CHAPTER 39

# On the maximum runup of cnoidal waves.

Costas Emmanuel Synolakis †, A.M., ASCE and Manas Kumar Deb ‡.

This is a study of the maximum runup of cnoidal waves on plane beaches. An approximate theory is described for determining the maximum runup of non-breaking cnoidal waves. It is shown that the linear and nonlinear theory predict mathematically identical maximum runup heights. An asymptotic result is derived for the maximum runup of solitary waves, which are one limiting form of cnoidal waves. A series of laboratory experiments is described to support the theory. Other numerical results are presented that suggest that the runup of cnoidal waves is significantly higher than the runup of monochromatic waves with the same waveheight and wavelength. Preliminary laboratory data are also presented which suggest that, for certain cnoidal waves, the maximum runup is not a monotonically varying function of the normalized wavelength.

## 1. Introduction

The problem of determining the runup and reflection of cnoidal waves on plane beaches usually arises in the study of the coastal effects of tsunamis. Tsunamis are long water waves of small steepness generated by impulsive geophysical events on the ocean floor or at the coastline. Cnoidal waves are believed to model important aspects of the coastal effects of tsunamis well.

The process of long wave generation and propagation is now well understood. The process of long wave runup and reflection is not. However, there is consensus that one suitable physical model for this process is the formalism of a long wave propagating over constant depth and encountering a sloping beach. The studies of long wave runup have concentrated either on solitary waves or on monochromatic waves, i.e., at the two extremes of cnoidal waves. (For a comprehensive review of studies on solitary wave runup see Synolakis (1986).) To date, there appear to have been only three studies on cnoidal wave runup, one unpublished study by Pedersen and Gjevik (1983), the study of Ohyama (1987) – in japanese – and the study of Synolakis et al (1988).

† Assistant Professor, School of Engineering, University of Southern California, Los Angeles, California 90089-0242

‡ Graduate Research Assistant, School of Engineering, University of Southern California, Los Angeles, California 90089-0242

Despite the quality of the analytical work, fundamental unresolved questions about the runup of long waves still persist. For example, in the study of solitary waves, the empirical relationship between the normalized runup and the normalized wave height that has been established in the classic laboratory investigations of Hall and Watts (1955) remained, until recently, unexplained. The results of the available numerical solutions have not been confirmed with detailed surface profiles from laboratory data, and, as a consequence, there is little conclusive information about the relative importance of dispersion and nonlinearity during runup. Until Synolakis (1987), there was no realization of the differences in the runup behaviour of breaking and nonbreaking long waves, and this had led to comparisons between numerical data for nonbreaking waves with laboratory data for breaking waves. There is still little understanding of the criteria that determine whether a wave breaks during runup or rundown, or of the reflection process. Compared to recent advances in understanding periodic wave runup on natural beaches, (Huntley et al 1977, Guza and Thornton, 1982 and Holman, 1986) the state of the art in cnoidal runup has been very limiting.

In the present study an exact solution to an approximate theory is described for determining the runup of long waves up plane beaches. The basic solution details and the evolution of the amplitude for solitary waves have been presented in Synolakis (1987). Some results for cnoidal waves have been presented in Synolakis et al (1988). In this paper we will summarize the predictions of the linear and nonlinear theory and we will show that the linear and nonlinear theory predict mathematically identical maximum runup heights. We will summarize the asymptotic results that lead to the runup law presented ibid and we will present numerical results for the runup of cnoidal waves. These results hint on a rich and unexpected behavior in the runup of long waves.

## 2. Basic equations and solutions

Consider a topography consisting of a plane sloping beach of angle $\beta$, as shown in figure 1. The origin of the coordinate system is at the initial position of the shoreline and $x$ increases seaward. The topography $\tilde{h}_0(\tilde{x})$ is described as follows :

$$\tilde{h}_0(\tilde{x}) = \tilde{x}\tan\beta \text{ when } \tilde{x} \le d\cot\beta \text{ and } \tilde{h}_0(\tilde{x}) = d \text{ when } \tilde{x} > d\cot\beta. \tag{1}$$

d is the undisturbed water depth in the constant depth region. Dimensionless variables are introduced as follows : $\tilde{x} = xd, \tilde{H} = Hd, \tilde{\eta} = \eta d$, $\tilde{h}_0 = h_0 d$, $\tilde{u} = u\sqrt{gd}$, and $\tilde{t} = t\sqrt{d/g}$. Consider a propagation problem described by the shallow water wave equations:

$$h_t + (hu)_x = 0, \tag{2a}$$
$$u_t + uu_x + \eta_x = 0, \tag{2b}$$

where $h(x) = h_0(x) + \eta(x,t)$.

