CHAPTER 14

Are solitary waves the limiting waves in long wave runup ?

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This is a study of the maximum runup of single long waves on plane beaches. Laboratory data are presented that suggest that solitary waves attain the higher runup distances than other single long waves with identical generation characteristics, such as energy or momentum. These results suggest that solitary waves may provide a limiting condition for long wave runup on plane beaches.

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1. Introduction.

The problem of determining the runup and reflection of long single waves usually arises in the study of the coastal effects of tsunamis. Tsunamis are long water waves of small steepness generated by impulsive geophysical events on the ocean floor or at the coastline. Solitary waves and cnoidal waves are believed to model important aspects of the coastal effects of tsunamis well and in the past twenty years there has been a large number of analytical, numerical and laboratory investigations studying the runup of solitary waves. There has been comparatively less attention paid to the runup of cnoidal waves. However, consensus has emerged that one suitable physical model for this process is the formalism of a long wave propagating over constant depth and encountering a sloping beach.

A comprehensive review of these results may be found in Synolakis (1986, 1987, 1988), where some basic questions –such as the analytical substantiation of empirical relationships– have been resolved. However a basic question still persists and it refers to the actual shape of tsunamis in nature which are rarely Boussinesq–solitary wave–like or cnoidal wave like. Therefore most analytical results have only been used to calibrate numerical codes which solve the equations of motion with more "realistic" and "tsunami–like" profiles and actual topographies.

Unfortunately there is little knowledge as to what constitutes a "realistic" tsunami profile for specifying boundary or initial conditions in numerical solutions. No data exist on the surface elevation of a tsunami at generation and little data exist on the elevation of tsunamis far from the source, although this is likely to change with the field implementation of the THRUSH program (Bernard, 1988).

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The question then arises of whether it is useful to develop numerical schemes correct to many orders of approximation when realistic initial or boundary data do not exist. One could argue that the details of the wave at generation may not be very important in the eventual propagation process and the coastal effects; it is well established that long waves fisssion after sufficient propagation distances into series of solitary waves. However, then the process can be described relatively well analytically.

In this paper, I will report some preliminary data that address this question in particular. I will describe a series of laboratory experiments which suggest that although the details of the wave at generation are important in estimating the coastal effects at finite distances, it is possible to use solitary wave theory to obtain a limiting condition for long wave runup.

2. Basic phenomenology and dimensional analysis.

Consider the two–dimensional topography consisting of a constant depth region of depth d encountering a plane sloping beach of angle β. This is the model most frequently used in calibrating numerical codes and it is only appropriate to check its sensitivity to different initial conditions. The topography is shown in figure 1.

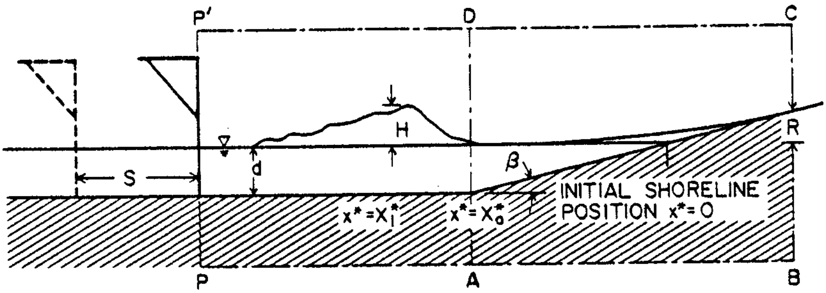

Figure 1. A definition sketch for long wave runup.

Waves will be generated with a vertical plate moving horizontally along the path $x = \xi(t)$, $0 < t < \mathbf{T}$, and where $\mathbf{T}$ is the generation time. The origin of the coordinate system is at the initial position of the wavemaker and x increases seaward. The problem can be stated as finding the maximum runup $\mathbf{R}$ that a wave motion generated with trajectory $\xi(t)$ at a distance L from the generator in a fluid of depth d, density ρ and viscosity μ will attain on a sloping beach of angle β. It is assumed that the flow boundary is hydrodynamically smooth. The following table lists the independent and dependent variables. The appropriate units are included in parentheses.

DIMENSIONAL ANALYSIS †

$\mathbf{R}$	is the maximum runup	(L),
d	is the depth of the constant depth region,	(L),
$\mathbf{T}$	is the characteristic generation time	(T),
$\mathbf{S}$	is the stroke of the generator	(L),
$\mathbf{L}$	is the propagation distance	(L),
β	is the angle of the sloping beach	(-),
g	is the acceleration of gravity	(L/T^2),
ρ	is the density of the fluid	(M/L^3),

and

μ	is the viscosity of the fluid	(M/LT).

