rj} depends on several factors, including the spacing between the wave gauges. The motivation of introducing nonuniform weighting is to make use of this information so that the errors associated with wave gauges that provide reliable estimates are weighted more than the errors associated with wave gauges that do not provide useful information for estimating a_{Lj} and a_{Rj} . The criteria for choosing the weights $\{W_{j,p}\}$ will be discussed in §3. The minimum of Eq. (7) occurs at the point where E_j is stationary with respect to the real and imaginary parts of a_{Lj} and a_{Rj} . At this point the following relations hold

$$\sum_{p=1}^{P} W_{j,p} \epsilon_{j,p} e^{-i\phi_{j,p}} = 0$$

$$\sum_{p=1}^{P} W_{j,p} \epsilon_{j,p} e^{i\phi_{j,p}} = 0.$$
(8)

These are two complex equations for the two complex amplitudes a_{Li} and a_{Ri} .

Substituting Eq. (6) into Eq. (8) yields the two equations

$$a_{Lj}S_{j} + a_{Rj}\sum_{p=1}^{P}W_{j,p} e^{-2i\phi_{j,p}} = \sum_{p=1}^{P}W_{j,p} A_{j,p} e^{-i\phi_{j,p}}$$

$$a_{Lj}\sum_{p=1}^{P}W_{j,p} e^{2i\phi_{j,p}} + a_{Rj}S_{j} = \sum_{p=1}^{P}W_{j,p} A_{j,p} e^{i\phi_{j,p}},$$
(9)

where $S_j \equiv \sum_{p=1}^{P} W_{j,p}$. Rather than work with the absolute phases $\phi_{j,p}$, it is more useful to consider phase differences between wave gauges. Let

$$\Delta \phi_{j,p} \equiv \phi_{j,p} - \phi_{j,1} = k_j (x_p - x_1) \tag{10}$$

denote the phase difference between wave gauges 1 and p for frequency ω_j . Then the solution of Eq. (9) can be expressed as

$$a_{Lj} = \left[S_j \sum_{p=1}^{P} W_{j,p} A_{j,p} e^{-i\Delta\phi_{j,p}} - \sum_{p=1}^{P} W_{j,p} A_{j,p} e^{i\Delta\phi_{j,p}} \sum_{q=1}^{P} W_{j,q} e^{-2i\Delta\phi_{j,q}} \right] \frac{e^{-i\phi_{j,1}}}{D}$$
(11*a*)
$$a_{Rj} = \left[S_j \sum_{p=1}^{P} W_{j,p} A_{j,p} e^{-i\Delta\phi_{j,p}} - \sum_{p=1}^{P} W_{j,p} A_{j,p} e^{-i\Delta\phi_{j,p}} \sum_{q=1}^{P} W_{j,q} e^{2i\Delta\phi_{j,q}} \right] \frac{e^{i\phi_{j,1}}}{D}$$
(11*b*)

where

$$D = S_j^2 - \sum_{p=1}^P W_{j,p} e^{2i\Delta\phi_{j,p}} \sum_{q=1}^P W_{j,q} e^{-2i\Delta\phi_{j,q}}$$
(12a)

The denominator D is a real quantity and can be simplified to

$$D = S_j^2 - \left(\sum_{p=1}^{P} W_{j,p} \cos 2\Delta\phi_{j,p}\right)^2 - \left(\sum_{p=1}^{P} W_{j,p} \sin 2\Delta\phi_{j,p}\right)^2$$
(12b)

$$= 4 \sum_{p=1}^{P} \sum_{q < p} W_{j,p} W_{j,q} \sin^2 \Delta \phi_{j,pq}$$
(12c)

where

$$\Delta\phi_{j,pq} \equiv \Delta\phi_{j,p} - \Delta\phi_{j,q} = \phi_{j,p} - \phi_{j,q} = k_j(x_p - x_q) \tag{13}$$

is the phase difference between wave gauges p and q at frequency ω_j . Equation (12b) is the most efficient form to compute D since it requires the fewest operations to evaluate; however, Eq. (12c) is more useful for showing the behaviour of the denominator. Since the weighting coefficients $\{W_{j,p}\}$ are positive,

it is clear that the denominator can be zero at frequency ω_j only if the wave gauges are placed such that $\sin \Delta \phi_{j,pq} \equiv 0$ for all p and q. Clearly this is a condition to be avoided if possible, and this will be discussed further in §3.

