CHAPTER 26

Spectral Evolution of Directional Finite Amplitude Dispersive Waves in Shallow Water

James M. Kaihatu and James T. Kirby [1]

Abstract

Two different aspects of nearshore wave modelling are discussed. The first section details a model valid in deeper water than the usual shallow water wave models. We derive a mild-slope equation in which nonlinearity is retained to second order in ϵ and dispersion is retained to all orders. The resulting parabolic model is then simplified for expedient calculation. This simplified model is compared to the data of Whalin (1971) with favorable results. The second section concerns the role of vector-sum interactions as compared to colinear, near-resonant interactions. We use the angular spectrum model of Kirby (1990) to determine which sort of interaction is dominant in the nearshore wavefield. A steady wave solution of the model and a case of unsteady wave evolution were investigated. Two simplified data sets with different amounts of directional spread were then run through the model. All three tests indicate that vector-sum interactions contribute significantly to wave field evolution.

Introduction

A number of frequency-domain models of nearshore wave evolution have appeared in the literature. Many of these models are based on the Boussinesq equations of Peregrine (1967) or closely-related variants; examples include Freilich and Guza (1984) and Liu, et.al. (1985). All these models calculate the wavefield in the frequency domain by factoring out the time periodicity of each frequency component, and treat the nonlinear terms in the governing equations by three-wave resonant interaction theory, making explicit the energy exchange between frequencies. Directional characteristics of the wavefield, if considered at all, are usually treated by the parabolic approximation (see Liu, et.al. (1985) for an example of this).

This paper is divided into two parts. The first part contains the development of a finite-depth spectral evolution model which can be used in deeper water than the Boussinesq-type models. A slightly simplified version of this model is compared to data, and the

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effect of the simplification discussed. The second part of the paper details an investigation of directional spectral interaction. Specifically, we use the angular spectrum model of Kirby (1990) to determine whether detuned vector-sum interactions are as important in the wavefield as resonant (or near-resonant) colinear interactions.

Finite Depth Evolution Model

Spectral models based on the Boussinesq equations (Peregrine (1967)) have existed for some time (Frielich and Guza (1984); Liu, et.al. (1985)). They work well within their range of validity ($kh << 1$, where k is the wavenumber and h the water depth). However, the shoaling mechanism for these models in their lowest-order variants is Green's Law:

$$\frac{A}{A_o} = (\frac{h}{h_o})^{\frac{1}{4}} \tag{1}$$

which overpredicts the shoaling of higher-frequency spectral components. Some dispersiveness has been added to the shoaling terms of Boussinesq-type models (e.g., the "dispersive shoaling model" of Freilich and Guza (1984); Madsen, et.al. (1992)). This added dispersiveness is of at most $O(kh)^2$, and is thus only a small correction to the shoaling term.

Theoretical Formulation

In this paper we seek a model which is valid for all orders of kh, and retain terms to $O(\epsilon)^2$, where ϵ is the ratio of wave amplitude to water depth. The derivation follows the method of Bryant (1973, 1974). A similar model, but truncated to two components, was detailed by Keller (1988).

We begin from the boundary value problem:

Governing equation ($-h < z < 0$):

$$\nabla^2 \phi = 0 \tag{2}$$

Combined free surface boundary condition (to $O(\epsilon^2)$) at $z = 0$:

$$\phi_z = -\frac{1}{g}\left(\phi_{tt} + \frac{1}{2}(\nabla_h \phi)_t^2 + \frac{1}{2}(\phi_z)_t^2 - \frac{1}{2g}(\phi_t)_{zt}^2 + \nabla_h \cdot (\phi_t \nabla_h \phi)\right) \tag{3}$$

Bottom boundary condition at $z = -h$:

$$\phi_z = -\nabla_h h \cdot \nabla_h \phi \tag{4}$$

We then assume the solution for ϕ in terms of a superposition of components:

$$\phi = \sum_{n=1} f_n(k_n, h, z)\tilde{\phi}_n(x, y, k_n, \omega_n, t) \tag{5}$$

where the f_n are the depth dependencies for each mode:

$$f_n = \frac{\cosh k_n(h+z)}{\cosh k_n h} \tag{6}$$

and the k_n are found using the linear dispersion relation:

$$\omega_n^2 = g k_n \tanh k_n h \tag{7}$$

Following the method of Smith and Sprinks (1975), we obtain the following mild-slope equation modified by nonlinearity:

$$\begin{aligned} &\tilde{\phi}_{tt} - \nabla_h \cdot \left((CC_g)_n \nabla_h \tilde{\phi}_n\right) + \omega_n^2 \left(1 - \frac{C_{gn}}{C_n}\right) \tilde{\phi}_n = \\ &\quad \frac{1}{2}\left(\sum_l \sum_m \left[\frac{\omega_l^2 + \omega_m^2}{g^2}(\tilde{\phi}_{l_t}\tilde{\phi}_{m_t})_t - \frac{\omega_l^2\omega_m^2}{g^2}(\tilde{\phi}_l\tilde{\phi}_m)_t\right]\right. \\ &\left. - \sum_l \sum_m \left[(\nabla_h \tilde{\phi}_l \cdot \nabla_h \tilde{\phi}_m)_t + \nabla_h \cdot (\tilde{\phi}_{l_t} \nabla_h \tilde{\phi}_m) + \nabla_h \cdot (\tilde{\phi}_{m_t} \nabla_h \tilde{\phi}_l)\right]\right)_n \end{aligned} \tag{8}$$

where the l, m and n are related according to three-wave interaction conditions. Introducing time periodicity and slow x-variation in amplitude:

$$\tilde{\phi} = \frac{-ig}{2\omega_n} A_n e^{i(\int k_n dx - n\omega t)} + \frac{ig}{2\omega_n} A_n^* e^{-i(\int k_n dx - n\omega t)} \tag{9}$$

yields an elliptic equation for the evolution of the complex amplitude A_n. Using the parabolic approximation:

$$((CC_g)_n A_{n,x})_x \quad << \quad 2i(kCC_g)_n A_{n,x} \tag{10}$$

and making use of a reference phase function (Kirby and Dalrymple (1983)) to redress the lack of phase dependence on y,

$$A_n = a_n e^{(\int k_{n_o}(x)dx - \int k_n(x)dx)} \tag{11}$$

where $k_{n_o}(x)$ is a y-averaged wavenumber, gives us:

$$\begin{aligned} &2i(kCC_g)_n a_{nx} - 2(kCC_g)_n(k_{n_o} - k_n)a_n + i(kCC_g)_{nx} a_n + ((CC_g)_n (a_n)_y)_y = \\ &\frac{1}{4}\left(\sum_{l=1}^{n-1} R(\omega_n, \omega_{n-l}, -\omega_l, k_{n-l}, k_l) a_l a_{n-l} e^{i(\int k_{l_o}(x) + k_{n-l_o}(x) - k_{n_o}(x) dx)} + \right. \\ &\left. 2\sum_{l=1}^{N-n} R(\omega_n, \omega_{n+l}, \omega_l, k_l, k_{n+l}) a_l^* a_{n+l} e^{i(\int k_{n+l_o}(x) - k_{l_o}(x) - k_{n_o}(x) dx)}\right)_n \end{aligned} \tag{12}$$

where:

$$R(\omega_n, \omega_{n+l}, \omega_l, k_l, k_{n+l}) =$$

$$\frac{g}{\omega_l \omega_{n+l}} \left(\omega_n^2 k_l k_{n+l} + (k_{n+l} - k_l)(\omega_{n+l} k_l + \omega_l k_{n+l})\omega_n \right)$$

$$- \frac{\omega_n^2}{g} \left(\omega_l^2 - \omega_l \omega_{n+l} + \omega_{n+l}^2 \right) \qquad (13)$$

and all wavenumbers are calculated from full linear theory. Preliminary testing of (12) has demonstrated that the interaction coefficients R require a large amount of computer time to calculate. Thus we simplified the model somewhat by taking the coefficients to the shallow-water limit (Ukai, et.al. (1990); Abreau, et.al. (1992); Mase and Kirby (1992)), and using the shallow water wavenumber as the reference phase function in (11). Thus we obtain:

$$2i(kCC_g)_n a_{nx} - 2(kCC_g)_n(k'_{n_o} - k_n)a_n + i(kCC_g)_{nx} a_n +$$
$$((CC_g)_n(a_n)_y)_y = \frac{3gn^2 k_1'^2}{4} \left(\sum_{l=1}^{n-1} a_l a_{n-l} + 2 \sum_{l=1}^{N-n} a_l^* a_{n+l} \right)_n \qquad (14)$$

where $k'_n = \frac{n\omega}{(gh)^{\frac{1}{2}}}$.

Comparison to Data

To see whether this simplification causes any loss of accuracy, one-dimensional versions of (12) and (14) were compared to flume data from Mase and Kirby (1992); the reader is referred to their paper for details on the experiment. Experimental results for Case 1 are compared to the two one-dimensional models at a depth $h = 15cm$ (Figure 1), and it is apparent that, save for some slight deviation at the tail of the spectrum, both models compare equally well to the data. However, (14) required roughly 12 percent of the computing time of (12). This indicates that retaining full dispersion in the shoaling terms is the dominant improvement of these models over the Boussinesq equations, and that little is lost in simplifying the nonlinear coefficients as we have done here. The one-dimensional version of (14) is equivalent to that of Mase and Kirby (1992), except for a different dispersion term. Kaihatu and Kirby (1991) have shown that (12) fits experimental shoaling data better than the "consistent model" of Freilich and Guza (1984), particularly in the higher frequency ranges.