In dimensional terms, one can write

$$\mathbf{R} = \mathcal{F}(d, \mathbf{S}, \mathbf{T}, \rho, \mu, L, g, \beta). \tag{1}$$

If the equations of motion were solvable for arbitrary plate trajectories, specification of the independent variables on the right hand side of equation (1) would be sufficient to define the runup exactly. Unfortunately, no analytical of numerical methods exist to calculate the evolution of a wave through breaking to its maximum runup and it is necessary to conduct laboratory experiments to determine the functional form of $\mathcal{F}$.

Buckingham's $\pi-$theorem suggests that six independent groups can be constucted. The following is one possible grouping of variables ‡,

$$\frac{\mathbf{R}}{d} = \mathcal{F}\left(\beta, \frac{(\mathbf{S}/\mathbf{T})d}{\mu/\rho}, \frac{\mathbf{S}}{\mathbf{T}\sqrt{gd}}, \mathbf{T}\sqrt{\frac{g}{d}}, \frac{L}{d}\right). \tag{2}$$

† In this list, I have replaced the trajectory time history $\xi(t)$ with the maximum stroke $\mathbf{S}$ and the generation time $\mathbf{T}$. During the experiments, $\xi(t)$ will be varied and $\mathbf{S}$ and $\mathbf{T}$ will be kept constant to determine the effect of the details of the trajectory.

‡ As is customary, the variable have been assembled in dimensionless groups that have the form of standard fluid mechanics parameters such as the Reynolds number $Re = (\mathbf{S}/\mathbf{T})d/(\mu/\rho)$ and the Froude number $Fr = \mathbf{S}/(\mathbf{T}\sqrt{gd})$.

For fixed propagation distances L, inviscid flow $\mu = 0$, and fixed slopes β, this list reduces to †

$$\frac{\mathbf{R}}{d} = \mathcal{F}\left(\frac{\mathbf{S}}{\mathbf{T}\sqrt{gd}}, \mathbf{T}\sqrt{\frac{g}{d}}\right). \tag{3}$$

One can now speculate on the possible form of the function $\mathcal{F}$. For a given depth, and given generation number $Fr = \mathbf{S}/(\mathbf{T}\sqrt{gd})$, as the generation time $\mathbf{T}\sqrt{g/d}$ increases the normalized runup $\mathbf{R}/d$ should increase, because then the stroke increases in proportion and higher waves are generated. However, this increase should not be observed for very large generation times because then the waves become unstable, break at generation and fission to smaller waves. It is likely that for each given value of the generation number, there exists some limiting generation time beyond which no further increases in runup would be possible. This would imply runup saturation and it would provide a limiting condition for long wave runup.

Whether this speculation is correct and whether different $\xi_1(t)\ldots\xi_n(t)$ with identical $\mathbf{S}/(\mathbf{T}\sqrt{gd})$ and $\mathbf{T}\sqrt{g/d}$ result in different $\mathbf{R}/d$ is unknown and will be explored in the next sections.

3. Experimental results.

Experiments were conducted in the 40m wave tank facility of the W.M. Keck laboratories of the California Institute of Technology. The facility is equipped with a programmable hydraulic wave generator and a wave measuring system. A ramp of slope 1 : 19.85 was installed at one end of the tank to model a uniformly sloping beach. The ramp surface was made of anodized aluminum panels and it was hydrodynamically smooth over the range of flows studied.

The experimental procedure is described in detail elsewhere (Synolakis, 1986, 1987). The experiments consisted of specifying a normalized plate trajectory shape $\xi(t)/d$ and of then varying $\mathbf{S}$ and $\mathbf{T}$ and of measuring $\mathbf{R}$ until a sufficiently large data set was generated to draw some preliminary conclusions.

3.1 Ramp trajectory. Type R waves.

The simplest possible plate motion is a ramp trajectory defined by

$$\begin{aligned} \xi(t) &= \left(\frac{\mathbf{S}}{\mathbf{T}}\right)t, \ 0 < t < \mathbf{T}, \\ \xi(t) &= 0, \ t > \mathbf{T}. \end{aligned} \tag{4}$$

† I have resisted the temptation to use an Irribaren like parameter in this analysis. This type of long waves change substantially as they propagate over constant depth and it is difficult to define the wave steepness consistently. If further analysis is desired, a tabulation of the data may be found in Synolakis (1986).