Further manipulation of the numerators in Eq. (11) leads to the following formulae for a_{Lj} and a_{Rj}

$$a_{Lj} = \sum_{p=1}^{P} C_{j,p}^* A_{j,p}$$
(14*a*)

$$a_{Rj} = \sum_{p=1}^{P} C_{j,p} A_{j,p}$$
(14b)

where

$$C_{j,p} = 2iW_{j,p} \frac{e^{i\phi_{j,1}}}{D} \sum_{q=1}^{P} W_{j,q} \sin \Delta \phi_{j,pq} e^{i\Delta \phi_{j,q}}$$
(15)

This form emphasizes that a_{Lj} and a_{Rj} are simply linear combinations of the $\{A_{j,p}\}$. For P = 2 the results of Goda & Suzuki (1976) are reproduced, and for P = 3 the decomposition formulae of Mansard & Funke (1980, 1987) are obtained.

3 Error analysis

The sensitivity of the decomposition formulae Eq. (11) or (14) to errors in measuring the Fourier coefficients $\{A_{j,p}\}$ at a given frequency ω_j depends on the choice of the weighting coefficients $\{W_{j,p}\}$ as well as the spacing of the wave gauges relative to the wave length associated with ω_j . Consequently, the weights and the wave gauge locations should be chosen appropriately to maximize the reliability of the decomposition estimates. To illustrate this, suppose that the elevation records obtained from the wave gauges can be expressed as the sum of two one-dimensional linear waves travelling in opposite directions plus a residual or error signal $\mathcal{E}_p(t)$:

$$\eta_p(t) = \sum_{j=-N/2}^{N/2} \left\{ A_{Lj} e^{i(k_j x_p + \omega_j t)} + A_{Rj} e^{i(-k_j x_p + \omega_j t)} \right\} + \mathcal{E}_p(t) .$$
(16)

The residual signal $\mathcal{E}_p(t)$ accounts for:

1) noise/nonlinearities in the wave gauges and data acquisition hardware.

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- 2) nonlinear hydrodynamic effects (e.g. deviations from the linear dispersion relation).
- 3) two-dimensional wave motion (such as cross modes in a wave channel).
- 4) viscous effects.

With

$$\mathcal{E}_p(t) = \sum_{j=-N/2}^{N/2} \varepsilon_{j,p} \, e^{i\omega_j t} \,, \tag{17}$$

Eq. (16) can be expressed as

$$\eta_p(t) = \sum_{j=-N/2}^{N/2} \left\{ A_{Lj} \, e^{i\phi_{j,p}} + A_{Rj} \, e^{-i\phi_{j,p}} + \varepsilon_{j,p} \right\} e^{i\omega_j t} \,. \tag{18}$$

This is equivalent to Eq. (1) with $A_{j,p} = A_{Lj} e^{i\phi_{j,p}} + A_{Rj} e^{-i\phi_{j,p}} + \varepsilon_{j,p}$. Substituting this expression for $A_{j,p}$ into the decomposition formulae Eq. (14) yields following estimates of the Fourier coefficients of the left and right travelling waves

$$a_{Lj} = A_{Lj} + \sum_{p=1}^{P} C_{j,p}^* \varepsilon_{j,p}$$
 (19a)

$$a_{Rj} = A_{Rj} + \sum_{p=1}^{P} C_{j,p} \varepsilon_{j,p}$$
 (19b)

The "exact" coefficients are obtained if the residual signal is zero; otherwise, the error $\varepsilon_{j,p}$ at wave gauge p is amplified by the coefficient $C_{j,p}$. The amplification of errors associated with wave gauge p can be represented by

$$\left|\frac{\partial a_{Lj}}{\partial \varepsilon_{j,p}}\right| = \left|\frac{\partial a_{Rj}}{\partial \varepsilon_{j,p}}\right| = |C_{j,p}| = \frac{2W_{j,p}}{D} \left|\sum_{q=1}^{P} W_{j,q} \sin \Delta \phi_{j,pq} e^{i\Delta \phi_{j,q}}\right| .$$
(20)

Since the residual signal will not, in general, be perfectly correlated between different wave gauges, large errors associated with different gauges will not cancel. If the residuals $\varepsilon_{j,p}$ are uncorrelated between wave gauges, a measure of the reliability of the decomposition as a function of frequency can be estimated by summing the terms in Eq. (20) over all wave gauges. The worst case occurs when D = 0; for P = 2 (Goda & Suzuki 1976) this occurs for $\Delta \phi_{j,12} = n\pi$, i.e., for $|x_2 - x_1| = n\lambda_j/2$ for any integer n. For arbitrary P this occurs if $\sin \Delta \phi_{j,pq} =$ for all p and q, i.e., if $2|x_p - x_q|/\lambda_j$ is an integer for each combination of p and

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q. Care should be taken when choosing the wave gauge positions to ensure that this criterion is not approached near frequencies of interest.

