CHAPTER 33

Nonlinear Transformation of Irregular Waves in Shallow Water

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1. Introduction

Propagation and shoaling of irregular wave trains in shallow water is a nonlinear process, where substantial cross spectral energy transfer can take place in relatively short distances. This process involves the generation of bound sub- and super-harmonics and near-resonant triad interactions, which are defined as the energy exchange between three interacting wave modes.

In the literature it is common practice to distinguish between bound waves and resonant free triads. The theory for bound waves is based on the assumption of a one way transfer of energy to generate higher and lower harmonics which are phase locked to the primary wave train. In reality, however, a feed back of energy to the primary frequencies will occur leading to near-resonant interactions.

This phenomena has previously been treated by e.g. Freilich and Guza (1984) on the basis of the classical Boussinesq equations. It turns out, however, that the accuracy of the linear dispersion relation for higher wave numbers is of major importance for the exchange of energy even in shallow water and for this reason we recommend as governing equations a special form of the Boussinesq equations. These were derived by Madsen et

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al. (1991a) on a horizontal bottom and here we shall extend these equations to a mildly sloping bottom.

The paper will contain a Fourier analysis of the inherent linear shoaling properties, transfer functions for bound sub and super-harmonics and a discussion of two cases of triad interactions.

2. New Boussinesq Equations

The applicability of various forms of the Boussinesq equations expressed in terms of e.g. the bottom velocity, the surface velocity, the depth-averaged velocity and the depth-integrated velocity was discussed by Madsen et al. (1991a). With the objective of improving the linear dispersion characteristics a new set of equations were derived in two horizontal dimensions. As the first step an improved linear dispersion relation was obtained by combining a polynomial expansion of Stokes first order theory with Pade's approximant. As the second step the classical Boussinesq equations were modified by invoking the linear long wave approximation and using the method of operator correspondence.

The equations presented in this paper represent an extension of the approach by Madsen et al. (1991a), by including first derivatives of the sea bed. The result will be a set of two-dimensional equations which incorporate excellent linear dispersion characteristics and are applicable to irregular wave propagation on a slowly varying bathymetry from deep to shallow water.

The starting point for the derivation is the classical Boussinesq equations derived by Peregrine (1967). These equations, which are valid on a variable bathymetry, are reformulated in terms of depth-integrated velocity variables, i.e. flux components, and are simplified by neglecting higher derivatives and products of derivatives of the still water depth h. First derivatives of h are considered small but are included in the formulation.

It is a classical procedure to simplify higher order terms in the Boussinesq or KdV equations by introducing the linear long wave approximations (see e.g. Mei, 1983; Madsen et al., 1991a). As an example P_{xxt} type terms can be replaced by S_{xxx} type terms by the use of this method. In shallow water it makes no difference, but in deeper water the form of the Boussinesq terms is critical for the accuracy of the linear dispersion relation. Instead of replacing P_{xxt} with S_{xxx} type terms we use a different approach. Spatial differentiations of the linear long wave equations lead to expressions containing the terms P_{xxt}, P_{xyt}, Q_{yyt} and Q_{xyt}. Since these expressions are effectively zero in shallow water we add them to the original Boussinesq equations and obtain the following new set of equations:

$$S_t + P_x + Q_y = 0 \tag{2.1a}$$

$$P_t + \left(\frac{P^2}{d}\right)_x + \left(\frac{PQ}{d}\right)_y + gdS_x + \psi_1 = 0 \tag{2.1b}$$

$$Q_t + \left(\frac{Q^2}{d}\right)_y + \left(\frac{PQ}{d}\right)_x + gdS_y + \psi_2 = 0 \tag{2.1c}$$

where subscripts x, y and t denote differentiation with respect to space and time, d is the total water depth, h is the still water depth, S is the surface elevation, P and Q are the depth-integrated velocity components, and ψ_1 and ψ_2 are the new Boussinesq terms defined by:

$$\begin{aligned}\psi_1 = &-(B+\tfrac{1}{3})\ h^2\ (P_{xxt} + Q_{xyt}) - Bgh^3\ (S_{xxx} + S_{xyy})\\ &- hh_x\left(\frac{1}{3}P_{xt} + \frac{1}{6}Q_{yt} + 2BghS_{xx} + BghS_{yy}\right)\\ &- hh_y\left(\frac{1}{6}Q_{xt} + BghS_{xy}\right)\end{aligned} \tag{2.2a}$$

$$\begin{aligned}\psi_2 = &-\left(B+\frac{1}{3}\right) h^2\ (Q_{yyt} + P_{xyt}) - Bgh^3\ (S_{yyy} + S_{xxy})\\ &- hh_y\left(\frac{1}{3}Q_{yt} + \frac{1}{6}P_{xt} + 2BghS_{yy} + Bghs_{xx}\right)\\ &- hh_x\left(\frac{1}{6}P_{yt} + BghS_{xy}\right)\end{aligned} \tag{2.2b}$$

Except for the slope terms proportional to h_x and h_y these expressions are identical to the Boussinesq terms presented by Madsen et al. (1991a). B is the linear dispersion parameter, which will be determined in the following section. Further details concerning the derivation, and a description of the numerical method used to solve them, will appear in Madsen and Sørensen (1992b).

3. Linear dispersion relation and shoaling properties

A Fourier analysis of the linearized one-dimensional version of the new Boussinesq equations will be made with the objective of studying the linear dispersion relation and the linear shoaling gradient embedded in the new equations.

As a starting point for the analysis, the one-dimensional wave equation corresponding to (2.1a-c) combined with (2.2a-b) is derived. By using (2.1a) linear terms containing P are eliminated and secondly (2.1a) and (2.1b) are cross-differentiated and subtracted. This leads to:

$$\mathbf{L} = \mathbf{M} + \mathbf{N}_{xx} \tag{3.1a}$$

where

$$\boldsymbol{L} = \left[S_{tt} - ghS_{xx} + Bgh^3S_{xxxx} - \left(B + \frac{1}{3}\right) h^2S_{xxtt}\right] \tag{3.1b}$$

$$\boldsymbol{M} = \left[gS_x + (2B + 1)hS_{xtt} - 5Bgh^2S_{xxx} \right] h_x \tag{3.1c}$$

$$\boldsymbol{N} = \left[\frac{1}{2} gS^2 + \frac{P^2}{d} \right] \tag{3.1d}$$

In the following linear analysis the non-linear operator **N** is neglected and we shall look for solutions to (3.1a) on the form

$$S\ (x,t) = A(x)\, e^{i(\omega t - \varphi(x))} \tag{3.2}$$

where ω is the cyclic frequency, A is the local wave amplitude and φ is the phase function, which is related to the local wave number by

$$\varphi_x = k(x) \tag{3.3}$$

The water depth, the wave number and the wave amplitude are considered to be slowly varying functions of x and consequently products of derivatives and higher derivatives of these quantities will be neglected in the following.

The linear dispersion relation is obtained by inserting (3.2) into (3.1) and neglecting all x-derivatives of h, k and A,

$$-\omega^2 + ghk^2 + Bgh^3k^4 - \left(B + \frac{1}{3}\right) k^2h^2\omega^2 = 0 \tag{3.4}$$

Alternatively this can be formulated as:

$$\frac{c^2}{gh} = \frac{1 + Bk^2h^2}{1 + (B + \frac{1}{3})\ k^2h^2} \tag{3.5}$$

where c is the wave celerity defined by $c = \omega/k$.

As shown by Madsen et al. (1991a) various classical formulations of the Boussinesq equations all lead to (3.5) with different values of B: using the surface velocity as dependent variable leads to B = -1/3, the bottom velocity leads to B = 1/6 and the depth-averaged or depth-integrated velocity leads to B = 0. By far the best agreement with Stokes first order theory is obtained by using the value B = 1/15, which is determined by matching (3.5) with a Taylor expansion of the Stokes first order celerity combined with Pade's expansion technique. This value was origionally suggested by Witting (1984). Madsen et al (1991a) analyzed the accuracy of (3.5) for the various possible values of B and concluded that the new Boussinesq equations combined with B = 1/15 provide excellent linear dispersion characteristics for values of h/L_0 as large as 0.5.