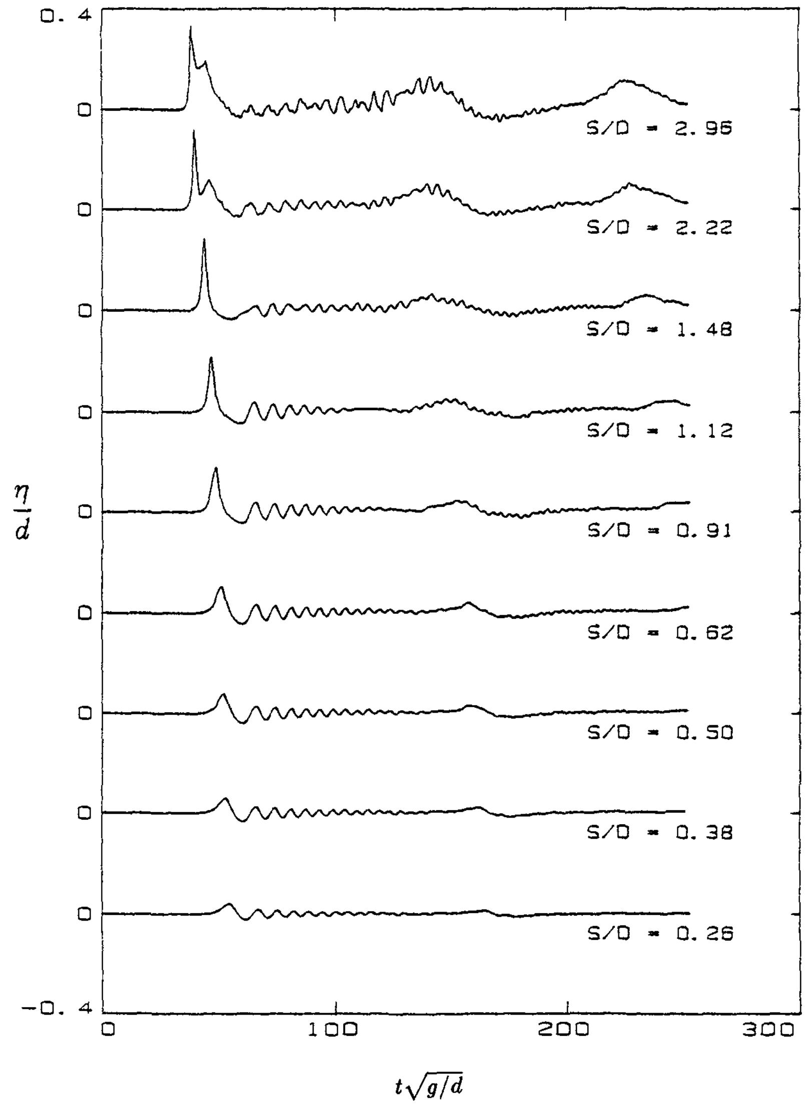

Figure 2 The wave hierarchy generated with a ramp trajectory with generation time $\mathbf{T}\sqrt{g/d} = 7.2$ and different strokes. The profiles were measured at approximately 20 depths away from the wave generator. The profiles are not syncronized in time and their relative positions do not necessarily reflect differences in propagation speeds.

In this case the entire wave generation process is described by three parameters only, the depth, the stroke and the generation time. For identification purposes in the subsequent discussion, these waves will be referred to as type R waves.

The experiments covered a range of depths from 12.60 to 20.24cm, a range of dimensionless times $\mathbf{T}\sqrt{g/d}$ from 3.481 to 15.747 and a range of generation numbers $\mathbf{S}/(\mathbf{T}\sqrt{gd})$ from 0.034 to 0.846. Small generation numbers $(\mathbf{S}/(\mathbf{T}\sqrt{gd}) < 0.11)$ resulted in waves that never broke and in small runup heights $(\mathbf{R}/d < 0.22)$ larger generation numbers $(\mathbf{S}/(\mathbf{T}\sqrt{gd}) < 0.37)$ resulted in waves that broke as they climbed up the beach and in larger runup heights $(\mathbf{R}/d < 0.80)$. Larger numbers $(Fr < 0.65)$ produced waves that broke before reaching the beach, reformed and then broke again as they climbed up the beach. Relatively large generation numbers $(Fr < 0.84)$ produced waves that were generated broken and propagated broken without ever reforming. These waves were essentially bores of finite volume and they had the highest runup distances $\mathbf{R}/d \approx 1$. Further increases in the generation number did not result in larger runup; there was runup saturation.

Figure 2 presents the surface elevation time history of one series of such waves generated with similar generation times but with generation numbers differing by an order of magnitude. All waves shown assume the shape of a leading solitary wave followed by an oscillatory tail. The second hump seen at about 100 dimensionless time units behind the leading wave is the reflected wave generated by the beach.

The relationship of the runup of the waves generated with a ramp motion (4) with the generation number is presented in figure 3. All the wave shown are waves that broke only as they climbed up the slope. As expected, the relative runup increases with the generation number. Also as anticipated, for fixed generation numbers, the relative runup increases with the generation time.

3.2 Solitary wave–like trajectory. Type S waves.

In this set of experiments waves were generated using different solitary wave–like trajectories. A solitary wave trajectory is a plate motion that produces a solitary wave of given H/d at a given depth d. This trajectory is the solution of the equation

$$\xi(t) = \frac{H}{kd}\tanh k(ct - \xi(t)), \tag{5}$$

where $k = \sqrt{(3/4)H/d^3}$ and $c = \sqrt{g(H+d)}$ (Goring, 1978).