We use the data of Whalin (1971) to verify the two-dimensional model equation (14). This particular data set has been used on numerous occasions to evaluate numerical models (e.g. Liu, et.al. (1985); Rygg (1988); Madsen, et.al. (1992)). The experiment consisted of a long channel with bathymetry resembling a cylinder tilted along its longitudinal axis; the experimental layout is shown in Figure 2. Sinusoidal waves of one, two and three second periods were generated at three different amplitudes for each period; higher harmonics were allowed to grow from zero amplitude. The amplitudes of the first

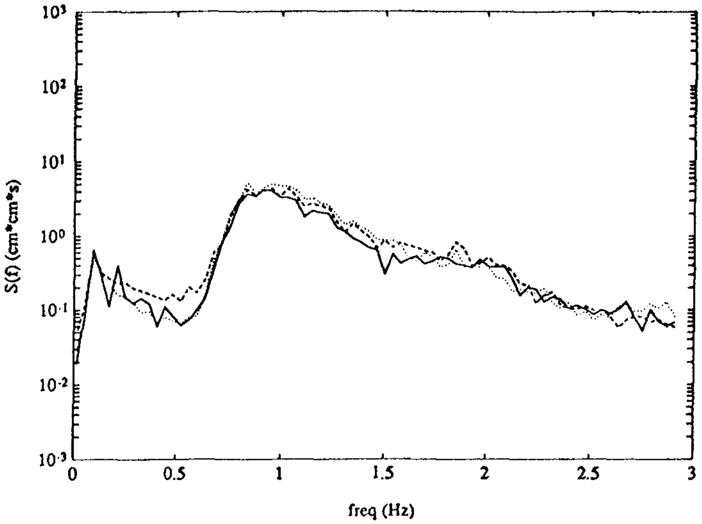

Figure 1: Comparison of Case 1 shoaling data of Mase and Kirby to equations (12) and (14) at $h = 15cm$. Data is solid line, equation (12) is dashed line and equation (14) is dotted line.

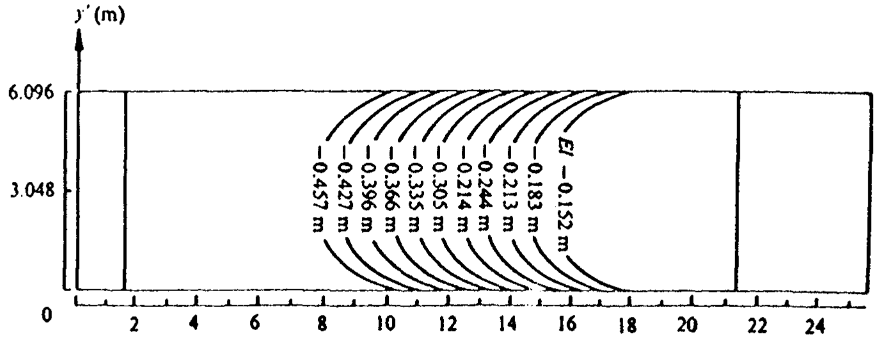

Figure 2: Topography of Whalin's (1971) experiment. From Liu, et.al. (1985)

three Fourier components were recorded along the centerline of the channel.

In this study we used the 3 second case for model-data comparison. This was done to facilitate a fair comparison between (14) and the Kadomstev-Petviashivili (KP) model of Liu, et.al. (1985). The KP model is essentially a lowest-order parabolic Boussinesq model, with a non-dispersive shoaling term and frequency dispersion retained up to $O(kh)^2$. Figures 3(a,b,c) show the Fourier amplitudes of the models compared to data along the centerline of the tank. For the most part the present model performs better than the KP model, especially in reproducing the higher harmonics. This is possibly due to the improved dispersion characteristic of the present model, where each mode is shoaled using full linear theory.

Discussion

To conclude this section, it appears that the simplified fully-dispersive model (14) performs well in both combined refraction-diffraction and in pure shoaling. One drawback of the model is the lowest-order parabolic formulation. This not only restricts the model to small angle of incidence, also gives equal weight to all interactions no matter the direction of approach. This can be remedied by using the angular spectrum approach (Kirby (1990)), as explained in greater detail in the next section.

