CHAPTER 23

EXTENSION OF MILD SLOPE EQUATION FOR WAVES PROPAGATING OVER A PERMEABLE SUBMERGED BREAKWATER

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ABSTRACT

A wave equation is presented for predicting reflected and transmitted waves for a permeable submerged breakwater. This equation includes the mild slope equation derived by Berkhoff(1972), which is a vertically integrated refraction-diffraction equation. Therefore, the equation derived can also predict the combined effects of refraction and diffraction. Numerical calculations with a dissipation term due to breaking are performed to obtain reflection and transmission coefficients as well as wave height distributions. Through the comparisons with the experimental results, the validity of the model is confirmed.

1. INTRODUCTION

Detached breakwaters and groins have been constructed on the coasts of Japan to prevent beach erosion. Recently, defense works against beach erosion, which is the combination of submerged breakwaters with wide crown width and large groins, has been planned on the Niigata west coast. It is very important for coastal engineers to estimate an effectiveness of the breakwaters. However, many of us are faced with a problem of how to estimate a distribution of wave height in the region existing such coastal structures.

Several analytical approaches have been done to predict the wave height of reflected and transmitted waves for permeable breakwaters (e.g., Ijima et. al,1971, Sollit and Cross,1972). Analytical solution obtained by them are valid only for rectangular permeable structures on a uniform bottom. These techniques cannot be applied to structures with arbitrary cross-section and require long algebraic operations to estimate the values of reflection and transmission coefficients. More recently, Sulisz(1985) has developed a numerical technique to predict wave reflection and transmission through a rubble mount breakwater of arbitrary cross section. The technique is ,however, so complicated that it cannot be applied in the three dimensional problem. Therefore, a simple calculation method for wave transformation is required for coastal engineers.

Berkhoff(1972) has derived the mild slope equation which is a vertically integrated refraction-diffraction equation and is easy to be applied for a three dimensional problem. It should be noted that the mild slope equation can be applicable to waves propagating over bottom slopes as steep as 1/3 and even to waves propagating across a

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step(Booij,1983). In this paper, a theoretical approach utilizes the linearized equation of motion in a permeable structure. A wave equation is derived by vertically integrating the equation of motion. The derivation of the wave equation is based on the assumptions that the slope of the permeable structure is adequately gentle, and that evanescent wave components of larger decay rate are negligible. The wave equation derived under the above assumptions is a two-dimensional elliptic type equation and includes the mild slope equation by Berkhoff.

2. DERIVATION OF EXTENDED MILD SLOPE EQUATION

2.1 Wave transformation within a porous medium

The analytical approach in this study begins with unsteady equations of motion for flow in a porous medium. After Madsen and White (1975), the linearized equations of mass and momentum in a porous medium are expressed as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$\frac{\tau}{\lambda}\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-f\frac{\sigma}{\lambda}u \tag{2}$$

$$\frac{\tau}{\lambda}\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-f\frac{\sigma}{\lambda}v \tag{3}$$

$$\frac{\tau}{\lambda}\frac{\partial w}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-f\frac{\sigma}{\lambda}w-g \tag{4}$$

where (u,v,w) is the average discharge velocity components in the direction of x-, y- and z-axis, respectively. These are conceptual quantities which are averaged over finite and total volumes. The quantity p is the pressure averaged over finite pore volumes, λ is the porosity of the porous medium, f is the linearized friction factor, and τ is a inertia coefficient and expressed in terms of a virtual mass coefficient C_M as:

$$\tau=1+\kappa(1-\lambda)=\lambda+C_M(1-\lambda) \tag{5}$$

Hence, we assume that the flow in a porous medium is irrotational. Then the velocity potential φ for virtual discharge velocity is defined as:

$$(u,v,w) = \nabla\varphi \tag{6}$$

Substituting Eq.(6) into the momentum equations (2)-(4) and integrating yields:

$$\frac{\tau}{\lambda}\varphi_t+\frac{p}{\rho}+gz+f\frac{\sigma}{\lambda}\varphi=C(t) \tag{7}$$

where C(t) is a integration constant. Equation (7) is the linearized unsteady Bernoulli equation. In order to remove the integration constant C(t), we introduce a new velocity potential defined by

$$\varphi'=\varphi-e^{-\frac{f\sigma}{\tau}t}\left[\frac{\lambda}{\tau}\int^t C(t)e^{\frac{f\sigma}{\tau}t}\,dt\right] \tag{8}$$