The correct stroke and generation times for solitary waves are $\mathbf{S} = \sqrt{(16/3)Hd}$ and $\mathbf{T} = (2/kc)(3.80 + H/d)$. † If different values are specified, then the generator

† It has been verified ibid and in Synolakis (1986, 1987) that these values produce true Boussinesq solitary waves $\forall\ 0.08 < H/d < 0.68$. However, depending on the response of the generation system fine tuning may be necessary to obtain perfectly clean waves. Specifically, $\mathbf{T}$ may have to be adjusted by 5%.

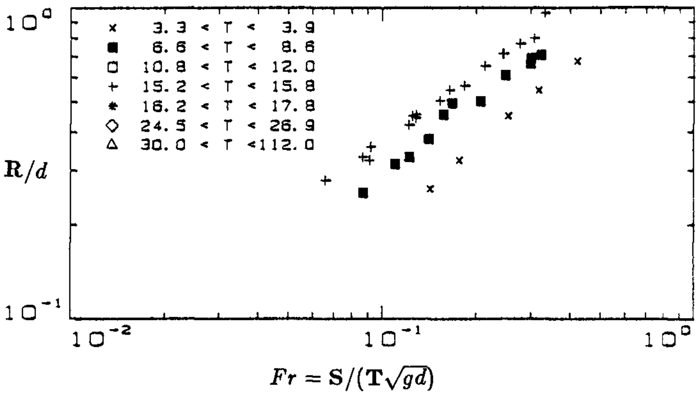

Figure 3 The variation of the normalized runup with the generation number. Type R waves. All waves shown break only as they climb up the beach.

produces a single long wave followed by an oscillatory tail. A *solitary wave-like* trajectory is a solution of (5) but with different **S** and **T** than for solitary waves.

Figure 4 shows one wave hierarchy that is produced when the generation time is kept constant at $\mathbf{T} = 15.0$ and the generation number is varied. These waves were generated at depths ranging from $d = 11.74$cm to $d = 25.53$cm. Waves generated with $\mathbf{S}/d < 0.7$ are nonbreaking, while waves produced with $\mathbf{S}/d > 2.0$ are breaking/reforming waves or bores.

Many other similar hierarchies were generated. The experiments covered a range of dimensionless times $\mathbf{T}\sqrt{g/d}$ from 3.388 to 55.966 and a range of generation numbers $\mathbf{S}/(\mathbf{T}\sqrt{gd})$ from 0.01 to 1.10. Small generation numbers ($\mathbf{S}/(\mathbf{T}\sqrt{gd}) <$ 0.10) resulted in waves that never broke and in small runup heights ($\mathbf{R}/d < 0.22$) while larger generation numbers ($\mathbf{S}/(\mathbf{T}\sqrt{gd}) < 0.40$) resulted in waves that broke as they climbed up the beach and in larger runup heights ($\mathbf{R}/d < 0.75$). Larger numbers ($Fr < 0.70$) produced waves that broke before reaching the beach, reformed and then broke again as they climbed up the beach. Relatively large generation numbers ($Fr < 0.80$) produced waves that were generated broken and propagated broken without ever reforming. These waves attained the highest runup distances $\mathbf{R}/d \approx 1$. It is really surprizing how well these values correspond to the values obtained with the ramp trajectory in section 3.1. Runup saturation was again observed; inreasing the generation number beyond $Fr = 0.80$ did not result into higher runup heights.