To choose the weighting coefficients $\{W_{j,p}\}$ information must be available about the relative magnitudes of the residuals $\varepsilon_{j,p}$. If these error terms are the same for each wave gauge (at a particular frequency ω_j), then uniform weighting is appropriate. However, if this residual signal varies from gauge to gauge, variable weighting is appropriate. This may be the case if the primary source of the residual signal is the deviation from the linear dispersion relation due to finite amplitude effects. In this case it may be better to concentrate the weighting near one wave gauge (say gauge number 1) and reduce the weighting for distant gauges where the phase deviates from the linear prediction $k_j(x_p - x_1)$. This information might be obtainable from a cross-spectral analysis between the wave gauge elevation records. It might also be possible to make use of the sensitivity analysis in this section to help choose the weights. Other techniques for choosing the weights will depend on the particular sources of the errors and their statistical properties.

Preliminary results have been obtained by using an *ad hoc* scheme based on heuristic reasoning. For each frequency ω_j being treated, a "goodness" function $G(\Delta \phi_{j,pq})$ is defined that quantifies the desirability of the phase difference associated with the spacing between gauges p and q. Multiples of one-half the wave length are undesirable, and a large spacing relative to the wavelength is also undesirable. A function that reflects these characteristics is:

$$G(\Delta\phi_{j,pq}) = \frac{\sin^2 \Delta\phi_{j,pq}}{1 + (\Delta\phi_{j,pq}/\pi)^2} .$$
⁽²¹⁾

A large value of G indicates a better wave gauge spacing for frequency ω_j than a smaller value of G. The weighting coefficient $W_{j,p}$ for wave gauge p can then simply (and somewhat arbitrarily) be defined as

$$W_{j,p} = \sum_{q=1}^{P} G(\Delta \phi_{j,pq}) .$$
(22)

4 Results

To illustrate the use of the decomposition theory presented, a simulated wave field in 2 m water depth was created consisting of 4096 points per wave gauge record with a 0.05 s time step. A Pierson-Moskowitz type spectrum was chosen with a spectral peak at 0.47 Hz (corresponding to a wavelength of 6.64 m in 2 m of water). The RMS height of the right travelling wave is 0.5 m, and the Fourier

components of the left travelling wave are exactly 10 % of the Fourier components of the right travelling wave, although with random phase shifts applied. Hence, the RMS height of the left travelling wave is 0.05 m, and the amplitude reflection coefficient is 0.1 for all frequencies. In addition, a 0.01 m RMS uncorrelated random noise signal was added on top of these two waves to simulate a wide variety of errors, noise, and other effects that cannot be simulated directly.

Eight wave gauge records were simulated at the locations x = 0, 0.016, 0.052, 0.130, 0.301, 0.679, 1.551 and 3.341 m. These locations were chosen so that the minimum and maximum wave gauge spacings could resolve the minimum and maximum energy containing wavelengths of interest in the wave spectrum. A simple geometric telescoping factor was used to locate the intermediate gauges.



The energy spectra resulting from using only 2 wave gauges (located at x = 0 m and x = 1.511 m) are presented in Fig. 2. The data was partitioned into 15 segments each with 512 points with a 50% overlap. A Welch window was applied to each segment. The spectra of both the left and the right travelling simulated waves are compared with the estimated spectra obtained from the decomposition technique described here. The left/right amplitude ratio is also shown. As discussed above, it *should* be 0.1, but instead it deviates considerably from this value except near the peak of the spectrum since the wave gauge spacing was chosen to optimize the accuracy in this region. Uniform weights were used $(W_{j,p} = 1)$.

The results for 3 wave gauges (located at x = 0, 0.679 and 1.511 m) and



uniform weights are presented in Fig. 3. These results are considerably better than for only 2 wave gauges since they are not susceptible to the singularity that the 2-wave gauge arrangement is.