### 2.1 Linear theory

The system of equations (2) can be linearized by retaining the first order terms only, resulting into $\eta_{tt} - (\eta_x h_0)_x = 0$. When $h_0(x) = x\tan\beta$, a well known solution of this equation is $\eta(x,t) = B(k,\beta)J_0(2k\sqrt{x\cot\beta})e^{-ikct}$, where $B$ is the amplification factor, $k$ is the wavenumber and $c = 1$. Keller and Keller (1964) presented another steady state solution for the combined topography defined by (1). For an incident wave of the form $\eta(x,t) = A_i e^{-i(x+ct)}$, they determined the amplification factor $B(k,\beta,A_i) = 2e^{-ik\cot\beta}A_i/[J_0(2k\cot\beta) - iJ_1(2k\cot\beta)]$.

Since the governing equation is linear and homogenous, any standing wave solutions can be used to obtain travelling wave solutions by linear superposition. For example, when the incident wave is of the form $\eta(X_0,t) = \int_{-\infty}^{\infty}\Phi(k)e^{-ikct}dk$, then the wave transmitted to the beach is given by :

$$\eta(x,t) = 2\int_{-\infty}^{\infty}\Phi(k)\frac{J_0(2k\sqrt{xX_0})e^{-ik(X_0+ct)}}{J_0(2kX_0) - iJ_1(2kX_0)}dk, \tag{3}$$

where $X_0 = \cot\beta$. This solution is only valid when $x \geq 0$. To obtain the details of the solution when $x < 0$ one must solve the nonlinear set (2).

### 2.2 Nonlinear theory

To solve the nonlinear set (2) for the sloping beach case, $h_0(x) = x\tan\beta$, Carrier and Greenspan (1958) introduced the following hodograph transformation,

$$x = \cot\beta\left(\frac{\sigma^2}{16} - \frac{\psi_\lambda}{4} + \frac{u^2}{2}\right),\ t = \cot\beta\left(\frac{\psi_\sigma}{\sigma} - \frac{\lambda}{2}\right)$$

$$u = \frac{\psi_\sigma}{\sigma},\ \text{and}\ \ \eta = \frac{\psi_\lambda}{4} - \frac{u^2}{2}. \tag{4}$$

This change of variables reduces the set of equations (2) to a single linear equation,

$$(\sigma\psi_\sigma)_\sigma = \sigma\psi_{\lambda\lambda}. \tag{5}$$

The transformation is such that in the hodograph plane, i.e., the $(\sigma,\lambda)$ space, the shoreline is always at $\sigma = 0$ ; this can be deduced easily by setting $\sigma = 0$ in (4); then $x = -\eta\cot\beta$, which is an equality only valid at the shoreline tip.

Equation (5) can be solved with standard methods. Defining the Fourier transform of $\psi(\sigma,\lambda)$ as $\Psi(\sigma,\tilde{k}) = \int_{-\infty}^{\infty}\psi(\sigma,\lambda)e^{-i\lambda\tilde{k}}d\lambda$,and, if $\Psi(\sigma_0,\tilde{k}) = F(\tilde{k})$, then the bounded solution at $\sigma = 0$ and $\sigma = \infty$ takes the form :

$$\psi(\sigma,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(\tilde{k})\frac{J_0(\tilde{k}\sigma)}{J_0(\tilde{k}\sigma_0)}e^{i\tilde{k}\lambda}d\tilde{k}. \tag{6a}$$

If an initial condition is available instead, one may use Hankel transform methods (Carrier (1966)).

The process of specifying explicitly an initial or a boundary condition to equation (5) is nontrivial. Even if initial or a boundary data are available in the $(x,t)$ space, the process of deriving the equivalent conditions in the $(\sigma,\lambda)$ space is very complex. These difficulties have restricted the use of the Carrier and Greenspan formalism and this is rather unfortunate, because some of the problems described can be circumvented. (Carrier, 1966.) Another method has been described by Synolakis (1986, 1987) to specify a boundary condition *including reflection*. We will summarize it here.

## 2.3 Approximate solution of the nonlinear theory

Carrier (1966) pointed out that far from the shoreline nonlinear effects are small. The transformation equations can then be simplified by neglecting terms $\sim O(u^2)$. To the same order, $\psi_\lambda \ll \frac{\sigma^2}{16}$ and $\frac{\psi_\sigma}{\sigma} \ll \frac{\lambda}{2}$. Using these approximations the transformation equations (4) reduce to $u = \frac{\psi_\sigma}{\sigma}$, $\eta = \frac{\psi_\lambda}{4}$, $x = \frac{\sigma^2}{16}\cot\beta$ and $t = -\frac{\lambda}{2}\cot\beta$. These equations are uncoupled and allow direct transition from the $(\sigma,\lambda)$ space to the $(x,t)$ space.