Proceeding with the Fourier alalysis and collecting terms to the next order in (3.1) includes the terms proportional to the first derivatives of h, k and A. The frequency ω is eliminated by the use of (3.4) and differentiation of this equation also makes it possible to eliminate terms proportional to k_x/k. After algebraic manipulations we get the expression:

$$\frac{A_x}{A} = -\gamma\ \frac{h_x}{h} \tag{3.6}$$

where γ is the linear shoaling gradient. The expression for γ reads:

$$\gamma = \frac{1}{4}\left[1 + (4B - 1)k^2h^2 + \left(6B^2 - \frac{2}{3}B\right)k^4h^4 + \right.$$
$$\left.\left(4B^3 + \frac{1}{3}B^2 + \frac{1}{9}B\right)k^6h^6 + \left(B^4 - \frac{1}{9}B^2\right)k^8h^8\right] \tag{3.7}$$
$$\left[1 + 2Bk^2h^2 + \left(B^2 + \frac{1}{3}B\right)k^4h^4\right]^{-2}$$

The reference linear shoaling coefficient based on Stokes first order theory is derived by using the concept of conservation of energy flux. After differentiation and algebraic manipulations this leads to:

$$\gamma^{Stokes} = \frac{2kh(sinh2kh) + 2k^2h^2\ (1-cosh2kh)}{(2kh + sinh2kh)^2} \tag{3.8}$$

A comparison between (3.7) and (3.8) as a function of h/L_0 requires that (3.7) is combined with the Boussinesq dispersion relation (3.4), while (3.8) is combined with Stokes dispersion relation. The result is presented in Fig. 1 and it can be concluded that the standard Boussinesq equations with B = 0 lead to major discrepancies for h/L_0 larger than 0.10, while B = 1/15 has a remarkable effect and results in an excellent agreement with Stokes first order theory for h/L_0 as large as 0.50.

4. Bound waves in shallow water

Irregular wave trains travelling in shallow water can generate and sustain a considerable amount of bound harmonics, which travel phase-locked to the primary wave train. At locations, where drastic changes of the wave heights occur e.g. due to diffraction or wave breaking, the bound waves can be released and proceed as free waves. This may cause harbour resonance, drift motion of moored vessels and surf beats. The phenomena of bound waves has been discussed in numerous papers in connection with the reproduction of regular and irregular waves in physical wave flumes (e.g. Barthel et al., 1983 and Sand and Mansard, 1986). It has been concluded that linear boundary conditions often are insufficient and should be replaced by second order boundary conditions including the effect of bound sub- and super-harmonics. This problem is important for physical waves flumes and equally relevant for numerical models solving non-linear equations.

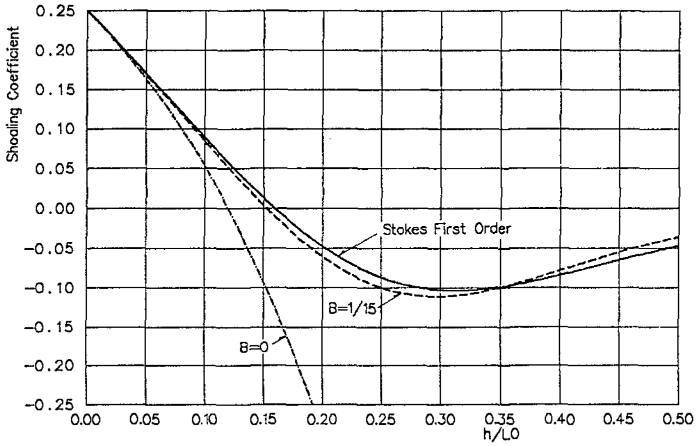

Fig. 1 Linear shoaling gradient, γ defined by (3.6), (3.7) and (3.8).