Figure 5 presents the variation of the runup of the waves generated with the

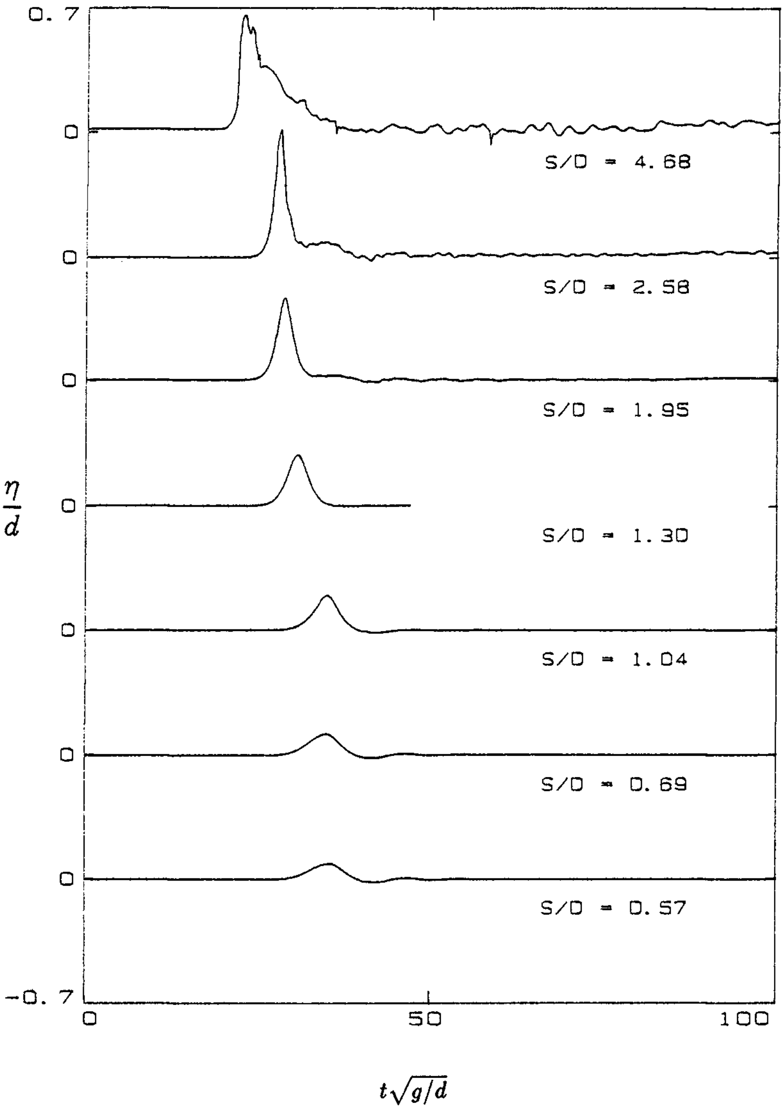

Figure 4 The wave hierarchy generated with a ramp trajectory with generation time $\mathbf{T}\sqrt{g/d} = 7.2$ and different strokes. The profiles were measured at approximately 20 depths away from the wave generator. The profiles are not syncronized in time and their relative positions do not necessarily reflect differences in propagation speeds.

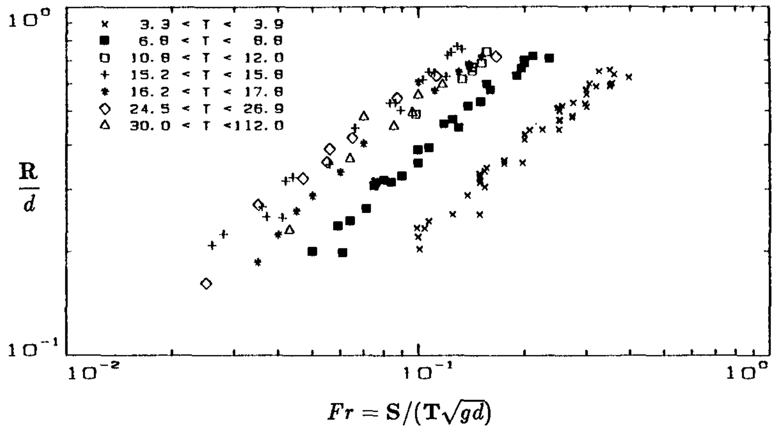

Figure 5 The variation of the normalized runup with the generation number. Type S waves. All waves shown break only as they climb up the beach.

solitary wave–like motion, i.e., equation (5) with the generation number.

3.3 Comparison with solitary waves.

It is interesting to compare the data described in section (3.1) and (3.2) with data for Boussinesq solitary waves. A comprehensive study of solitary wave runup has been presented in Synolakis (1987,1987). Figure 6 presents the variation of the normalized runup with the generation number for type R waves, type S waves and for solitary waves. It appears that for a given generation number and regardless of the generation time and the trajectory used, the solitary waves always attain the highest runup distances. This observation hints that solitary waves may indeed be a limiting condition, at least in single long wave runup.

It is important to note that bores of finite volume produce normalized runup distances than are even higher than those produced by solitary waves. This is anticipated and it is not contradictory to the previous discussion; it is not possible to create a solitary wave that is simoultanously a bore. The highest generation number that produces a solitary wave is $Fr = 0.189$. On the other hand it is possible to create bores with generation numbers as high as $Fr = 0.85$. Even then, there is saturation and further increases in the generation number cannot produce higher runup.

These three observations, the fact that details of the plate motion are important in determining the runup, the fact that solitary waves provide a limiting condition for the runup of long waves that break on the beach and the fact that the bore runup is saturated, suggest that there might exist some dimensionless number such that solitary waves assume the highest value of this number. This question will be explored next.