One method for specifying a boundary condition in the physical space is to use the solution of the equivalent linear problem, as given by equation (3), at the seaward boundary, where $x = X_0 = \cot\beta$, i.e., the point $\sigma = \sigma_0 = 4$ in the $(\sigma,\lambda)$ space. Then equations (4) imply that $\eta(X_0,t) = \frac{1}{4}\psi_\lambda(4,\lambda)$. The boundary condition $F(\tilde{k})$ in the $(\sigma,\lambda)$ space is determined from (3) by repeated application of the Fourier integral theorem. Assuming that $\psi(\sigma_0,\lambda) \to 0$ as $\lambda \to \pm\infty$, then the solution of equation (5) follows,

$$\psi(\sigma,\lambda) = -\frac{16i}{X_0}\int_{-\infty}^{\infty}\frac{\Phi(\kappa)}{\kappa}\frac{J_0(\frac{\sigma\kappa X_0}{2})e^{-i\kappa X_0(1-\frac{\lambda}{2})}}{J_0(2\kappa X_0) - iJ_1(2\kappa X_0)}d\kappa\,. \tag{6b}$$

## 3. Comparison of the linear and of the nonlinear theory

The maximum runup according to the linear theory is the maximum value attained by the wave amplitude at the initial position of the shoreline ,i.e., at $x = 0$, or

$$\eta(0,t) = 2\int_{-\infty}^{\infty}\frac{\Phi(k)e^{-ik(X_0+ct)}}{J_0(2kX_0) - iJ_1(2kX_0)}dk. \tag{7}$$

In the nonlinear theory the maximum runup is given by the maximum value of the amplitude at the shoreline $\eta(x_s,\lambda)$.($x_s$ is the $x-$coordinate of the shoreline and it corresponds to $\sigma = 0$.) Using (4), one obtains that :

$$\eta(x_s,t) = \frac{\psi_\lambda}{4} - \frac{{u_s}^2}{2} = 2\int_{-\infty}^{\infty}\frac{\Phi(\kappa)e^{-i\kappa X_0(1-\frac{\lambda}{2})}}{J_0(2\kappa X_0) - iJ_1(2\kappa X_0)}d\kappa - \frac{{u_s}^2}{2}; \tag{8}$$

$u_s = dx_s/dt$ is the velocity of the shoreline tip. At the point of maximum runup $u_s$ becomes zero. Setting $u_s = 0$ and $\sigma = 0$ in the transformation equations (4) reduces the transformation equations to $u = 0$, $\eta = \frac{\psi_\lambda}{4}$, $x = -\eta \cot\beta$, and $t = -\frac{\lambda}{2}\cot\beta$. Substitution of these values in equation (8) reduces it to equation (7), proving that the maximum runup predicted by the linear theory is identical to the maximum runup predicted by the nonlinear theory. At the minimum rundown point the shoreline tip attains zero velocity also, and the same argument applies again. This result was first noted by Carrier (1971) in a problem where reflection was negligible. As shown by Synolakis (1987) it is valid in general, even when reflection is important. It is largely unexpected, because the linear and nonlinear theory differ most at the initial position of the shoreline.

## 4. The cnoidal wave solutions

Cnoidal waves are exact periodic solutions of the KdV equation $\eta_t+(1+\frac{3}{2}\eta)\eta_x+\frac{1}{6}\eta_{xxx} = 0$. A cnoidal wave propagating over constant depth is given by Svendsen (1974) as

$$\eta(x,t) = y_t - 1 + H\mathrm{cn}^2(2K(\frac{x}{L}+\frac{t}{T})|m). \tag{9}$$

$y_t$ is the distance of the trough from the bottom, and $H, L$ and $T$ are the dimensionless wave height, wavelength and period respectively. $K(m)$ is the first elliptic integral and $m$ is the elliptic parameter. If one defines the Ursell number as $\mathrm{U} = HL^2$, then $\mathrm{U} = (16/3)mK^2$. The function $\mathrm{cn}(z|m)$ is the Jacobian elliptic function and it is given by Abramowitz and Stegun (1972) as:

$$\mathrm{cn}(z|m) = \frac{2\pi}{\sqrt{m}K}\sum_{i=0}^{\infty}\frac{q^{i+\frac{1}{2}}}{1+q^{2i+1}}\cos[(2n+1)\frac{\pi z}{2K}]$$

