CHAPTER 19

Development and Application of a Mathematical Model of Wave Action on Steep Slopes

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Abstract

This paper summaries the initial stages in the development of a mathematical model of wave action on slopes. The model calculates water surface elevation and depth averaged velocity on the slope, and uses this data to estimate the level of wave run-up. The model has been validated by comparing its results with those from similar models, and from a physical model. Examples are presented of these comparisons, which were found to be in good agreement in most cases.

1 Introduction

A mathematical model has been developed to estimate wave induced flows on sloping faces of breakwaters, coastal revetments and seawalls. Of particular interest is the application of the model to slopes armoured with high porosity single layer armour units. The model calculates instantaneous surface elevations and uprush/downrush velocities for waves normally incident to a slope. The nature of the armour layer is described in terms of an empirical roughness coefficient which must be calibrated against flume measurements and, if possible, also against prototype measurements. Part of a field measurement programme on a breakwater in Jersey has recently been completed, see Stephens (1988), and it is anticipated that its results will be used to calibrate the mathematical model in the near future.

The mathematical model once calibrated will be of value in preliminary assessment of hydraulic characteristics of an armoured slope. It is envisaged that one of its applications will be in examining various options prior to more extensive physical model testing. The use of this numerical model, together with physical modelling techniques, may lead to considerable improvements in efficiency and cost over the alternative of simply testing a number of potential designs in a physical model.

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2 Description of the mathematical model

The basic technique used in the mathematical model is to solve numerically the non-linear shallow water wave equations. Boundary conditions allow reflection and run-up at a steep slope to be incorporated. The set of equations which are solved contain a term to describe frictional dissipation on a rough slope. The model is based on a method originally developed for beach slopes which is reported in Hibbard & Peregrine (1970) and Packwood & Peregrine (1981). Clearly many structural slopes will be considerably steeper than typical beach slopes, and the extension of the original method to include steeper slopes has been described by Kobayashi et al (1987) and is also addressed in a companion paper by Thompson (1988).

The layout and co-ordinate system used in the mathematical model are shown in Figure 1. It should be noted that the x' co-ordinate is taken to be positive in the landward direction, with x' = 0 at the toe of the slope. The z' co-ordinate is taken to be positive upwards with z' = 0 at the toe of the slope. The water depth is denoted by h' and the depth averaged velocity by u'. The local angle of the slope is Θ'. The structure slope, s' is defined by $\frac{ds'}{dx'} = \Theta'$. The slope is assumed horizontal for x' < 0.

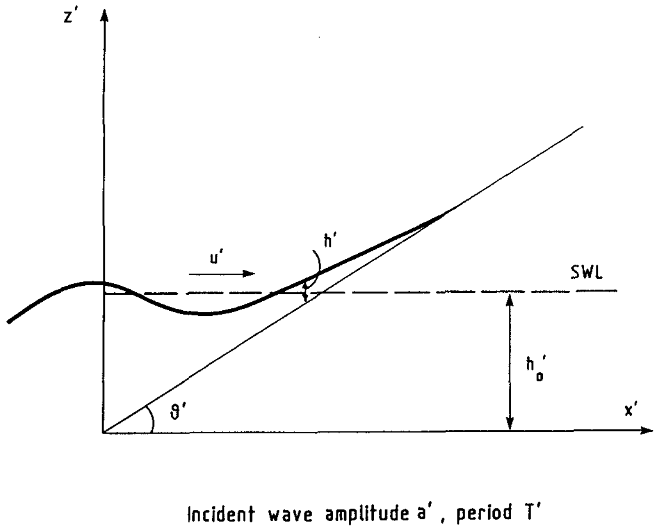

Figure 1 Mathematical model layout

The incident wave train is specified at the toe of the slope, where the water depth below SWL is given by h_o. It is assumed that the slope is impermeable, waves are non-breaking and that no overtopping occurs. Vertical pressure is taken to be hydrostatic and the vertical fluid acceleration is assumed to be negligible. This is a reasonable assumption for a relatively mild slope, i.e $\tan \Theta' << 1$.