Directional Spectra

This section of the paper discusses the role of directional spectral interaction in an evolving wave field. Specifically, we wish to address two recently-published viewpoints concerning whether detuned, vector-sum interactions play as large a role in the evolution of nearshore waves as near-resonant, colinear interactions. Vector-sum interactions place the energy exchanged between two spectral components approaching at different directions at the sum frequency and the vector-sum-wavenumber direction, while the colinear interactions only take place between components travelling in the same direction. This sort of interaction is seemingly stronger, since the two interacting components are very nearly in resonance. However, Freilich, Guza and Elgar (1990, hereafter referred to as FGE) maintain that vector-sum interactions may be as important as colinear interactions, especially when the interactions come from very energetic spectral peaks approaching at different directions. They used field data taken at Torrey Pines Beach, California, where directional arrays were placed at $10m$ and $4m$ water depth. Figure 4 is taken from their paper, and shows a contoured frequency-direction plot of waves at the $4m$ depth. Freilich, et.al. contended that the spectral peak located at $(0.06Hz, -4^o$ from beach normal, marked "A"), interacted with the peak at $(0.10Hz, +10^o$ from beach normal, marked "B"), to produce a peak at $0.16Hz$ approaching at a direction that was within one degree of the vector-sum wavenumber direction (marked "C").

In direct opposition to this, Abreau, et.al. (1992) argue that only resonant colinear interactions are important. They formulated a collision integral to obtain a wave Boltzmann equation where only colinear interactions were allowed. They compared their results to the data of Freilich, et. al. (1990), and demonstrated qualitatively good agreement. Abreau, et.al. stated that the peak at $f = 0.16Hz$ in the field data was not due to noncolinear interaction between highly energetic peaks as stated by FGE, but rather to near-resonant, colinear interactions between significantly less energetic

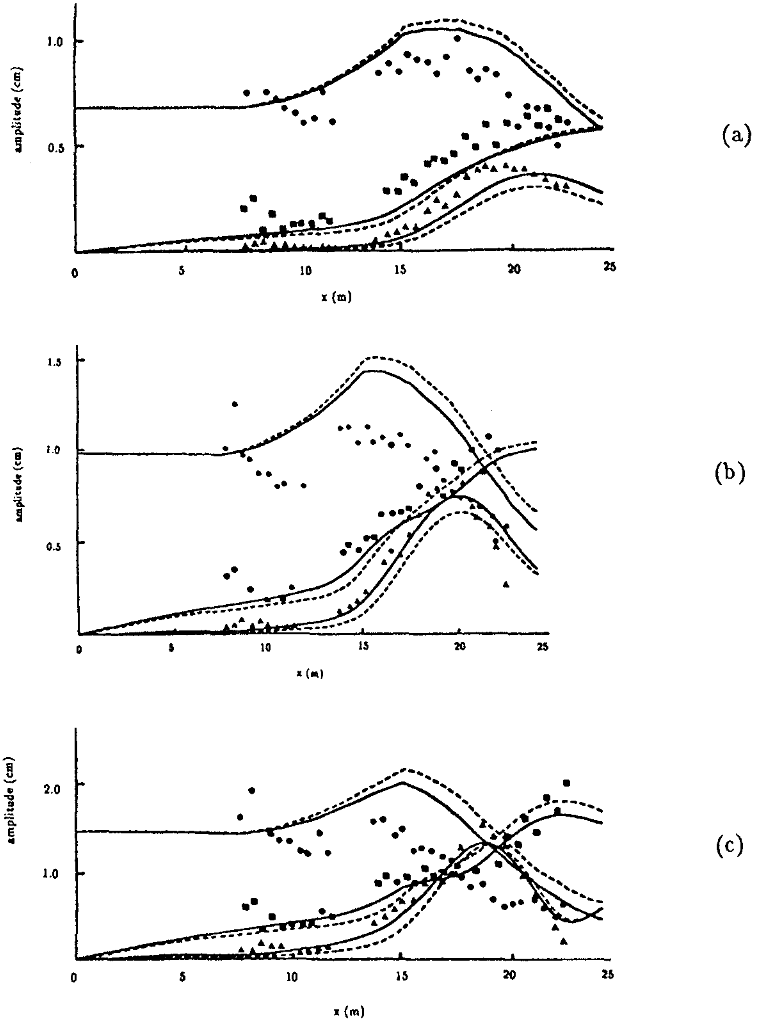

Figure 3: Comparison of equation (14) and KP model of Liu, et. al. (1985) to data of Whalin (1971) along centerline of tank. Solid line is equation (14), dashed line is KP model, and discrete points are data: dots are first harmonic Fourier amplitudes, squares are second harmonic amplitudes and triangles are third harmonic amplitudes. (a): $a_o = 0.68cm$, (b): $a_o = 0.98cm$, and (c): $a_o = 1.46cm$