$q = e^{-\pi K'/K}$, and $K$ and $K'$ are the real and imaginary quarter periods of the elliptic functions respectively. Cnoidal waves have two important limiting cases. As $m \to 1$, it can be shown that

$$\eta(x,t) = H\mathrm{sech}^2\sqrt{\frac{3}{4}H}(x+ct). \tag{10}$$

$c = \sqrt{1+H}$ is the wave celerity. Equation (10) is the Boussinesq profile for a solitary wave. As $m \to 0$, then

$$\eta(x,t) = \frac{H}{2}\cos 2\pi(\frac{x}{L}+\frac{t}{T}),$$

the profile of a monochromatic periodic wave. For comparison, figure 1 shows a cnoidal wave, a solitary wave and a sinusoidal wave with the same normalized height $H$ propagating over constant depth. The cnoidal and sinusoidal waves also have the same wavelength $L$.

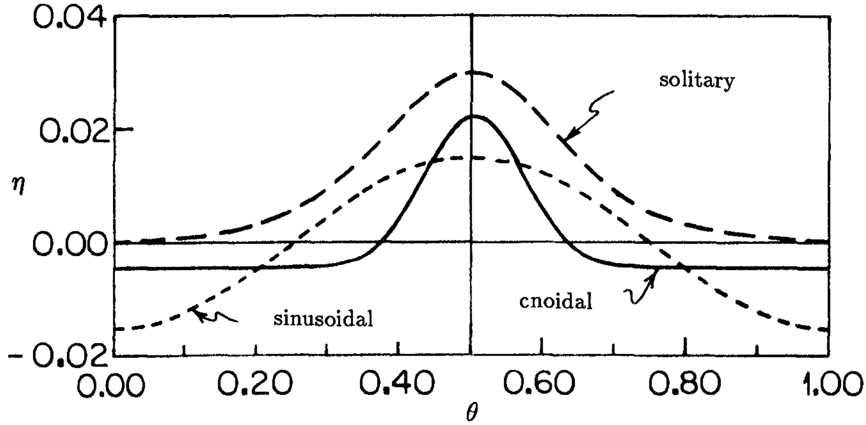


**Figure 1** The amplitude profile of a solitary wave ($\eta = H\text{sech}^2\theta$), a sinusoidal wave ($\eta = \frac{H}{2}\cos\theta$) and a cnoidal wave ($\eta = y_t - 1 + H\text{cn}^2(2K\theta|m)$) with the same $H = 0.027$ propagating over constant depth as a function of the dimensionless phase $\theta$. The cnoidal wave and the sinusoidal wave also have the same wavelength $L$.

The results of the previous section will now be applied to derive a result for the maximum runup of cnoidal waves climbing up a sloping beach.

### 4.1 The solitary wave solution

A solitary wave centered at $x = X_1$ at $t = 0$ has the following surface profile $\eta(x,0) = \frac{H}{d} sech^2\gamma(x - X_1)$, where $\gamma = \sqrt{3H/4d}$. The function $\Phi(k)$ associated with this profile is derived in the appendix. It is given by $\Phi(k) = (2/3)k\text{cosech}(\alpha k)e^{ikX_1}$ where $\alpha = \pi/2\gamma$. Substituting $\Phi(k)$ into equation (7) and defining as $R(t)$ the dimensless surface elevation at the initial position of the shoreline, then

$$R(t) = \frac{4}{3}\int_{-\infty}^{\infty} k\text{cosech}(\alpha k)\frac{e^{ik(X_1 - X_0 - ct)}}{J_0(2kX_0) - iJ_1(2kX_0)}dk. \tag{11}$$

This integral can be calculated with standard methods of applied mathematics; its convergence and evaluation is discussed in Synolakis (1987,1988). The integration result is :

$$R(t) = 8H\sum_{n=1}^{\infty}\frac{(-1)^{n+1}ne^{-2\gamma(X_1 - X_0 - ct)n}}{I_0(4\gamma X_0 n) + I_1(4\gamma X_0 n)}. \tag{12}$$

The series can be simplified further by using the asymptotic form for large arguments of the modified Bessel functions. When $4X_0\gamma \gg 1$, then :

$$R(t) = 8\sqrt{\pi X_0}H(3H)^{\frac{1}{4}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{\frac{3}{2}}e^{-2\gamma(X_1 + X_0 - ct)n}. \tag{13}$$