The results for all 8 wave gauges and uniform weights are presented in Fig. 4. These results are even better than for 3 wave gauges, although the improvement is perhaps not striking as the difference between the 2 wave gauge and the 3 wave gauge case.

However, in Fig. 5 the results for using 8 wave gauges and the *variable* weight scheme of § 3 are presented. A marked improvement is seen, with the left/right amplitude ratio very flat near 0.1 as it should be. It should be mentioned that the variable weight scheme described in § 3 yields exactly the same results for 2 wave gauges as for the uniform weight case. For 3 wave gauges, only a very slight improvement is obtained by using variable weighting coefficients, and the results do not differ appreciably from those displayed in Fig. 3.

The results in the time domain of the decomposition of with 8 wave gauges and variable weights are presented in Fig. 6. Here, only a small segment of the time record is displayed, but the relative amplitudes of the left and right waves as well as the noise signal can be seen. It should be noted here that the main source of the inaccuracy of the decomposition is due to the presence of the noise added to the left and right simulated waves. However, this noise signal does not cause significantly degrade the decomposition.









Figure 6. Left and right travelling waves (RMS amplitudes in parentheses).

5 Conclusions

A method based on linear wave theory has been presented to decompose one-dimensional wave fields into left and right travelling components using using an arbitrary number of wave gauges. Results were presented for simulated wave gauge data to show that an increased accuracy is possible if more than three wave gauges are used, especially for broad band wave spectra. Results were also presented to indicate that a further improvement in accuracy is possible by an appropriate choice of the least squares weighting coefficients.

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LIST OF SYMBOLS

a_{Lj}	estimate of the j^{th} Fourier coefficient of the left travelling wave.
a_{Rj}	estimate of the $j^{\rm th}$ Fourier coefficient of the right travelling wave.
A_{Lj}	hypothetical $j^{\rm th}$ Fourier coefficient of the left travelling wave assuming there is no noise and that linear theory holds exactly.
A_{Rj}	hypothetical $j^{\rm th}$ Fourier coefficient of the right travelling wave assuming there is no noise and that linear theory holds exactly.
$A_{j,p}$	$j^{\rm th}$ Fourier coefficient of the wave amplitude time series recorded at wave gauge $p.$
$C_{j,p}$	weighting coefficient for expressing a_{Lj} and a_{Rj} as linear combinations of the $\{A_{j,p}\}$.
E_{j}	merit function whose minimum yields the amplitudes of the left and right travelling waves, $a_{Lj} \& a_{Rj}$, at frequency j .
$\mathcal{E}_p(t)$	residual elevation signal at wave gauge p due to noise, nonlinear, viscous, and other effects.
g	gravitational acceleration.
h	still water depth.
i	$\sqrt{-1}$
k_{j}	wave number of the j^{th} Fourier component: $2\pi/\lambda_j$
P	number of wave gauges used to record the composite wave spectrum $% \mathcal{A} = \mathcal{A}$
S_j	sum of the least squares weighting coefficients: $\sum_{p=1}^{P} W_{j,p}$.

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duration of wave gauge records $\{\eta_p(t)\}.$
least squares weighting coefficient for probe p at frequency ω_j .
location of wave gauge p .
phase difference between probes 1 and p for frequency ω_j : $\Delta \phi_{j,p} = \phi_{j,p} - \phi_{j,1} = k_j(x_p - x_1)$.
phase difference between probes p and <i>q</i> for frequency ω_j : $\Delta \phi_{j,pq} = \Delta \phi_{j,p} - \Delta \phi_{j,q} = \phi_{j,p} - \phi_{j,q} = k_j(x_p - x_q).$
the error in matching the j^{th} Fourier coefficient $A_{j,p}$ at wave gauge p using the least squares algorithm.
$j^{\rm th}$ Fourier coefficient associated with the residual elevation signal $\mathcal{E}_p(t).$
wave elevation recorded at location x_p .
wavelength of the j^{th} Fourier component.
absolute phase of the $j^{\rm th}$ Fourier component at the $p^{\rm th}$ wave gauge: $k_j x_p$
frequency of the j^{th} Fourier component: $2\pi j/T$

SPECIAL SYMBOLS

()* complex conjugate of the enclosed quantity.