For simplicity, the prime will be omitted hereafter. The imcompressible equations of motion can be expressed in terms of the velocity potential defined in Eq.(8)

$$\nabla^2\varphi+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{9}$$

$$-\frac{\tau}{\lambda}\varphi_t+\frac{p}{\rho}+gz+f\frac{\sigma}{\lambda}\varphi=0 \tag{10}$$

where ∇ is the gradient operator in the horizontal plane. The set of equations (9) and (10) govern the flow within a porous medium, should

be solved under appropriate boundary conditions. Boundaries of the domain are the bottom at a depth z=-h and a free surface, z=ζ. The dynamic free surface condition is obtained by setting P=0 at z=ζ. In the above linearized Bernoulli equation.

$$\frac{\tau}{\lambda}\varphi_t + g\zeta + f\frac{\sigma}{\lambda}\varphi = 0 \quad : z=0 \tag{11}$$

where φ and its derivative with respective to t are evaluated at the still water level z=0. The linearized kinematic boundary condition at a free surface and a bottom are expressed by

$$\varphi_z = \lambda\zeta_t \quad : z=0 \tag{12}$$

$$\varphi_z = -\nabla\varphi\nabla h \quad : z=-h \tag{13}$$

Our task is to seek a solution to the Laplace equation (9) which satisfies the above boundary conditions Eqs.(11) to (13).

It is assumed that a velocity potential φ is expressed by:

$$\varphi = \hat{\varphi}e^{-i\sigma t} = f(z)\eta(x, y)e^{-i\sigma t} \tag{14}$$

In case of constant water depth or very gentle slope, substitution of Eq.(14) into Eq.(9) yields

$$f = A\cosh k(h+z) \tag{15}$$

where k is the complex wave number satisfying a dispersion relation which will be described just later. Elimination of ζ in Eqs.(11) and (12) and substitution of Eq.(14) together with Eq.(15) into the boundary condition gives a dispersion relation as follows:

$$\sigma^2(\tau + if) = gk\tanh kh \tag{16}$$

Equation (16) has a countable infinite number of complex root. Since evanescent wave components may decrease their magnitudes rapidly, we will neglect them. If τ=1 and f=0, then we obtain the dispersion relation for linear free surface waves. In order to obtain a wave equation for the mild bottom slope, we apply the following Green's identity (see, Mei,1983).

$$\int_{-h}^{0}\left\{\Psi_0 L(\Psi_1) - \Psi_1 L(\Psi_0)\right\}dz = \left[A(z)\left(\Psi_0\Psi_1' - \Psi_0'\Psi_1\right)\right]_{-h}^{0} \tag{17}$$

where L is the self-adjoint differential operator which is expressed as

$$L = \frac{d}{dz}\left[A(z)\frac{d}{dz}\right] + B(z) \tag{18}$$

Applying Green's formula for $\hat{\varphi}$ and f and integrating from the bottom to the still water level yields

$$\nabla(C^*C^*_g\nabla\eta) + k^2C^*C^*_g\eta = 0 \tag{19}$$

where

$$C^*C^*_g = \frac{g}{k}\tanh kh\frac{1}{2}\left\{1 + \frac{2kh}{\sinh 2kh}\right\} \tag{20}$$

Equation (19) is the same in the expression as the mild slope equation derived by Berkhoff(1972). However, it is different that the wave number k in Eq.(19) is a complex number. If we put τ=1 and f=0, Eq.(19) reduces to the mild slope equation by Berkhoff. In case of constant water depth, Eq.(19) reduces to the Helmholtz equation which has a solution with an exponential decay in the direction of incident wave propagation.

2.2 Wave transformation over a permeable submerged breakwater

Let us consider the wave transformation over a submerged permeable breakwater as shown in Fig.1. In this figure, h is the still

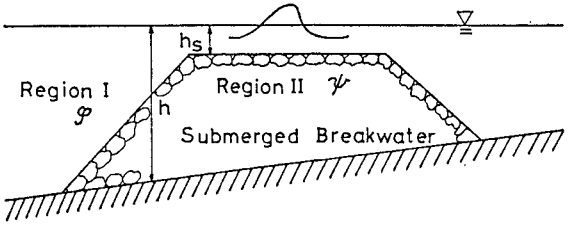

Fig. 1 Definition sketch.

water depth, and hs is the depth above the interface of a submerged breakwater. The domain is divided into two regions, i.e., regions outside and inside the submerged breakwater, denoted as Region I and Region II, respectively. Irrotationality for the flow in pore of the breakwater permits to define the velocity potential ψ in Region II.