The governing equations may then be expressed in dimensionless form as:-

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) = 0 \qquad (1)$$

$$\frac{\partial}{\partial t} (hu) + \frac{\partial}{\partial x} (hu^2 + \tfrac{1}{2}h^2) = -\Theta h - f|u|u \qquad (2)$$

where

$$h = h'/a', \qquad u = u'/(ga')^{\frac{1}{2}}$$

$$t = t'/T', \qquad x = x'/T'(ga')^{\frac{1}{2}},$$

$$\Theta = T'(g/a')^{\frac{1}{2}} \tan \Theta' \quad \text{and} \quad f = \tfrac{1}{2}T'(g/a')^{\frac{1}{2}}f'.$$

here f' is an empirically determined constant friction factor associated with the slope, a' is the incident wave amplitude and T' is the incident wave period. For all the mathematical model tests described in this paper the incident wave train is taken to be sinusoidal.

In addition to the governing equations, initial and boundary conditions also need to be specified. At time $t = 0$ the fluid is assumed undisturbed giving initial conditions

$$h = h_o - s, \quad u = 0 \quad \text{at } t = 0, \quad x \geqslant 0$$

The seaward boundary condition is derived from a characteristics based argument in which it is assumed that waves reflected from the structure do not modify the incident wave. The derivation of the seaward boundary condition used in the model is described in Beardsley et al (1988). At the landward boundary it is assumed that both the water depth on the slope and its velocity are zero at the leading edge of the wave.

The governing equations (1) and (2) are presently solved using a Lax Wendroff finite difference scheme. This has the advantage of being relatively easy to apply, and is numerically stable provided the Courant condition,

$$(gh_o)^{\frac{1}{2}} \frac{\Delta t}{\Delta x} \leqslant 1 \qquad ,$$

where Δt and Δx are the time and space steps, is satisfied. This scheme has been found to provide a reasonably efficient and robust method for solving the differential equations for waves of moderate steepness. For steeper waves there will be inaccuracies introduced by using this finite difference scheme, and the adoption of an alternative method will be necessary. The alternatives which have been examined so far are all Roe type schemes, see Priestley (1987), which are widely used in aerodynamic problems. The results from early tests with these schemes appear promising, and work is currently underway to investigate them further.

3 Calibration and validation of the mathematical model

Similar mathematical models to the one described here have been developed by both Kobayashi et al (1987) and Thompson (1988). Therefore a starting point in validating the model is to compare its results with those available from similar models. In addition, a series of physical model tests were carried out at Hydraulics Research to allow comparisons to be made between the mathematical model results and those from a physical model.

3.1 Comparisons with other mathematical models

On comparing the results from the present model with those of Thompson very good agreement was achieved both for the surface elevations and velocities calculated by the model. In general the differences between the two sets of model results was less than 5%. The outcome of these comparisons is discussed further in Beardsley et al (1988).

The results from the present model were also compared with one of the cases given in Kobayashi et al (1987). Both models use a similar form of the equations and boundary conditions, and the same finite difference scheme. The only significant different between them will be in the numerical implementation and the accuracy of the computer on which the models are run.

The present model was run for the case $\Theta = 6.0$, $h_o = 5.8$, $\sigma \equiv T'(\frac{g}{H'})^{\frac{1}{2}} = 15$ and $Ur = 4.5$. It should be noted that these are all dimensionless variables, and that therefore the results are also given in terms of dimensionless values h and u. In fact the angle Θ given here is equivalent to a slope of 1:2.5. The results from the two model tests are compared in Figures 2 and 3 in terms of the dimensionless elevations and velocity. The present model was run using a dimensionless time step $\Delta t = 0.0025$ and dimensionless space step $\Delta x = 0.01$; these satisfy the stability constraint. The results were plotted after 5, 5.25, 5.5 and 5.75 wave periods had elapsed.

It can be seen that the predicted elevations between the two models are in close agreement, but that in general Kobayashi's model gives velocities which are higher in magnitude. This discrepancy is probably due to the different time steps which were used in running the models. Kobayashi uses 2000 time steps per wave period whereas in the present model a maximum of 400 per wave period were used. Some early runs of the present model were actually made with 250 time steps per wave period and increasing to 400 was found to have no significant effect on the velocity profiles. It is therefore likely that the discrepancies can only be due to machine rounding errors, which will have a small but cumulative effect on the solution.