In this section transfer functions for second order bound sub and super-harmonics will be presented on the basis of the new Boussinesq equations. The derivation is straight forward and is based on a perturbation solution to the wave equation given in (3.1a-d). As a start we consider the forcing due to a simple first order wave group made up of just two frequencies ω_n and ω_m at a constant depth. Each of the two wave components are considered to be solutions to the linearized problem $L\{S^{(1)}\} = 0$ where L is defined by (3.1b). The next step is to look for second order solutions to $L\{S^{(2)}\} = N_{xx}\{S^{(1)}\}$ where N is defined by (3.1d). The first order bichromatic wave train will force a second order wave train consisting of four new frequencies: one sub-harmonic and three super-harmonics. The four second order wave numbers are determined from combinations of k_n and k_m, and they do not satisfy the linear dispersion relation, which implies that these waves are bound or phase-locked to the first order wave train. For general irregular wave trains consisting of many wave components, the contributions from all pairs of frequencies inherent in the wave train can be summed up (see Sand and Mansard, 1986).

The second order transfer function derived from the new Boussinesq equations reads:

$$G_p = \frac{ghk_p^2\left(\tfrac{1}{2} + \omega_n\omega_m / (ghk_nk_m)\right)}{\left(\omega_p^2 - ghk_p^2 - Bgh^3k_p^4 + (B + 1/3)\, h^2\omega_p^2k_p^2\right)} \tag{4.1}$$

where $\omega_p = p\Delta\omega$ denotes the discrete sub or super-harmonic frequency receiving energy transfer from the primary frequencies $\omega_n = n\Delta\omega$ and $\omega_m = m\Delta\omega$. For sub-harmonics (4.1) should be used with n = m + p and $k_p = k_{m+p} - k_m$. For super-harmonics n = p-m and $k_p = k_{p-m} + k_m$. Further details and the complete formulation of second order boundary conditions for irregular waves can be found in Madsen and Sørensen (1992a).

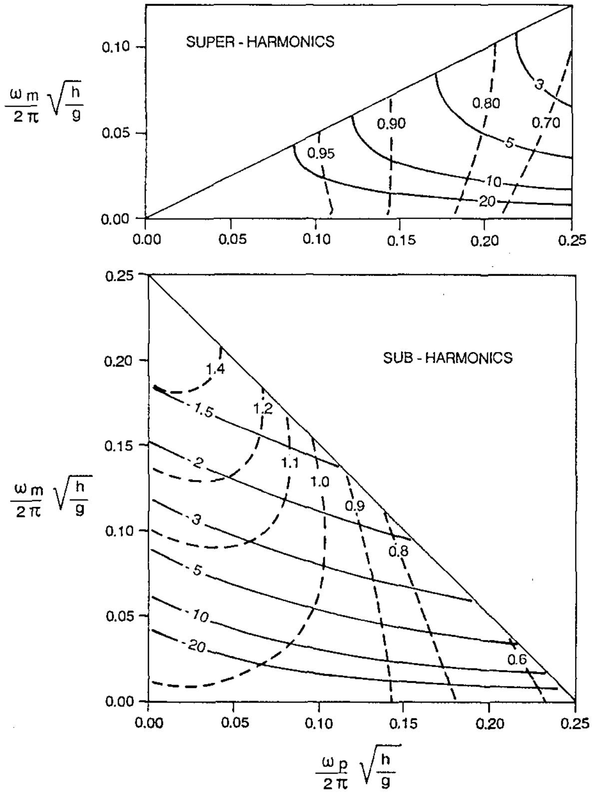

Fig. 2 Transfer Functions for super-harmonics and for sub-harmonics.
— Boussinesq transfer function determined by (4.1) B=1/15
--- Ratio between the Boussinesq transfer and the transfer function determined from the Laplace equation.