In Region I, governing equation and boundary conditions can be expressed in terms of the velocity potential φ as:

$$\nabla^2\varphi+\varphi_{zz}=0 \tag{21}$$

$$\varphi_{tt}+g\varphi_z=0 \quad : z=0 \tag{22}$$

$$\varphi_t+\frac{p}{\rho}-gh_s=0 \quad : z=-h_s \tag{23}$$

$$\varphi_z=-\nabla\varphi\nabla h_s+w_s \quad : z=-h_s \tag{24}$$

where ws is the average discharge velocity at z=-hs. Equation (22) is the boundary condition arising from the kinematic and dynamic condition at the free surface, and Eq.(23) is the dynamic condition at the interface z=-hs. Equation (24) is the modified kinematic condition including the discharge at the interface of the permeable bed.

In Region II, governing equation and boundary conditions are as follows:

$$\nabla^2\psi+\psi_{zz}=0 \tag{25}$$

$$\frac{\tau}{\lambda}\psi_t+\frac{p}{\rho}-gh_s+f\frac{\sigma}{\lambda}\psi=0 \quad : z=-h_s \tag{26}$$

$$\psi_z=-\nabla\psi\nabla h_s+w_s \quad : z=-h_s \tag{27}$$

$$\psi_z=-\nabla\psi\nabla h \quad : z=-h \tag{28}$$

where Eqs.(26) and (27) are the dynamic and kinematic conditions at the interface z=-hs, respectively, Eq.(28) is the kinematic condition for an impermeable bed. Since the pressure and the vertical discharge velocity at the interface must be continuos, the following continuity

conditions should be required.

$$p_I = p_{II} \quad : z = -h_s \tag{29}$$

$$\varphi_z = \psi_z \quad : z = -h_s \tag{30}$$

From Eqs.(23),(26) and (29), we have

$$\varphi_t = (\tau/\lambda)\psi_t + (f\sigma/\lambda)\psi \quad : z = -h_s \tag{31}$$

The velocity potentials φ and ψ may be expressed as foilows:

$$\varphi = -\frac{ig}{\sigma} F_1(z)\eta(x, y)e^{-i\sigma t} \tag{32}$$

$$\psi = -\frac{ig}{\sigma} F_2(z)\zeta(x, y)e^{-i\sigma t} \tag{33}$$

Substituting Eqs.(32) and (33) into Eqs.(21) and (25) respectively, and invoking the mild slope assumption, we obtain

$$F_1(z) = A\cosh k(h_s + z) + B\sinh k(h_s + z) \tag{34}$$

$$F_2(z) = D\cosh k(h + z) \tag{35}$$

where A, B and D are integration constants. The free surface boundary condition and the continuity condition for mass flux at the the interface together with Eqs.(32) to (35) give a dispersion relation by

$$\sigma^2 = gk\frac{\tanh kh_s + q\tanh k(h - h_s)}{1 + q\tanh k(h - h_s)\tanh kh_s} \tag{36}$$

where $q = \lambda/(\tau + if)$. When $\lambda = 1$, $\tau = 1$ and f=0, or $\lambda = 0$, the dispersion relation Eq.(36) reduces to that for the linear wave over an impermeable bottom.

Applying Green's formula for F1 and $\hat{\varphi}$, and for F2 and $\hat{\psi}$,

$$\int_{-h}^{-h_s}\left\{F_2\frac{\partial^2\hat{\varphi}}{\partial z^2} - \frac{\partial^2 F_2}{\partial z^2}\hat{\varphi}\right\}dz + \int_{-h_s}^{0}\left\{F_1\frac{\partial^2\hat{\varphi}}{\partial z^2} - \frac{\partial^2 F_1}{\partial z^2}\hat{\varphi}\right\}dz = \left[F_2\frac{\partial\hat{\varphi}}{\partial z} - \frac{\partial F_2}{\partial z}\hat{\varphi}\right]_{-h}^{-h_s} + \left[F_1\frac{\partial\hat{\varphi}}{\partial z} - \frac{\partial F_1}{\partial z}\hat{\varphi}\right]_{-h_s}^{0} \tag{37}$$

and substituting Eqs.(32) to (35) into Eq.(37), we get the extended mild slope equation.