3.2 Comparison with physical models

Tests were conducted in the 'Deep Wave Flume' facility at Hydraulics Research. A smooth impermeable slope was constructed at a gradient of 1:2. The flume bed profile seaward of the test slope was at a gradient of 1:50. In all of the tests the static water level was

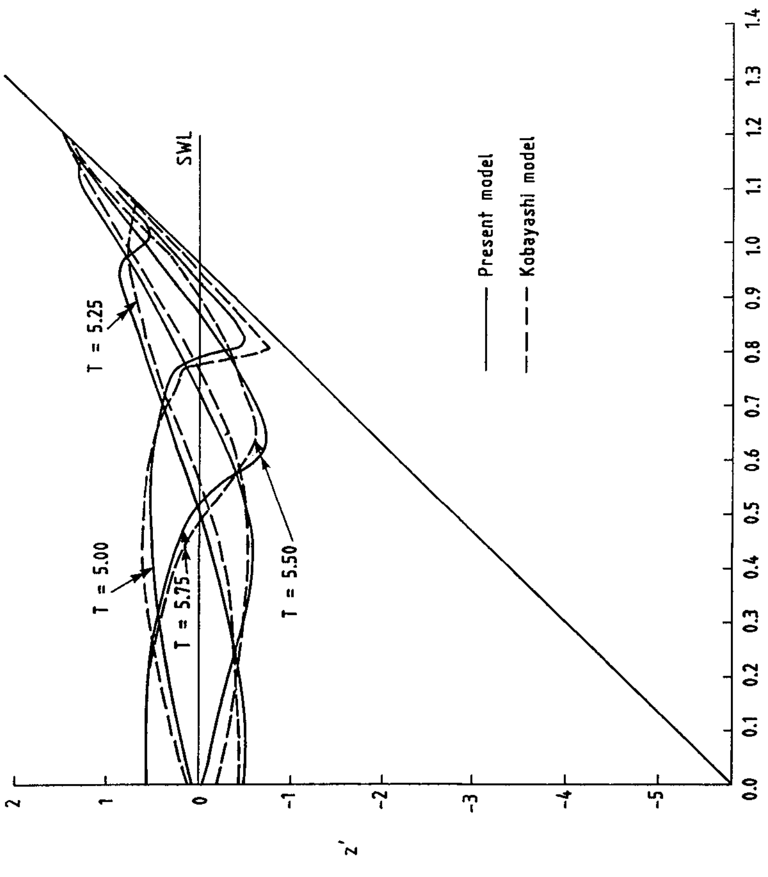

Figure 2 Comparison of model elevation with those of Kobayashi

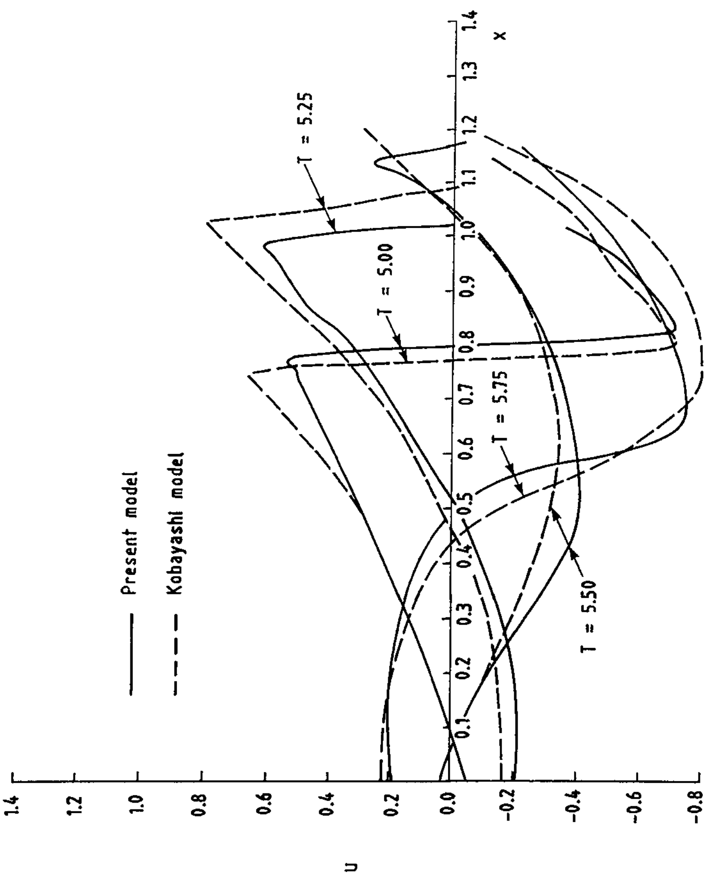

Figure 3 Comparison of model velocities with those of Kobayashi

set to be 0.3 metres above the toe of the 1:2 slope. A series of parallel vertical wave probes were positioned at regular intervals up the slope (see Fig 4) to measure instantaneous surface elevation. Measurements were also made of wave run-up and reflection, and the tests were recorded using a video camera.