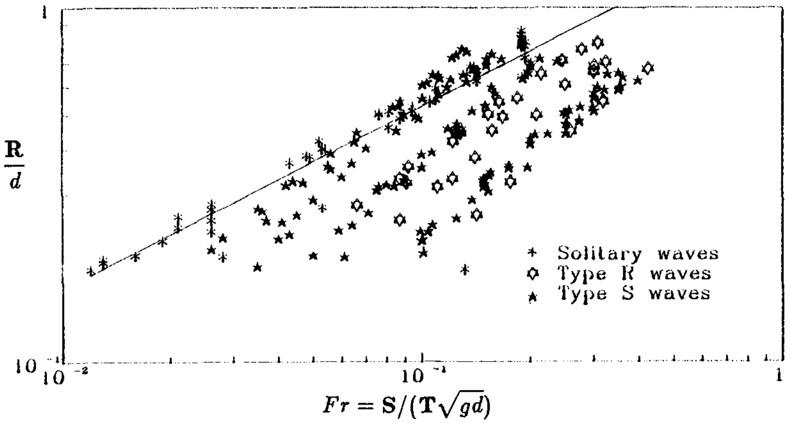

Figure 6 The variation of the normalized runup with the the generation number. Breaking type R, S waves and solitary waves. The solid line is the line of best fit for the solitary wave data and it is given by $\mathbf{R}/d = 1.739Fr^{0.518}$.

4. The runup number.

In attempting to determine a single parameter to describe the process, it is instructive to consider the kinematic analogy of a material particle moving up a frictionless inlined plane. Suppose that a particle of mass m starts climbing up a ramp of slope $\tan\beta$ with horizontal momentum mV. As the particle climbs up, the component of the body force in its direction of motion acts to reduce its momentum. Newton's law implies that $dV/dt = -g\sin\beta$. The maximum elevation the particle will reach is $\mathbf{R} = V^2/2g$. The same result can be arrived at by using an energy argument. It is therefore likely that two important motion invariants are the particle's energy and momentum.

The kinematic analogy cannot be carried further; as a wave climbs up a beach reflection continuously reduces its momentum, while wave breaking dissipates its energy. Friction dissipates both momentum and energy. However, it is well known that these processes are slowly acting in extremely long waves such as tsunamis. I will therefore proceed to derive momentum and energy scales that can be used in the derivation of motion invariants.

Consider a single long wave with horizontal momentum $\mathcal{M}_x$ per unit mass and per unit width incident upon the sloping beach. The wave attains a maximum runup height $\mathbf{R}$. The basic variables are $\mathbf{R}, g, \mathcal{M}_x$ and they can produce only one group, $\mathbf{R}g^{\frac{1}{5}}/\mathcal{M}_x^{\frac{2}{5}}$. If one uses the kinetic energy per unit mass and per unit depth as an independent variable, dimensional analysis suggests the group $\mathbf{R}g^{\frac{1}{3}}/\mathcal{E}^{\frac{1}{3}}$. †

The horizontal momentum per unit width and per unit mass is given by $\mathcal{M}_x = \int_{\mathcal{V}} u_x d\mathcal{V}$ where $\mathcal{V}$ is the volume occupied by the wave and u_x is the horizontal velocity. To calculate this integral detailed amplitude and velocity data are needed. However since the wavemaker displaces a volume $\mathbf{S}d$ per unit width and since $\mathbf{S}/\mathbf{T}$ is a measure of the velocity imparted to the motion, an estimate of the momentum of the wave motion is $\mathbf{S}^2 d/\mathbf{T}$. This leads to a dimensionless momentum invariant $\mathcal{R} = \mathbf{R}g^{\frac{1}{5}}/(\mathbf{S}^2 d/\mathbf{T})^{\frac{2}{5}}$. This parameter will be referred to as the runup number.

To determine whether this parameter is motion invariant, it is necessary to examine its variation with other independent parameters. Repeating the dimensional analysis of section 2 one obtains

$$\mathcal{R} = \frac{\mathbf{R}g^{\frac{1}{5}}}{(\mathbf{S}^2 d/\mathbf{T})^{\frac{2}{5}}} = \mathcal{F}\left(\beta,\ \frac{(\mathbf{S}/\mathbf{T})d}{\mu/\rho},\ \frac{\mathbf{S}}{\mathbf{T}\sqrt{gd}},\frac{\mathbf{S}}{d}\right). \tag{6}$$

Figure 7 presents the variation of the runup number with three of the parameters in equation (6) for type R waves. With the exception of bores of finite volume, the runup number does seem to vary very little. Figure 8 presents the variation of the runup number for type S waves. The runup number variation is greater than it is for type R waves; still it only ranges from $\mathcal{R} = 0.8$ to $\mathcal{R} = 1.2$.

Figure 9 presents the runup number variation with the generation number for all wave types previously described. ‡ The figure suggests that the runup number might not be a true motion invariant, in the sence that it varies from $\mathcal{R} = 0.7$ to $\mathcal{R} = 1.3$. (However, the scatter seen is within the margin of experimental error.) More important, the figure also suggests that solitary waves have the highest runup number $\mathcal{R} = 1.30$, implying that indeed solitary waves may provide a limiting condition for wave runup.

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† From this point on, I will discuss the influence of the momentum of generation only. The analysis presented in Synolakis (1986) suggests that the motion invariant derived from the momentum describes the process further better than the motion invariant derived from energy.