$$\nabla(a\nabla\eta) + k^2 a\eta = 0 \tag{38}$$

$$a = \frac{g}{k}\tanh kh_s\frac{1}{2}\left[1 + \frac{2kh_s}{\sinh 2kh_s}\right] + q\frac{g}{k}\tanh k(h - h_s)\tanh^2 kh_s \tag{39}$$

$$+ q\frac{g}{k}\frac{\tanh k(h - h_s)}{\cosh^2 kh_s}\frac{1}{2}\left[1 + \frac{2k(h - h_s)}{\sinh 2k(h - h_s)}\right] + q^2\frac{g}{k}\tanh kh_s\cdot\tanh^2 k(h - h_s)\frac{1}{2}\left[1 - \frac{2kh_s}{\sinh 2kh_s}\right]$$

If we put $\lambda = 0$, or h=hs in Eqs.(38) and (39), Eq.(38) reduces to the mild slope equation for an impermeable bed. So the wave equation (38) includes the mild slope equation derived by Berkhoff(1972). Furthermore, when hs=0, we get a wave equation similar to that for wave transformation within a permeable media in previous section, but have a dispersion relation slightly different from Eq.(16).

2.3 Modeling of wave breaking over a submerged breakwater.

Wave breaking over a submerged breakwater is very important mechanism of wave energy dissipation. The wave equation (38) should be modified to include a energy dissipation due to breaking. To do this, we add the energy dissipation term in Eq.(38) to be expressed by

$$\nabla(a\nabla\eta)+k^2a\eta=-i\sigma f_D\eta \tag{40}$$

where f_D is the energy dissipation factor, σ is the angular frequency and i is the imaginary unit. Since breaking waves over a wide submerged breakwater may recover, thc value of f_D for recovered waves must be required to be equal to zero. Referring the studies by Watanabe and Dibajnia(1988) and Isobe et. al(1988), we will adopt the following expression for f_D.

$$f_D=\alpha_D\tanh\left[\beta_D\frac{\overline{h_s}}{L_0}\right]\sqrt{\frac{g}{h_s}}\sqrt{\frac{\gamma-\gamma_r}{\gamma_s-\gamma_r}}s_a \tag{41}$$

where $\overline{h_s}$ and h_s are the water depth above the brcakwater crown and the local water depth abovc the interface of the breakwatcr, rcspectively. The quantity γ is the ratio of the wave amplitude to the water depth h_s, γ_s and γ_r show the values of γ for breaking waves over a uniformly sloping beach and for recovered waves over a step type beach, respectively. The quantity s_a is the averaged bottom slope defined by Izumiya and Isobe(1986). The proportionality constants α_D and β_D will be determined from the results of experiment on wave transformation.

3. EXPERIMENTS ON WAVE TRANSFORMATION OVER A PERMEABLE SUBMERGED BREAKWATER

3.1 Experimental equipments and procedure

In order to confirm the validity of the model, the experiments on the wave transformation over a submerged permeable breakwater were conducted. A uniform slope of 1/15 made of wooden board was installed in a wave flume, and a submerged breakwater composed of three kinds of gravels was placed on the slope. The cross section of the model breakwater is shown in Fig.2. The average weight of gravels, W1,W2 and W3, is 86.3 gf, 10.4 gf and 0.89 gf respectively. The average porosity of the submerged breakwater is 0.433. The experiments with 24 cases were made for various wave heights and periods with the crown water depth of 3 cm.

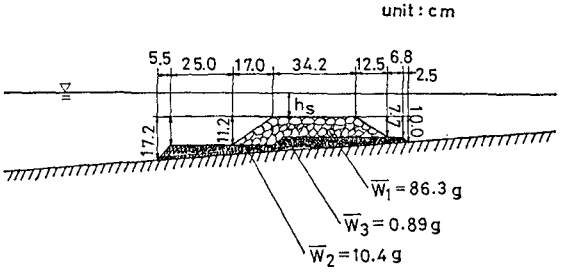

Fig. 2 Cross section of model submerged breakwater.