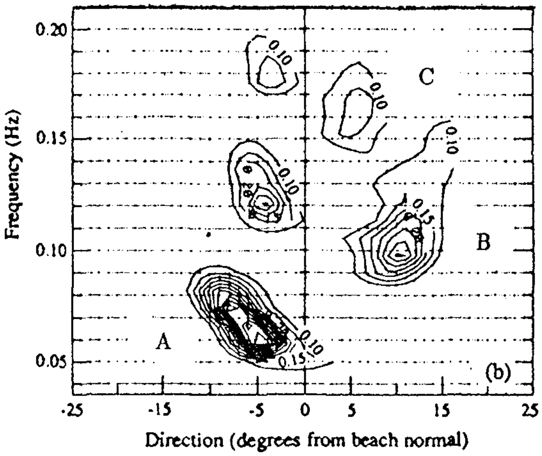

Figure 4: Data of Freilich, et.al. (1990) at $4m$ water depth, plotted as contours of $S(f,\theta)$. From Freilich, et.al. (1990).

components of the two directional spectra. They stated that this interaction occurred because of the directional overlap between the two spectra in deep water.

Angular Spectrum Boussinesq Model

We wish to ascertain whether the vector-sum interactions are as important as the colinear interactions in wavefield evolution. The Boussinesq equation is used to serve as the diagnostic tool. We have three options concerning the form of the Boussinesq model used: a time-dependent model, a parabolic model and an angular spectrum model. The time-dependent formulation (e.g., Wu and Wu (1982)) treats both the resonant and non-resonant interactions in a non-explicit manner, making it difficult to discern between them. Parabolic models (e.g., Liu, et.al. (1985)) are not helpful in this situation since they treat nonlinear interactions with equal weight regardless of direction. Thus we use the angular spectrum model of Kirby (1990).

The angular spectrum approach consists of decomposing the governing equation into $2M+1$ components of the longshore wavenumber λ $(= k\sin\theta)$ as well as N components of the frequency ω. Thus the free surface elevation, for instance, can be written as:

$$\eta_n(x,y,t) = \sum_{n=1}^{N}\sum_{m=-M}^{M} A_n^m(x)e^{i(n\int k\bar{\gamma}_n^m dx + m\lambda_0 y - n\omega t)} + c.c. \tag{15}$$

where :

$$\bar{\gamma}_n^m = (1 - (\frac{m}{n})^2(\frac{\lambda_0}{k})^2)^{1/2} \tag{16}$$

In contrast to the parabolic approach, where N coupled partial differential equations

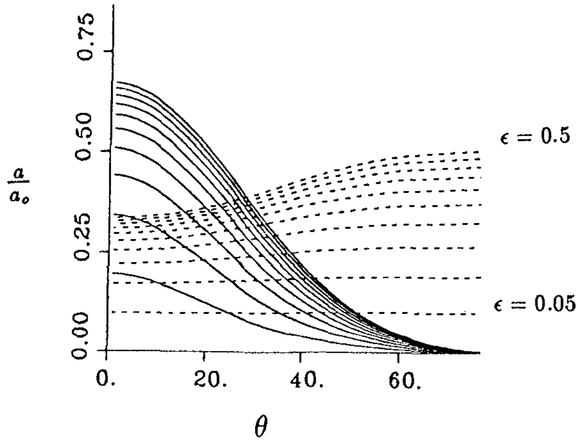

Figure 5: Steady-wave solution of angular spectrum Boussinesq model. Solid lines are amplitudes of vector-sum components, dashed lines are amplitudes of colinear second harmonics.

were solved, the angular spectrum approach computes $N \times (2M+1)$ ordinary differential equations. Further details of the derivation of the angular spectrum Boussinesq model can be found in Kirby (1990), along with the governing equations.

We have used the model in two preliminary investigations to determine the relative strength of the vector-sum interaction. For the case of a steady wave field, we generated permanent form solutions using five amplitudes: two primary harmonics of equal amplitude separated by an angle 2θ, two second harmonics travelling in the same direction as their respective primary harmonics, and one component at the sum frequency and moving at the vector-sum direction. Figure 5 shows a comparison of vector-sum amplitudes to those of the in-line second harmonic for permanent-form solutions and for several values of θ and ϵ. It is apparent that the vector-sum harmonic amplitude is larger than that of the in-line second harmonic for θ up to 25°. This corresponds to a separation angle of 50° between primary components, much larger than the 14° separation between the spectral peaks in the field data of FGE. We also used the model to investigate unsteady wave evolution. We created an initial condition for the model using the same five components as the steady wave example, but the higher harmonics were given zero amplitude. This condition was then run through the model. Figure 6 shows a comparison of the primary amplitude to the vector-sum harmonic and the in-line second harmonic amplitudes. We see that the vector-sum amplitude dominates the in-line second harmonic for much of the domain. We also see evidence of recurrence, whereby the system evolves back into its original state. Recurrence has been shown to be unstable to broad-banded noise (Elgar, et.al. (1990)), so results of this analysis is only moderately applicable to events in the ocean. However, these two investigations do show that vector-sum interactions make non-negligible contributions to wave field evolution.