‡ Figure 9 also includes data for type P waves, which are waves generated with a parabolic trajectory. In the interest of brevity these waves are not in discussed in detail here. Refer to Synolakis (1986) for tables of data and for a describtion of the experiments.

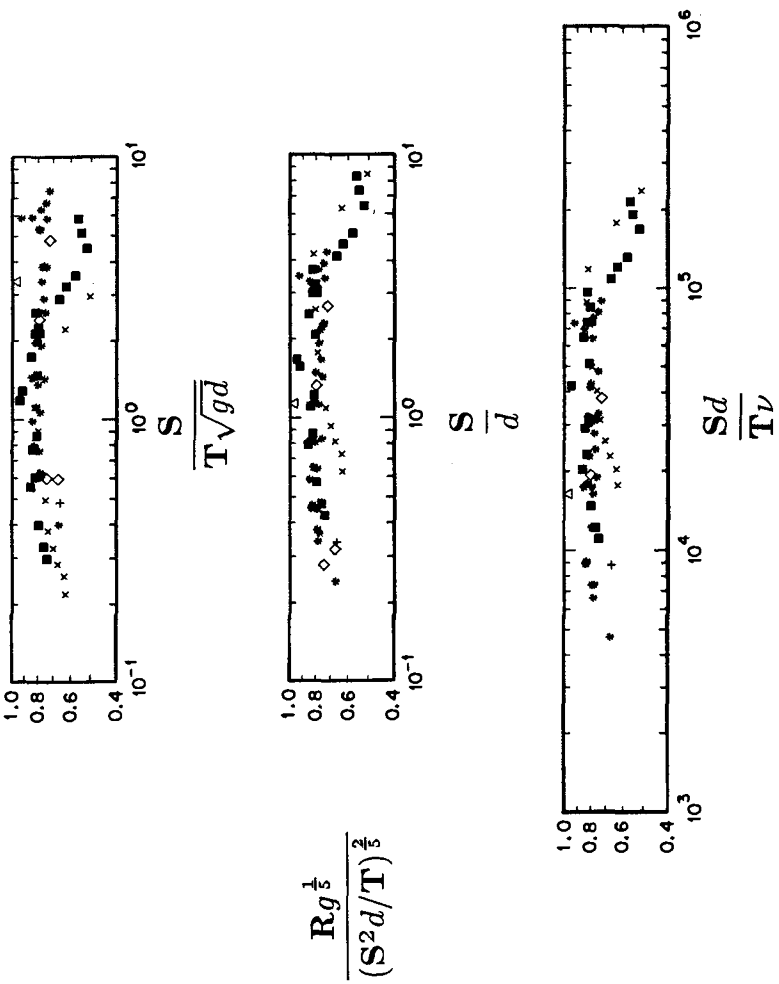

Figure 7 The variation of the runup number with the generation number (a), with the normalized stroke (b) and with the Reynolds number (c). Type R nonbreaking, breaking, breaking/reforming and bores of finite volume.