3.2 Wave breaking condition over a submergcd breakwater

A few investigation on wave breaking over a submerged breakwater has been made. For thls, a large number of experiments were carried out to obtain the empirical formula for the breaking condition. In case of wave breaking over a submerged breakwater, the breaklng water

depth is defined as a depth from the interface of a submerged breakwater to the still water level. From the experimental data of the breaking water depth h_B, wave period and the local average slope s_a, the following expression for the non-dimensional breaking depth is obtained.

$$\frac{h_B}{H_0}=0.43+\frac{0.1}{-0.77-log_{10}(H_0/L_0)}+0.22\left\{1-\frac{0.184\,s_a^2}{s_a^2-0.107\,s_a+0.054}\right\}\cdot\left(\frac{H_0}{L_0}\right)^{-0.37} \quad (42)$$

where Ho is the deepwater wave height and Lo is the deepwater wavelength. When the local average slope steeper than 0.1, s_a takes a value of 0.1. The relation (42) is obtained to be slightly modified from the expression for steep slopes proposed by Isobe et. al(1988). Using Eq.(42), we can evaluate the breaking water depth as a function of the deepwater steepness and the average bottom slope.

4. METHOD OF NUMERICAL CALCULATION

A two-dimensional problem in the vertical plane is analyzed to examine the validity of the model equation. Equation (38) is discretized into a finite difference form and is solved by using two boundary conditions, i.e., the offshore boundary condition and the non-reflective boundary condition at a shoreward boundary to be expressed by

$$ik_i\varphi+\varphi_x=ik_iH_ie^{ik_ix_0} \quad (43)$$

$$ik\varphi-\varphi_x=0 \quad (44)$$

where ki is the wave number at the offshore boundary, Hi is the incident wave height and xo is the location of the offshore boundary. These boundary conditions are also discretized into finite difference schemes. The procedure of numerical calculation is summarized as follows:

1) Solve the dispersion equation, (36), for the first eigen value.
2) Assume $f_D=0$, and solve Eq.(38) with the boundary conditions Eqs.(43) and (44).
3) Determine the breaking point using the breaking condition Eq.(42).
4) Calculate the wave heights with Eqs.(40), (41),(43) and (44).
5) Calculate the difference of wave heights for the successive iterations at each location.
6) Repeat 4) and 5) until the solution converges.

Through the above procedure, the converged solutions are obtained within 7 to 8 iterations.

5. COMPARISON OF THE MODEL RESULTS AGAINST THE EXPERIMENTAL DATA

The computed results with Eqs.(38) and (39) are compared with the experimental data to examine the applicability and the validity of the present model. In the calculation, The values of α_D, β_D, τ and f are determined by fitting experimental data for wave height. The results are $\alpha_D=0.8$, $\beta_D=90.0$, $\tau=1.1$ and f=10.0. Figure 3-(a) to (e) show the comparison between measured and calculated values of wave height. The solid lines indicate the wave height distributions calculated with the extended mild slope equation (38) including the dissipation term due to breaking. The closed circles show the experimental data. The agreement of wave heights outside the surf zone is very good. This means that the magnitude and the phase of reflected waves are well predicted by the present model. On the other hand, the

wave heights measured over a submerged breakwater are fairly smaller than the computed results. This is because that since the water depth above the breakwater crown, 3 cm, is shallow, the waves collapse on the breakwater with the strong nonlinearity of motion. So the difference between measured and computed wave heights occurred. Nevertheless, the transmitted wave heights calculated from this model are in good agreement with the experimental results.