This form of the solution is particularly helpful for calculating the maximum runup. The series in (12) is of the form $\sum_{n=1}^{\infty}(-1)^{n+1}n^{3/2}\chi^n$ ; its maximum value occurs at $\chi = 0.481$. This value defines the time $t_{\max} = (1/c)[X_1 - X_0 - 0.366/\gamma]$ when the wave reaches its maximum runup, and the value of the series at $t_{max}$ is $s_{max} = 0.15173$. Defining as $\mathbf{R}$ the maximum value of $R(t)$ and evaluating the term $8\sqrt{\pi\sqrt{3}}s_{max}$, then the following expression results for the maximum runup :

$$\mathbf{R} = 2.831\sqrt{\cot\beta}(H)^{\frac{5}{4}}. \tag{14}$$

The term *the runup law* has been coined for this equation. It is formally valid when the series converges and when $4X_0\gamma \gg 1$, i.e., $\sqrt{H} \gg 0.288\tan\beta$.

### 4.2 Validity of the solution

The solution described is valid for functions $\Phi(k)$ such that the Jacobian of the transformation (4) is never equal to zero. The Jacobian becomes zero when the surface slope $\partial\eta/\partial x$ becomes infinite. In the physical plane this point is usually interpreted as the point of wave breaking.

The Jacobian of the Carrier and Greenspan transformation is $\mathbf{J} = c({t_\sigma}^2 - {t_\lambda}^2)$. Taking the limit as $\sigma \to 0$, then $\mathbf{J} \to (u_\lambda - \frac{1}{2})^2$. Using the formalism used in calculating the runup integral (Eq. 11), one can show analytically that the limiting $H$ when $u_\lambda - \frac{1}{2}$ goes through zero, as :

$$H = 0.8183(\cot\beta)^{-\frac{10}{9}}. \tag{15$a$}$$

This is a weaker restriction than that presented in Pedersen and Gjevik (1983), who reported that waves break when

$$H > 0.479(\cot\beta)^{-\frac{10}{9}}. \tag{15$b$}$$

However, there are two basic differences between the two results. The Gjevik and Pedersen criterion (15) indicates the limiting $H$ when a solitary wave breaks during the backwash. Equation (15b) indicates when a wave first breaks during runup. Also, the Gjevik and Pedersen result (15b) was derived by using the sinusoidal wave profile that best fits the Boussinesq profile, while equation (15a) is based on the actual Boussinesq profile (Eq. 10).

### 4.3 Maximum runup

Although it has long been known that breaking and nonreaking periodic waves follow different runup variations, this behaviour has never been recognized in single–wave runup. In Synolakis (1987), data was presented that demonstrated conclusively that two different runup regimes exist, one for breaking and one for nonbreaking solitary waves.