Fig. 2 shows the transfer function for the super-harmonics ω_p generated by the interaction between ω_m and ω_{p-m}, and for the sub-harmonics ω_p generated by the interaction between ω_m and ω_{m+p}. G^+ and G^- determined by (4.1) are shown as full lines in Fig. 2, while the ratios between G and the transfer functions derived from the Laplace equation (Sand and Mansard, 1986) are shown as dotted lines. Generally, the Boussinesq equations tend to underestimate the super-harmonics while the sub-harmonics can be underestimated as well as overestimated. Discrepancies up to 40% are noticed in the shown truncated spectrum, but typical errors are less than 10%.

An application of the theory is presented in Fig. 3, where time series of surface elevations are generated from a JONSWAP spectrum with $\gamma = 3.3$. The water depth is 10 m, the significant wave height is 2.0 m and the peak period is 9.0 s. The linear wave train is generated with random phases and with energy in the interval 0.05 hz to 0.20 hz. The maximum frequency corresponds to h/L_0 = 0.25 which is within the range of application of the new Boussinesq equations. The bound sub-harmonics cover the interval the 0.001 hz to 0.15 hz, while the bound super-harmonics cover the interval from 0.10 hz to 0.40 hz.

The consequence of neglecting the bound wave components in the input time series will be the release of spurious free wave components of the same order of magnitude. In this situation the free sub-harmonics are by far the most critical, since they can penetrate, e.g. into harbours almost without being reduced in magnitude and result in harbour resonance or at least in a major overestimation of the local wave disturbance.

5. Triad Interactions

The theory of bound waves assumes an equilibrium situation where the non-linear wave train propagates without changing its form. In reality and especially in shallow water a substantial cross spectral energy transfer will take place due to triad interactions, which describe the exchange of energy between three interacting wave modes (see e.g. Freilich and Guza, 1984).

The simplest example of triad interactions occur, when first order monochromatic boundary conditions are applied in shallow water. This will unintentionally generate spurious free second order waves in addition to the bound second order waves, leading to an energy exchange between the primary wave frequency and its super-harmonics. A numerical example is presented in Fig. 4a, which shows the computed surface elevation at eight equidistant times within one wave period as a function of the distance from the open boundary. The corresponding spatial variation of the amplitudes of the first three harmonics is obtained by FFT-analysis of the computed time series and is shown in Fig. 4b.

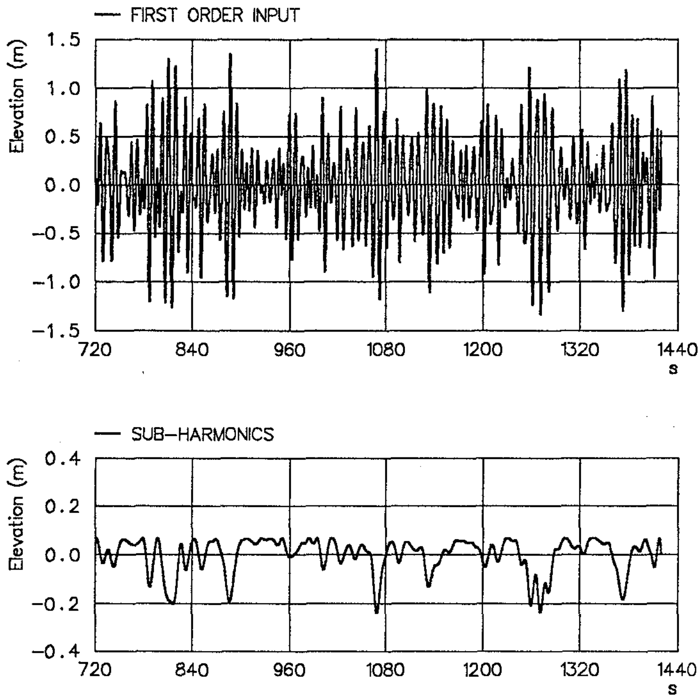

Fig. 3 First order irregular waves and their correspondingly bound sub-harmonics.
Boussinesq Eq. with B = 1/15. Water depth = 10 m.
JONSWAP SPECTRUM with H_s = 2.0 m. T_p = 9.0 s, γ = 3.3.