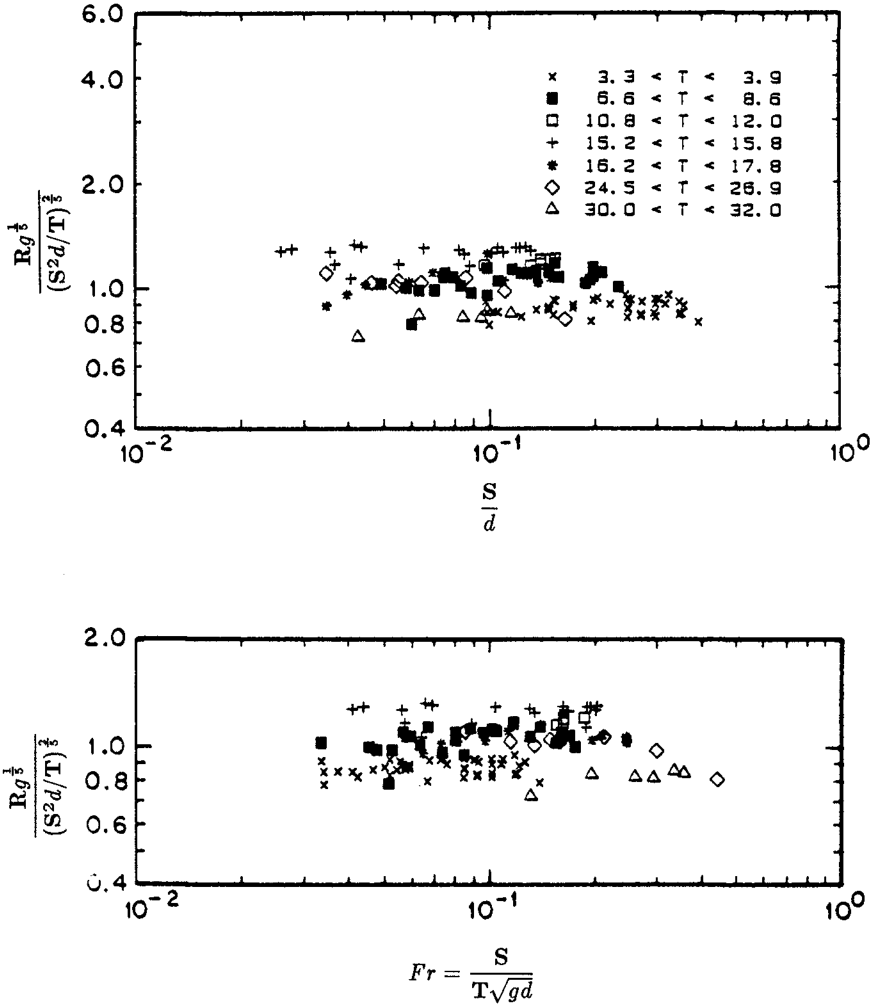

Figure 8 The variation of the runup number with the stroke (a) and with the generation number (b). Type S breaking waves.

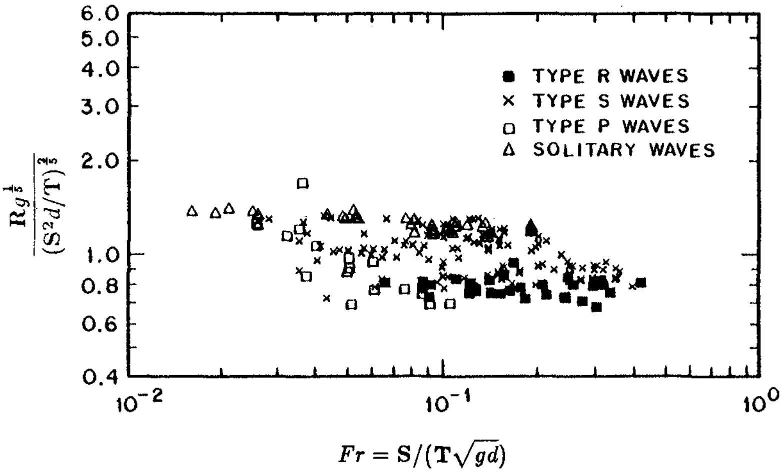

Figure 9 The variation of the runup number with the generation number. Type R, S, P waves and solitary waves.

5. Discussion.

The data presented clearly demonstrate that solitary waves are a limiting condition for wave runup on a sloping beach. For a given generation number the solitary waves attain the highest runup distances regardless of the generation times and trajectory type. Also solitary waves have the highest runup number of all long waves described in this study. One can only speculate why this should be the case.

Arbitrary plate motions produce either single long waves that may be nonbreaking, breaking, or bores followed by an oscillatory train or solitary waves. It is reasonable to expect that the runup of a wavetrain will be primarily affected by the runup of the leading wave that emerges from the wavetrain. † The momentum of this wave is less than the momentum that was imparted to the fluid during wave generation. Had all the momentum at generation been used to produce a solitary wave, then this wave would have a larger waveheight and it would attain higher runup distances than the leading wave emerging from a long wavetrain with the same momentum.

This is a very exciting result. It implies that it might be possible to determine

† This analysis is consistent with nonlinear dispersive theory that predicts that any long wavetrain of positive volume will fission into a series of solitons given a sufficiently large propagation distance.

the highest possible runup distance that a wave motion of a given momentum distribution may attain by simply calculating the runup of the solitary wave generated with the same total momentum. Whether this result is true for natural beaches ‡ and whether it is true for three-dimensional topographies remains to be established. However it does suggest that it might be possible to obtain reliable predictions of the coastal effects of tsunamis using solitary wave theory.

6. Conclusions.

There are three major conclusions based on the data on the 1 : 19.85 beach.

1) On a given beach, there are maximum runup values that a wave may attain depending on its breaking character, i.e., whether a wave is nonbreaking, breaking, breaking/reforming or a bore of finite volume.

2) The generation characteristics of a long wave determine its runup at finite propagation distances. In particular if the momentum scale is estimated using the generation time and stroke, the resulting dimensionless group, the runup number, might be a motion invariant. The runup number for the 1 : 19.85 beach is given by

$$\mathcal{R} = \frac{\mathbf{R} g^{\frac{1}{5}}}{(\mathbf{S}^2 d/\mathbf{T})^{\frac{2}{5}}} = 1.023 \pm 0.3.$$

The exact value of the generation number depends weakly on the generation time.

3) Breaking solitary waves have the highest possible runup number among all other single long waves, thereby providing a limiting condition for long wave runup.

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‡ Even though this discussion implies runup saturation, there is little evidence ((Huntley et al, 1977, Guza and Thornton, 1982) that there is runup saturation in periodic waves up natural beaches.