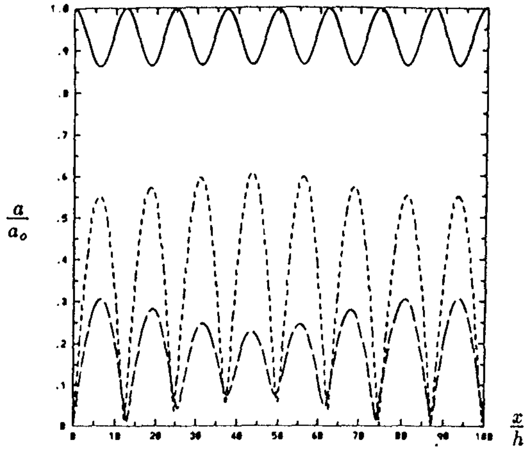

Figure 6: Recurrence test of angular spectrum Boussinesq model for $\epsilon = 0.1$, $(kh)^2 = 0.36$, $\theta = 10^o$. Solid line is amplitude of primary harmonic, short dashed line is amplitude of vector-sum component, and long-dashed line is amplitude of colinear second harmonic.

Results

We at first considered using the model directly on the field data, after the appropriate re-mapping into (f, λ) space. We encountered some difficulty in doing so, primarily due to the random-phase assignments to the components of the field data. This attempt is continuing; however, its successful completion would not demonstrate how vector-sum interactions contribute to wave field evolution. This is because the two directional spectra of FGE overlap in direction, making it difficult to determine whether energy at the sum frequency originated from vector-sum interaction at the two peaks or from colinear interaction by directionally overlapped components.

We then decided to create an artificial wave field, one which would demonstrate the importance of vector-sum interactions for both non-overlapping and overlapping spectra but with fewer components. Two cases, both with the same total variance, were considered: one with two non-overlapping spectra, and one where two spectra overlap in direction. Figure 7a shows two spectra in (f, λ) space, each consisting of one frequency component and six longshore wavenumber components. The frequencies and peak λ chosen correspond to the frequencies and directions of the spectral peaks in FGE. This condition was run through (17) using the same bathymetry as FGE. The result at $4m$ water depth is shown in Figure 7b. The in-line second harmonic for the $0.06Hz$ peak (marked "A"), the in-line third harmonic for the same peak (marked "B"), the in-line second harmonic for the $0.10Hz$ peak (marked "C"), and the vector-sum peak (marked "D") and vector-difference peak (marked "E") have all been generated by nonlinear interaction. It is clear that the vector-sum peak is of the same order of magnitude as the in-line second harmonic of the $0.06Hz$ spectral peak.

The second case is shown in Figure 8a. Two directional spectra with only one frequency component each are positioned at the same frequencies as the first case, but the number of λ components was increased to 20 to allow directional overlap. The result of the calculation is shown in Figure 8b. Here we see very little energy at the vector-sum peak (marked "D"). Energy in this location was generated by detuned interactions between the spectral peaks, and by colinear interactions between the overlapping components. It is apparent that the colinear interactions between the less-energetic directionally overlapping components did not transfer as much energy to peak "D" as noncolinear detuned interactions between two more energetic spectral peaks did in the non-overlapping case. This shows that, for these cases, the vector-sum detuned interactions are responsible for a significant part of nonlinear energy transfer.

Discussion

In this section we used the angular-spectrum Boussinesq model of Kirby (1990) to show that vector-sum detuned interactions play an important role in nonlinear energy transfer. This is shown in the two preliminary investigations of steady and unsteady wave evolution discussed herein. Actual input of the field data to the model has proved to be problematic, so we used two simplified wave fields. These simplified wave fields retain much of the character of the actual field data with fewer components. The results show that the vector-sum interactions can be as prominent in wavefield evolution as the colinear interactions. Model investigations using field data, however, are continuing.

It can be argued that the field data of FGE was actually in deeper water than is valid for lowest-order Boussinesq theory. Therefore, future work will focus on the use of a more dispersive Boussinesq-type model than that of Kirby (1990). This model will be similarly cast in the angular-spectrum format. Additional work will also include the use of a directional bispectrum, which is derivable from the model equation. This will allow us to investigate the strength of the different interactions.