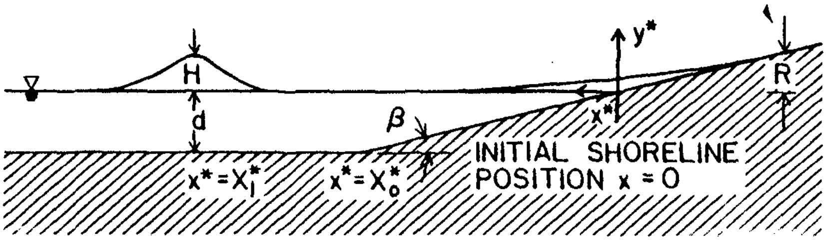


**Figure 2a.** A definition sketch for solitary wave runup.

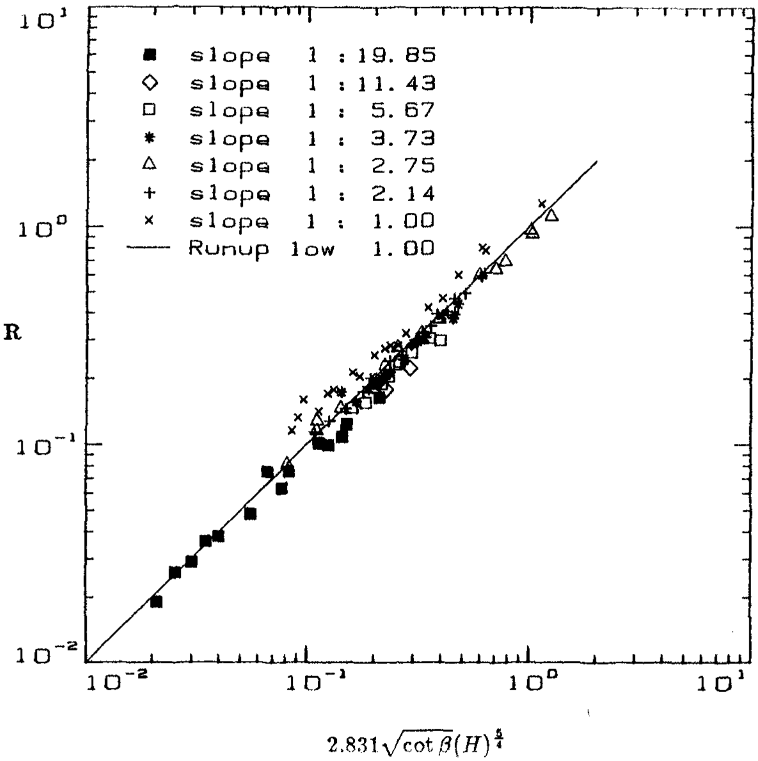


**Figure 2b.** The normalized runup **R** of solitary waves of height $H$ climbing up different beaches of slope 1 : cot $\beta$ plotted against the runup law (Eq. 13). All data refer to nonbreaking waves and they were derived in the laboratory. The data for the slopes 1 : 11.47, 1 : 5.67, 1 : 3.73, 1 : 2.14, and 1 : 1.00 are from Hall and Watts (1953), the data for the 1 : 2.75 slope are from Pedersen and Gjevik (1983), and the data for the 1 : 19.85 slope are from Synolakis (1987).

To verify this observation for other beaches, the runup law has been compared ibid with laboratory data. Comparisons of analytical results with laboratory data such as those of Hall and Watts (1953) are not as straightforward as often assumed. That study includes both breaking and nonbreaking wave data without identifying them as such, and therefore the empirical relationships derived there may not be directly applicable when determining the runup of nonbreaking waves. To perform a posteriori identification of the Hall and Watts data, the breaking criterion (Eq 15) was used, and the identified nonbreaking wave data are presented in figure 2b. The abscissa is the runup law (Eq. 14), and the ordinate is the maximum runup. The asymptotic result does appear to model the laboratory data very well. No data are presented for slopes smaller than 1:20, because no such data are yet available; on a 1:100 slope, the highest nonbreaking wave is a wave with $H \approx 0.003$.

## 4.4 The cnoidal wave solution

Consider an incoming cnoidal wave of the form (9). In this case the method of choice for the solution is a Fourier–Bessel series. The solution proceeds in a manner similar to the development in section 4.2.

The complete expression for a cnoidal wave is given by

$$\begin{aligned}\eta(x,t) &= y_t - 1 \\ &+ \frac{2\pi^2 H}{mK^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j+1}}{(1+q^{2i+1})(1+q^{2j+1})} \cos 2\pi(i+j+1)\left(\frac{x}{L} - \frac{t}{T}\right) \\ &+ \frac{2\pi^2 H}{MK^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{m+n+1}}{(1+q^{2n+1})(1+q^{2m+1})} \cos 2\pi(m-n)\left(\frac{x}{L} - \frac{t}{T}\right). \quad (16)\end{aligned}$$

Using the Keller and Keller (1964), and, after some algebra, we obtain the wave transmitted to the beach and eventually the wave amplitude at the initial shoreline :

$$\begin{aligned}\eta(0,t) &= y_t - 1 \quad (17) \\ &+ \frac{4\pi^2 H}{mK^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j+1}}{(1+q^{2i+1})(1+q^{2j+1})} \frac{\cos[\phi - k_{ij}(\cot\beta + c_0 t)]}{\sqrt{J_0^2(2k_{ij}\cot\beta) + J_1^2(2k_{ij}\cot\beta)}} \\ &+ \frac{4\pi^2 H}{mK^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{i+j+1}}{(1+q^{2i+1})(1+q^{2j+1})} \frac{\cos[\phi' - k'_{ij}(\cot\beta + c_0 t)]}{\sqrt{J_0^2(2k'_{ij}\cot\beta) + J_1^2(2k'_{ij}\cot\beta)}}.\end{aligned}$$

$k_{ij} = 2\pi(i+j+1)/L$, $k'_{ij} = 2\pi(i-j)/L$, $\phi = \arctan[\frac{J_1(2k_{ij}\cot\beta)}{J_0(2k_{ij}\cot\beta)}]$ and $\phi' = \arctan[\frac{J_1(2k'_{ij}\cot\beta)}{J_0(2k'_{ij}\cot\beta)}]$. Since $\cot\beta \geq 1$, the arguments of the Bessel functions are either zero or much larger than one, and the Bessel functions can be replaced by their asymptotic forms for large arguments. Simplifying (17) we obtain,

