

## COMPLEX ENVELOPE IDENTIFICATION OF WAVE GROUPS

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The complex envelope function is presented as the natural analysis technique for wave records where the identification of wave groups is a dominant interest. Algorithms have been developed and confirmed for separation of the complex envelope function, estimation of the dominant frequency and unwrapping of the phase function. Cross-correlograms and coherence spectra reveal a link between the envelope amplitude and phase traces that appears to be an intrinsic property of wave groups. Nevertheless, the majority of the information in typical wave records can be described as random, accounting for the relative success of the Gaussian random wave model.

## INTRODUCTION

There is widespread acceptance of wave grouping in coastal and ocean design and considerable recent research effort has been focussed on the statistics of wave groups and the development of alternative analysis techniques to accommodate wave grouping. There is sufficient evidence from field records that wave grouping does exist but the specific nature and extent of wave grouping remains largely unresolved. It has been suggested, for example, that wave grouping is a direct consequence of the Gaussian random wave model, that it is a consequence of finely-tuned focussing of the phases of linear or non-linear wave trains, and that it is essentially a non-linear phenomenon, such as envelope solitons.

The evidence must be sought initially from field data. Care must be taken, however, that the choice of analysis technique does not predetermine the data interpretation. Spectral analysis and zero-up-crossing identification of individual waves, for example, ignore the phase information in the record and implicitly favor the Gaussian random wave model description. This approach has proved to be useful and reasonably successful. It may ultimately prove appropriate but questions remain regarding the existence of useful phase information in the record.

A natural analysis technique for wave grouping would appear to be the complex envelope function  $A(t)$ , related to the wave record  $\eta(t)$  as

$$\eta(t) = \text{Real} [A(t) \exp(i\omega_0 t)] \quad (1)$$

where  $\omega_0 = 2\pi f_0$  is the dominant frequency. No information is lost from the original record and attention can be focussed on the envelope modulation. For typically narrow-banded sea states,  $\text{mod } A(t)$  is an excellent approximation to the wave envelope, as recognized by Rice (1943, 1945) and Longuet-Higgins (1951) in the theoretical establishment of the Rayleigh distribution as an estimate of the probability distribution of

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wave amplitude. Full advantage, however, has rarely been taken of the complete complex envelope function. It is the purpose of the present paper to evaluate its potential in the identification and perhaps quantification of wave groups. Several recent studies of wave grouping have adopted an envelope concept, all based on the application of a low pass moving average filter to  $\eta^2(t)$ , the square of the wave record. Funke and Mansard (1979) used a filter with half-width  $1/(2f_p\Delta t)$  points in defining a smoothed instantaneous wave energy history.  $f_p$  is the spectral peak frequency and  $\Delta t$  is the discrete time step of the data record. W.C. Thompson, Nelson and Sedivy (1983) and E. F. Thompson and Seelig (1983) extended the filter half-width to  $1/(f_p\Delta t)$  points. These studies have shown the utility of the envelope concept but have not taken advantage of the phase information in the record. In addition, filtering removes potentially valuable information from the record.

### COMPLEX DEMODULATION

The potential of complex demodulation in the analysis and interpretation of periodic data has been elucidated by Bingham, Godfrey and Tukey (1966) and Hasan (1982). In principle, complex demodulation is a generalization of harmonic analysis, its objective being to extract the slowly varying amplitude  $R(t;\omega)$  and slowly varying phase  $\Phi(t;\omega)$  of the signal component at frequency  $\omega$ . In general, the record will not consist solely of a perturbed sinusoid; it may be represented as

$$\begin{aligned}\eta(t) &= R \cos(\omega t + \Phi) + r(t) \\ &= \frac{1}{2} R \{ \exp[i(\omega t + \Phi)] + \exp[-i(\omega t + \Phi)] \} + r(t)\end{aligned}\quad (2)$$

where  $r(t;\omega)$  is the residual noise signal. Multiplying Equation 2 by  $\exp(-i\omega t)$  gives

$$\eta \exp(-i\omega t) = \frac{1}{2} R \exp(i\Phi) + \frac{1}{2} R \exp[-i(2\omega t + \Phi)] + r \exp(-i\omega t)\quad (3)$$