Acknowledgements

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References

Abreau, M., Larraza, A., and Thornton, E., Nonlinear transformation of directional wave spectra in shallow water, *Journal of Geophysical Research*, vol. 97, No. C10, 15579-15589, 1992.

Bryant, P.J., Periodic waves in shallow water, *Journal of Fluid Mechanics*, 59, 625-644, 1973.

Bryant, P.J., Stability of periodic waves in shallow water, *Journal of Fluid Mechanics*, 66, 81-96, 1974.

Elgar, S.L., Freilich, M.H., and Guza, R.T., Recurrence in truncated Boussinesq models for nonlinear waves in shallow water, *Journal of Geophysical Research*, vol. 95, No. C7, 11547-11556, 1990.

Freilich, M.H., and Guza, R.T., Resonance effects on shoaling surface gravity waves, *Philosophical Transactions of the Royal Society of London*, A311, 1-41, 1984.

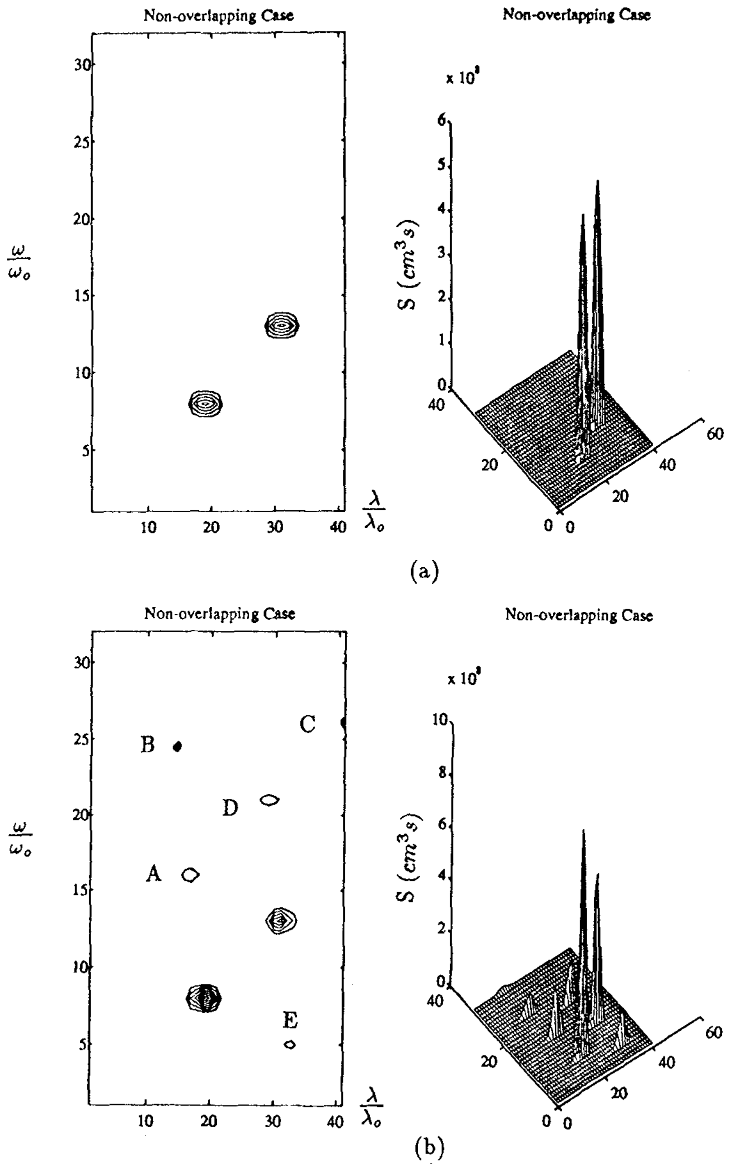

Figure 7: Simplified wave field - narrow directional spread. (a): Contour and surface plots of input condition in (f, λ) space at $10m$ water depth; (b): Contour and surface plots of result at $4m$ water depth.