$$\eta(0,t) \approx y_t - 1$$
$$+ \frac{4\pi^2 H}{mK^2}\sqrt{\frac{2\pi}{L}\cot\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\sqrt{(i+j+1)}q^{i+j+1}}{(1+q^{2i+1})(1+q^{2j+1})}\cos[\frac{2\pi}{L}(i+j+1)(\cot\beta - c_0 t) - \frac{\pi}{4}]$$
$$+ \frac{4\pi^2 H}{MK^2}\sqrt{\frac{2\pi}{L}\cot\beta}\sum_{\substack{i=0\\ i\neq j}}^{\infty}\sum_{j=0}^{\infty}\frac{\sqrt{|i-j|}q^{i+j+1}}{(1+q^{2i+1})(1+q^{2j+1})}\cos[\frac{2\pi}{L}(i-j)(\cot\beta - c_0 t) - \frac{\pi}{4}]$$
$$+ \frac{2\pi^2 H}{mK^2}\sum_{i=0}^{\infty}\frac{q^{2i+1}}{(1+q^{2i+1})^2}. \tag{18}$$

As $m \to 0$, it is possible to simplify this expression. However, since it is necessary to determine the behavior $\forall m$, we proceed with a direct evaluation of the series in Eq (18).

Figure 3 shows the maximum relative runup of cnoidal waves $\mathbf{R}/H$ with the same $H$ as a function of the dimensionless wavelength $L$ up three different sloping beaches. The figure suggests that the runup of a cnoidal wave is substantially higher than the runup of the monochromatic wave with the same height and wavelength. It is also seen that there exists a minimum in the functional variation of the cnoidal wave runup with the wavelength. This minimum is more pronounced in the runup variation of the larger waves, in Figures 3(b) and 3(c).

All the waves in figure 3 are nonbreaking. The linear theory cannot provide directly a breaking criterion. To determine when waves start to break, the Jacobian of the Carrier and Greenspan transformation was monitored continuously as described in Synolakis (1986). The implication is that these results are valid for waves climbing up very steep beaches or for very long waves. Most natural beaches are gently sloping, and it is not obvious if similar differences exist. However, Figure 3 does suggest that the common practice of choosing the dominant frequency of an incoming wave-spectrum and using the runup of the monochromatic wave with that dominant frequency for calculating wave runup may not be entirely appropriate when the incoming wave energy is dominated by low frequency swell.

To investigate if this behavior can be demonstrated in the laboratory, a series of preliminary experiments were conducted in the $40m$ wave tank of the Keck Laboratory of Hydraulics and Water Resources of the California Institute of Technology. At a distance of $20m$ from the wave generator a sloping beach of 45° angle was installed. The wave generation system and the generation algorithm is described in detail by Skjelbreia (1987).

The results of the laboratory experiments for the climb of cnoidal waves of $H = 0.1$ up a 45° beach are shown in Figure 3. The relative runup is shown to decrease and then increase again as the wavelength increases. This is as the theory suggests.