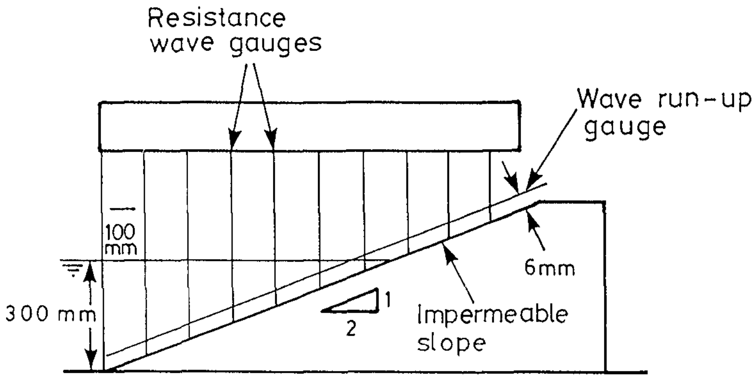

Figure 4 Layout for physical model tests

Two sets of tests were run; the first for the smooth, impermeable slope; and the second for the same slope covered with a single layer of Shed armour units. In each instance, short sequences of regular waves were run for a variety of combinations of wave height and period. The philosophy behind running these tests was that they should be used to provide calibration data for the mathematical model.

To run the mathematical model a friction factor f' needs to be specified. For this model f' is regarded as a calibration factor which needs to be determined experimentally. Once a value of f' has been determined for the particular slope characteristics, it would then be used for all subsequent tests with slopes of that type.

As a starting point in the calibration the mathematical model was set up to represent the experimental layout. The first case to be examined used a wave amplitude of 0.043m and a wave period of 2.0s at the seaward boundary. These incident conditions were taken from the results of a reflection analysis performed in the physical model at an equivalent position. The model was run using a time step of 0.0075s and a space increment of 0.018m; these satisfy the stability constraint.

For this first case a variety of friction factors were used in the mathematical model. After each run a visual comparison of the