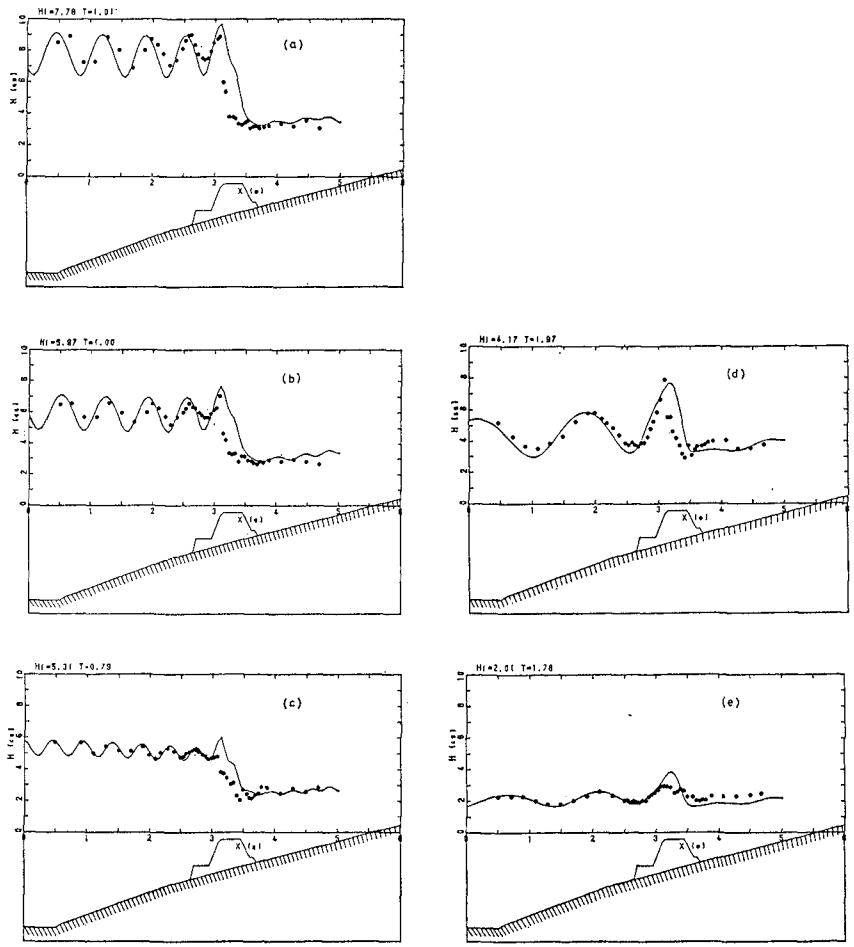

Fig. 3 Wave height transformations.

Figure 4 shows the comparison between measured and calculated values of reflection coefficient as a function of relative crown width B/Lo. It is found that although the computed results in the range (0.1 < B/Lo < 0.2) indicate values slightly larger than the experimental data, the agreement is fairly good. Figure 5 compares the the transmission coefficients with the measured and calculated results. From this figure, wa can find that the estimated values of the transmission coefficient agree well with the experimental data, and also find that the coefficient decreases with decrcasing relative crown depth hs/Ho. Figure 6 shows the comparison between measured and calculated values of energy loss coefficient as a function of the relative crown depth hs/Ho. It is seen from the figure that thcre is a scatter in the data in the range of hs/Ho >0.8, however, the agreement is fairly good in the range of hs/Ho<0.8.

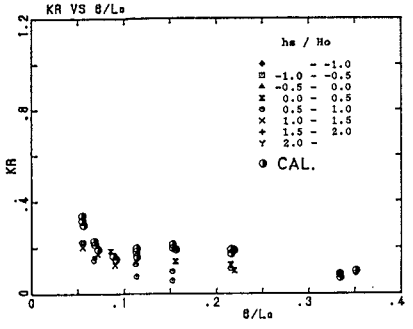

Fig. 4 Comparison between measured and calculated values of reflection coefficient.

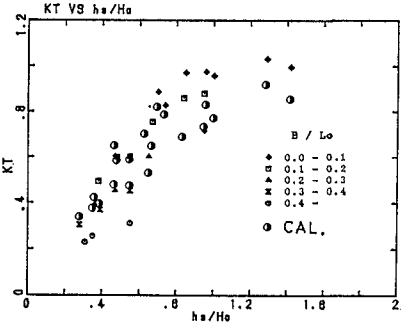

Fig. 5 Comparison between measured and calculated values of transmission coefficient.

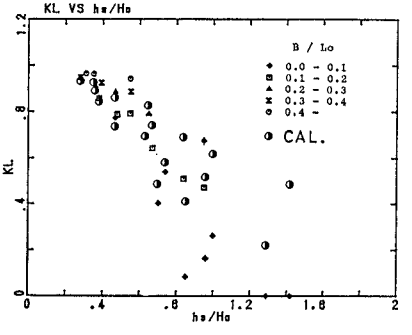

Fig. 6 Comparison between measured and calculated values of energy loss coefficient.

6. CONCLUDING REMARKS

A wave equation has been developed to evaluate the important characteristics of permeable breakwater. The wave equation obtained is applicable in and outside of a permeable structure, and has the advantage that the computation time is fairly shorter than those of existing methods. The validity of the model was confirmed through a comparison with the experimental data for a two-dimensional case. As the result, it was found that the present model can produce accurate results for practical purposes for the reflection, transmission and dissipation characteristics.

ACKNOWLEDGMENTS

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