The second and third items in Equation 3 oscillate at frequencies  $2\omega$  and  $\omega$  respectively, so that the smooth component (and hence  $R$  and  $\Phi$ ) can be extracted by linear filtering (Bloomfield 1976). This procedure is appropriate in real time, regardless of the nature of the  $\eta(t)$  data series. It will extract the slowly varying amplitude and phase at any nominated frequency  $\omega$ , and the complete spectrum can be established in this manner.

The value of complex demodulation however is its ability to focus on a single frequency and herein lies its utility in wave record analysis. The spectra of surface gravity waves are consistently narrow-banded, to the extent that some dominant frequency  $\omega_0$  can be anticipated. The precise definition of  $\omega_0$  requires some further consideration but it is clear that it will be near the spectral peak. The complex demodulate at frequency  $\omega_0$  is the complex envelope function  $A(t)$ . The residual  $r(t;\omega_0)$  in Equation 2 is consumed into the definition of  $A(t)$ , so that  $A(t)$  represents the influence of all frequencies other than  $\omega_0$ . This is Equation 1. A narrow-banded data series is clearly essential to the utility of the complex envelope function, as a wide band width would destroy the essential envelope interpretation of  $A(t)$ .

The complex envelope function appears to have been introduced by Rice (1944). His interest however was only in the amplitude function  $R(t)$ , which he termed the envelope. This Rice envelope function formed the basis of his establishment of the Rayleigh distribution as the probability distribution for the amplitudes of narrow-banded random noise. This result was adapted to surface gravity waves by Longuet-Higgins (1951) and has proved remarkably successful. Potential information in the phase function  $\Phi(t)$  was not considered.

### COMPLEX ENVELOPE FUNCTION

The filtering technique implied above is certainly appropriate but advantage can be taken of the computational efficiency of the FFT algorithm (Sobey & Colman 1983). This requires a specific assumption about the data series, namely that it represents a stationary process. This same assumption is also implicit in traditional spectral analysis of wave records leading to the variance spectrum. That it is a reasonable assumption is well established.

The initial step (Deutsch 1962) is to establish the Hilbert transform  $\hat{\eta}(t)$  of the data series  $\eta(t)$ :

$$\hat{\eta}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta(\tau)}{t-\tau} d\tau \quad (4)$$

The Fourier transform of  $\eta(t)$  is

$$F(\omega) = \int_{-\infty}^{\infty} \eta(t) \exp(-i\omega t) dt \quad (5)$$

and it follows from Equation 4 that  $\hat{F}(\omega)$ , the Fourier transform of  $\hat{\eta}(t)$ , is

$$\hat{F}(\omega) = \begin{cases} -iF(\omega) & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ +iF(\omega) & \text{for } \omega < 0 \end{cases} \quad (6)$$

Computationally, the Hilbert transform can be established from the inverse Fourier transform of  $\hat{F}(\omega)$ .

The record and the Hilbert transform are combined to establish the pre-envelope function:

$$z(t) = \eta(t) + i\hat{\eta}(t) \quad (7)$$

which in turn can be shown (Deutsch 1962) to be

$$z(t) = R(t; \omega) \exp[i\Phi(t; \omega)] \exp(i\omega t) \quad (8)$$

Multiplying through by  $\exp(i\omega t)$  recovers the complex demodulate at frequency  $\omega$ .