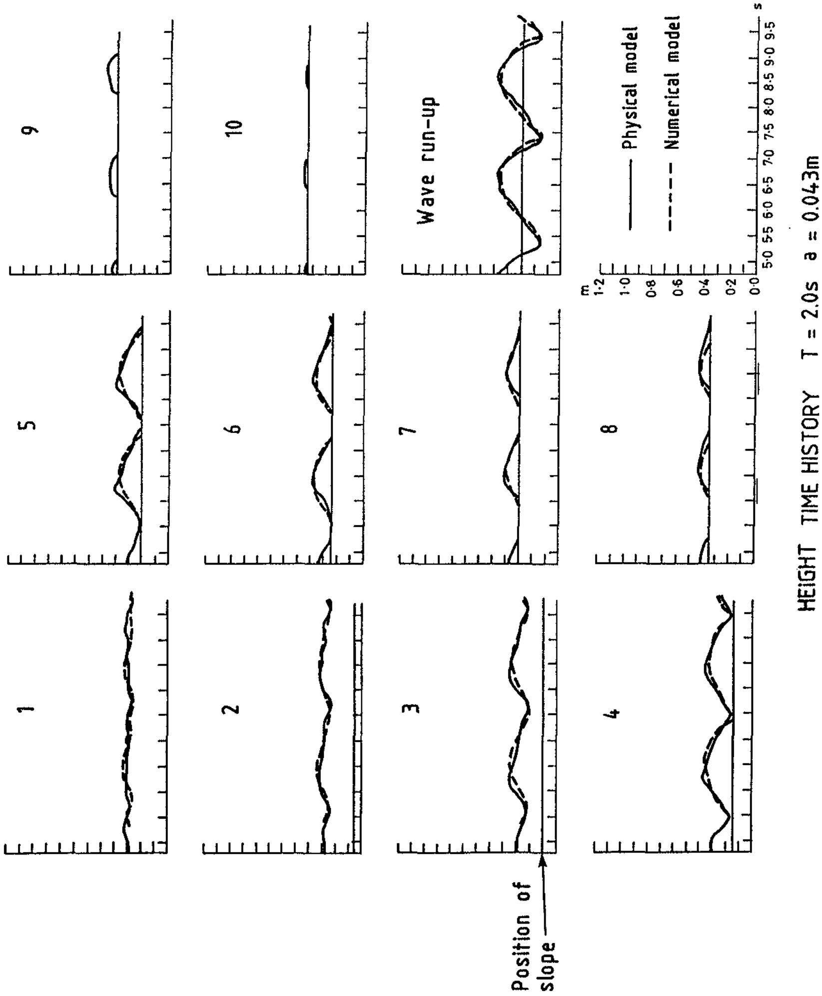

Figure 5 Comparison of elevations for physical and numerical models

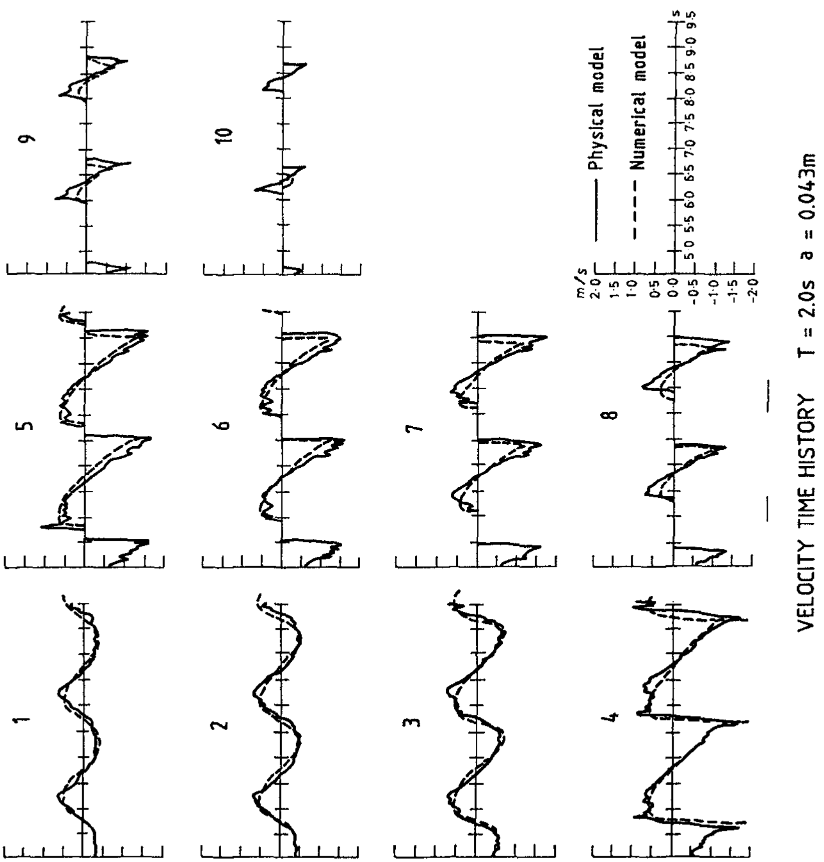

Figure 6 Comparison of velocities for physical and numerical models

elevation and velocity time histories at positions corresponding to the location of the wave probes in the physical model were made. Good agreement was achieved using a friction factor of 0.01. The comparisons for this case are shown in Figures 5 and 6, where results are plotted for each of the probe positiona shown in Figure 4. It can be seen from Figure 5 that the elevations predicted by the mathematical model are in good agreement with those from the physical model. The agreement between the velocities (Figure 6) is not as good as for the elevations, but is reasonable with the same specific trends being displayed.

The next stage in the calibration process is to make similar comparisons for other wave conditions which were tested in the physical model. However, rather than relying on a visual comparison to assess the best choice of friction factor, the maximum run-up levels and velocities at probe positions will be examined. Analysis of results obtained using this method is still in progress, but they appear to be promising, and give confidence that the mathematical model can be developed and used further to represent wave action on rough slopes.

4 Conclusions

The non-linear shallow water equations have been adapted to calculate wave action on steep slopes. A number of numerical techniques have been investigated, and initial comparisons made with physical model tests. The results from the mathematical model appear to give a reasonable representation of the hydrodynamic processes. Clearly, further research is needed to develop the model and extend and improve its range of applicability. However, the results reported here indicate that the model will eventually be a useful tool for the engineer involved in the design of coastal structures.

5 Acknowledgements

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6 References

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