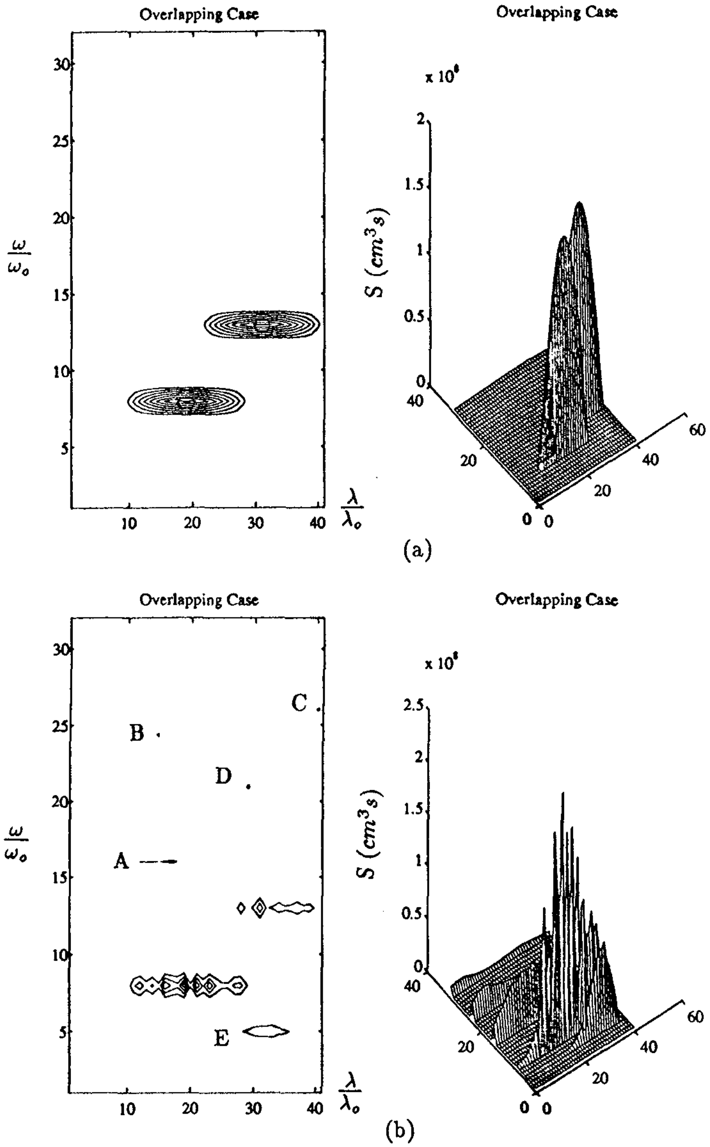

Figure 8: Simplified wave field - broad directional spread. (a): Contour and surface plots of input condition in (f, λ) space at $10m$ water depth; (b): Contour and surface plots of result at $4m$ water depth.

Freilich, M.H., Guza, R.T., and Elgar, S.L., Observations of nonlinear effects in directional spectra of shoaling gravity waves, *Journal of Geophysical Research*, vol. 95, No. C6, 9645-9656, 1990.

Keller, J.B., Resonantly interacting water waves, *Journal of Fluid Mechanics*, 191, 529-534, 1988.

Kirby, J.T., Modelling shoaling directional wave spectra in shallow water, *Proc. 22nd International Conference on Coastal Engineering*, 109-121, 1990.

Kirby, J.T. and Dalrymple, R.A., A parabolic equation method for the combined refraction-diffraction of Stokes waves by mildly-varying topography, *Journal of Fluid Mechanics*, 136, 453-466, 1983.

Kaihatu, J.M., and Kirby, J.T., Frequency domain water wave evolution in finite depth, *EOS Transactions, American Geophysical Union*, Suppl. p. 253, 1991.

Liu, P.L.-F., Yoon, S.B. and Kirby, J.T., Nonlinear refraction-diffraction of waves in shallow water, *Journal of Fluid Mechanics*, 153, 185-201, 1985.

Madsen, P.A., Sorenson, O.R., and Schaeffer, H.A., Nonlinear transformation of irregular waves in shallow water, *Proc. 23rd International Conference on Coastal Engineering*, to appear, 1992.

Mase, H., and Kirby, J.T., Hybrid frequency-domain KdV equation for random wave transformation, *Proc. 23rd International Conference on Coastal Engineering*, to appear, 1992.

Peregrine, D.H., Long waves on a beach, *Journal of Fluid Mechanics*, 27, 815-827, 1967.

Rygg, O.B., Nonlinear refraction-diffraction of surface waves in intermediate and shallow water, *Coastal Engineering*, 12, 191-121, 1988.

Smith, R. and Sprinks, T., Scattering of surface waves by a conical island, *Journal of Fluid Mechanics*, 72, 373-384, 1975.

Ukai, A., Yasuda, T., Ito, K., Spatial variation of wave group statistics and representative wave heights of swell, *Proc. 22nd International Conference on Coastal Engineering*, 812-825, 1990.

Whalin, R.W., The limit of application of linear wave refraction theory in a convergence zone, *Research Report H-71-3*, U.S. Army Engineer Waterways Experiment Station, Vicksburg, MS, 1971.

Wu, D.M., and Wu, T.Y., Three-dimensional nonlinear long waves due to moving surface pressure, *Proc. 14th Symposium on Naval Hydrodynamics*, 103-125, 1983.