As discussed in Madsen et al. (1991b) the energy transfer can be shown to depend strongly on the phase mismatch, which is defined by:

$$\delta^{+} \equiv k_m + k_{p-m} - k_p \tag{5.1a}$$

$$\delta^{-} \equiv k_{m+p} - k_m - k_p \tag{5.1b}$$

This calls for an accurate description of the linear dispersion relation and in connection with this it should be emphasized that even when the primary wave corresponds to very shallow water in terms of h/L_0, this is not necessarily the case for the higher harmonics. Hence, by using classical types of KdV or Boussinesq equations, the estimate of the phase mismatch may be rather inaccurate.

The new Boussinesq equations with B = 1/15 improve the accuracy considerably and as an example Fig. 5 shows a comparison with the measurements of Chapalain et al. (1992). The agreement between the measurements and the FFT analysis of the time domain simulations is seen to be excellent, except for the beat length of the third harmonic, which is slightly underestimated. More examples of harmonic generation forced by linear monochromatic and bichromatic boundary conditions on a constant depth can be found in Madsen and Sørensen (1992a).

The final example is a study of non-linear refraction-diffraction on a semi-circular shoal. This was studied experimentally by Whalin (1971) for wave periods of 1, 2 and 3 seconds. We shall concentrate on a discussion of the case of 1 second waves, which has not previously been treated by the use of Boussinesq equations. The value of h/L_0 varies from 0.29 at the toe of the shoal to 0.096 behind the shoal. An FFT analysis of time series in each grid point along the centre line has been made and the resulting spatial evolution of first and second harmonics is compared with Whalin's experimental data in Fig. 6. A considerable scattering in the data is seen in front of the shoal but behind the shoal the agreement between the data and the new Boussinesq equations with (B = 1/15) is acceptable. Reasonable agreement is also found between the new Boussinesq equations and the results obtained by Liu and Tsay (1984), who solved the non-linear Schrödinger equation. Finally the classical Boussinesq equations (i.e. B = 0) are seen to fail completely by predicting an unrealistic decrease of the first harmonic, a discrepancy which can be explained by the variation of the linear shoaling gradient in Fig. 2.

In fact this example demonstrates that Fig. 2 should be taken quite seriously as a measure of the range of application of different types of Boussinesq equations. A common mistake is that the accuracy of the wave celerity is taken as the practical measure, in which case a 5% error restricts the use of the standard Boussinesq equations to approximately $h/L_0 = 0.22$. However, according to Fig. 2 this is clearly much too optimistic in case of a variable bathymetry and the shoaling falsification increases rapidly for h/L_0 exceeding 0.10. This emphasizes the importance of using the new Boussinesq equations presented in this paper, in which case h/L_0 as large as 0.5 can be considered.