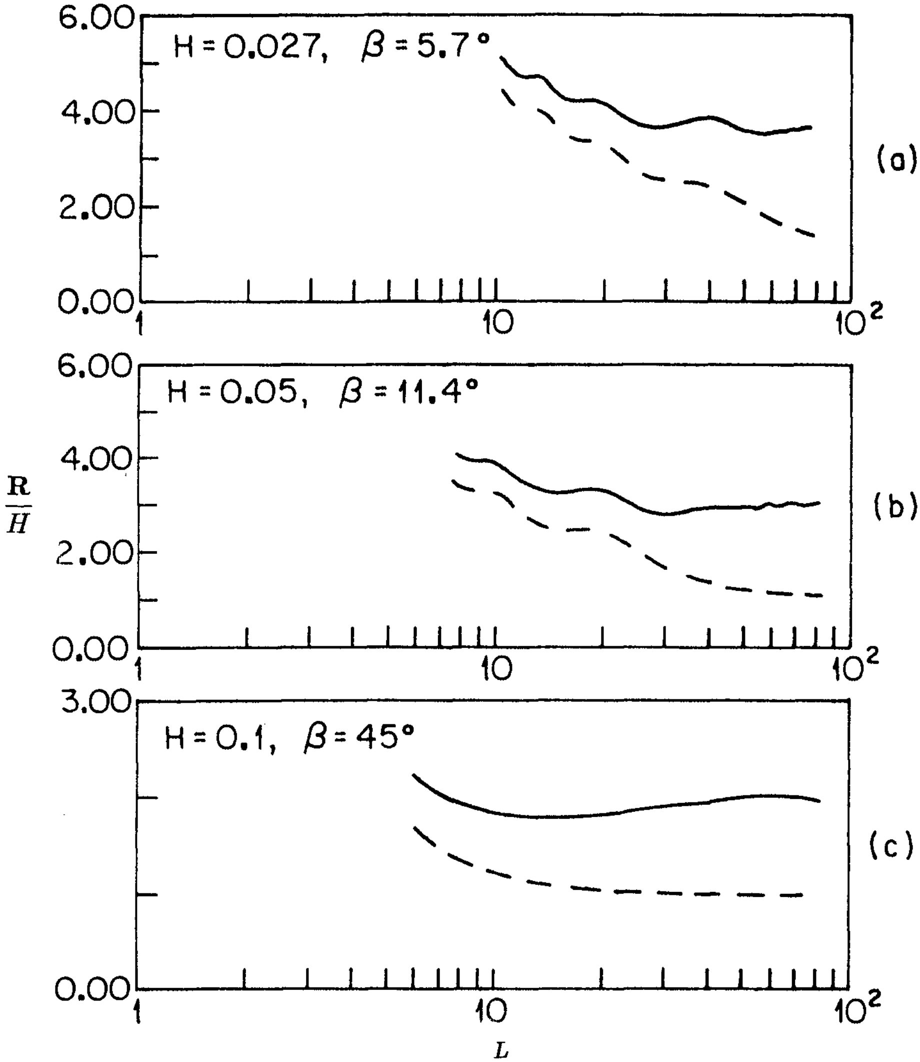


**Figure 3** The maximum relative runup of sinusoidal waves (dashed lines) and the maximum relative runup of cnoidal waves (solid lines) with the same height $H$ up three sloping beaches as a function of the dimensionless wavelength. (a) $H = 0.027, \beta = 5.7°$; (b) $H = 0.05, \beta = 11.3°$; (c) $H = 0.1, \beta = 45°$. Nonbreaking waves.

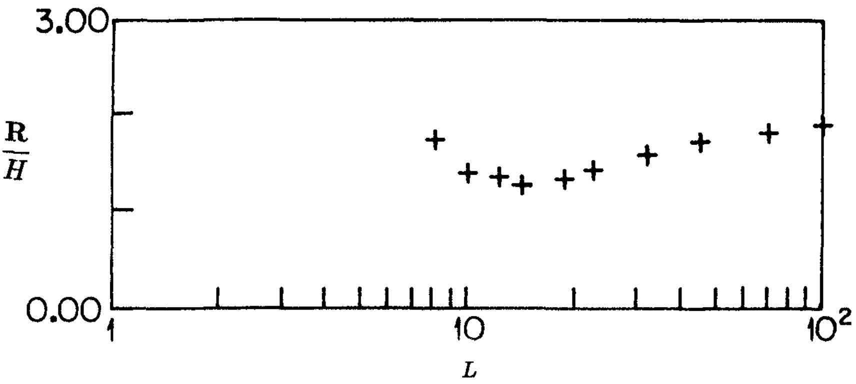


**Figure 4.** The maximum relative runup of nonbreaking cnoidal waves with $H = 0.1$ up a 45° beach as a function of the normalized wavelength. Laboratory data after Synolakis et al (1988).

This behavior is not entirely unexpected. The current paradigm in wave runup predicts that the relative runup decreases as the wave-steepness decreases, i.e., as the wavelength increases Ahrens and Titus (1985). On the other hand, there is empirical evidence (ibid) that as the nonlinearity of a given wave increases, the relative runup increases as the wavelength increases. (This is an intuitively pleasing behavior for cnoidal waves; as the nonlinearity increases and the wave has an increasing portion of its volume above the mean water level, the relative runup is expected to increase.) Figure 3(c) and Figure 4 suggest that both descriptions are correct, over different ranges of the wavelength.

We suggest two explanations why this behavior has been previously unrecognised. One, most existing runup data have been obtained from laboratory studies with monochromatic waves, where the effects of nonlinearity are difficult to quantify. Two, since most empirical runup relationships are based on synthesis of laboratory data from many different runup investigations, it is likely that any anomalies in the functional variations were attributed to differences in the quality of the data.

## 5. Conclusions

In summary, our analysis suggests the following conclusions.

1) The runup of nonbreaking waves predicted by nonlinear theory is mathematically identical to the runup predicted by linear theory for waves climbing up plane beaches.

2. The runup of solitary waves is described well by the runup law,

$$\mathbf{R} = 2.831\sqrt{\cot\beta}(H)^{\frac{5}{4}}.$$

3. Different criteria apply for wave breaking during runup and rundown.

4. The runup of cnoidal waves is significantly higher than the runup of the equivalent monochromatic waves.

5. The runup variation of cnoidal waves is not a monotonically increasing or decreasing function of the wave steepness.

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