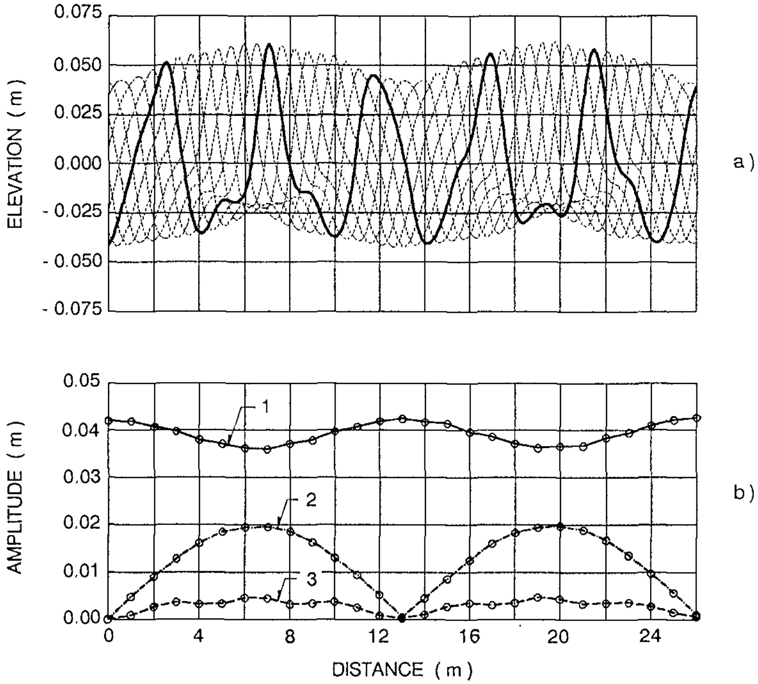

Fig. 4 Triad interactions due to first order monochromatic boundary conditions by time domain Boussinesq model with B = 1/15. Water depth = 0.40 m, T = 2.5 s, H = 0.084 m, grid size = 0.04 m, Time step = 0.01953 s.
a) Wave envelope
b) Spatial variation of the amplitudes for the first three harmonics.
1: f_1 = 0.40 hz, 2: f_2 = 0.80 hz, 3: f_3 = 1.20 hz.

6. Conclusion

The paper presents a new set of Bousssinesq equations applicable to irregular wave propagation on a slowly varying bathymetry from deep to shallow water. It can be concluded that the new equations are capable of describing the phenomenon of harmonic generation and triad interactions with an accuracy which is significant better than what can be obtained on the basis of the classical forms of the Boussinesq equations.

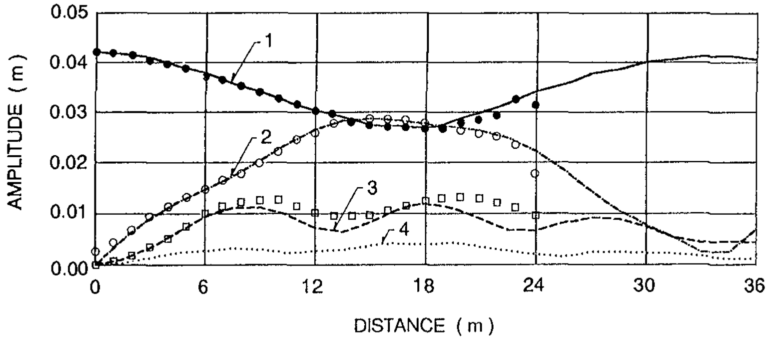

Fig. 5 Triad interactions due to first order monochromatic boundary conditions. Water depth = 0.40 m, T = 3.5 s, H = 0.084 m, Grid size = 0.06 m, Time step = 0.02734 s.
—— Time domain Boussinesq model with B=1/15.
•, °, ▫ Measurement by Chapalain, Cointe and Temperville (1991).

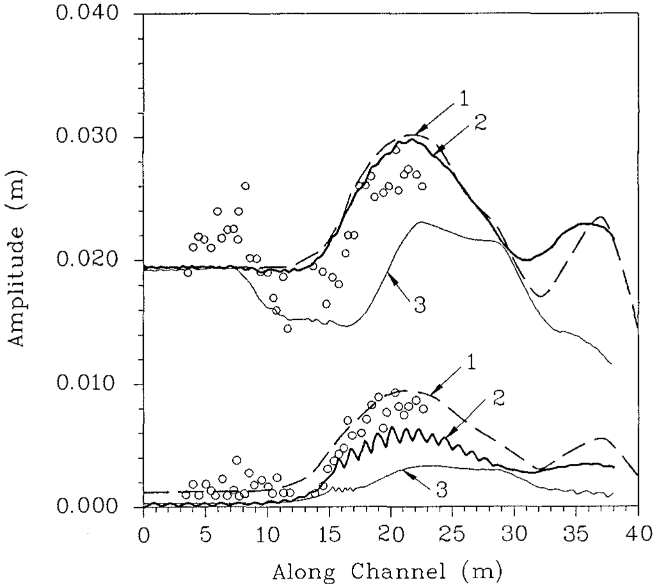

Fig. 6 Refraction-diffraction on a semi-circular shoal. Spatial variation of the amplitudes for the first three harmonics along the center line.
0: Experiment, Whallin (1971), 1: Liu and Tsay (1984),
2: Boussinesq B=1/15, 3: Boussinesq B=0.

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