To specifically recover the complex envelope function requires knowledge of the dominant frequency  $\omega_0$ . Physically,  $\omega_0$  must be reasonably close to the peak frequency  $\omega_g$ , which can be estimated from the variance spectrum (Sobey & Young 1986) as

$$\omega_p = \int \omega E^{\mathfrak{R}}(\omega) d\omega / \int E^{\mathfrak{R}}(\omega) d\omega \quad (9)$$

Adopting  $\omega_p$  as a first guess  $\omega_g$  at the dominant frequency, the correction of  $\omega_g$  towards  $\omega_0$  follows Bolt and Brillinger (1978). At the dominant frequency, Equation 8 is

$$z(t) = R(t; \omega_0) \exp[i\Phi(t; \omega_0)] \exp(i\omega_0 t) \quad (10)$$

where the complex envelope function is

$$A(t) = R(t; \omega_0) \exp[i\Phi(t; \omega_0)] \tag{11}$$

Multiplying both sides of Equation 10 by  $\exp(-i\omega_g t)$  gives

$$z(t) \exp(-i\omega_g t) = R(t; \omega_0) \exp\{i[\Phi(t; \omega_0) + (\omega_0 - \omega_g)t]\} \tag{12}$$

The estimate  $\omega_g$  may be corrected to  $\omega_0$  by estimating and removing any linear trend, identified as  $(\omega_0 - \omega_g)t$ , in the phase record. In practice, this procedure requires phase unwrapping as an initial step.

**UNWRAPPING THE PHASE FUNCTION**

Phase unwrapping refers to the modulo  $2\pi$  operation on phase angles. The phase returned by the FFT and any coded trigonometrical algorithm is in the range  $-\pi$  to  $\pi$  and is termed the principal phase  $\Phi_p(t)$ . Any principal phase angle may in fact be  $\Phi_p(t) + 2\ell\pi$ , where  $\ell$  is any signed integer, without changing the complex envelope function. The "true" phase is obtained by "unwrapping" the principal phase through addition or subtraction of multiples of  $2\pi$ ; this phase is called the unwrapped phase,  $\Phi_u(t)$ . The subscript  $u$  has been dropped but is implied in the subsequent discussion.

The procedure adopted to determine the signed integer  $\ell$  is closely analogous to that adopted by Read and Sobey (1985) in unwrapping the phase spectrum of  $\eta(t)$ . From the time derivative of the natural log of Equation 11, it follows that

$$\frac{d\Phi}{dt} = \text{Imag} \left[ \frac{1}{A} \frac{dA}{dt} \right] \tag{13}$$

Given the slope of the phase function, numerical integration will provide an estimate of  $\Phi_u$  at the new time. Integration proceeds by the trapezoidal rule, consistent with its adoption in the definition of the discrete Fourier and inverse Fourier transforms. A raw estimate of the principal phase  $\Phi_p$  is also available from the Hilbert transform/FFT procedure. In principle, the integer  $\ell$  is

$$\ell = (\Phi_u - \Phi_p)/2\pi \tag{14}$$

In practice,  $\ell$  will not be an integer as Equation 14 utilizes raw estimates of both  $\Phi_u$  and  $\Phi_p$ . To retain the same value of principal phase at each time step,  $\ell$  is adjusted to the nearest integer and the raw estimate of the unwrapped phase adjusted to

$$\Phi = \Phi_p + 2\ell\pi \tag{15}$$

It remains to estimate the right hand side of Equation 13. From the definition of the pre-envelope function,

$$A(t) \exp(i\omega_0 t) = \frac{1}{2\pi} \int_0^\infty 2 F(\omega) \exp(i\omega t) d\omega \tag{16}$$

Differentiating both sides with respect to time leads to

$$\frac{1}{A} \frac{dA}{dt} = \frac{\exp(-i\omega_0 t)}{A(t)} \left[ \frac{1}{2\pi} \int_0^\infty 2i\omega F(\omega) \exp(i\omega t) d\omega \right] - i\omega_0 \quad (17)$$

The term in square brackets can be computed from the inverse FFT algorithm, noting that the kernel is zero for negative frequencies. The slope  $d\Phi/dt$  follows from Equation 13.

The success of this algorithm is demonstrated by Figure 1. The record is a smooth paper wave with a dominant frequency of  $2\pi/10.24$  rad/s and envelope defined by

$$\begin{aligned} \text{mod } A(t) &= 1.5 + 1.0 \sin \Omega_1 t + 0.75 \sin \Omega_2 t \\ \text{arg } A(t) &= 2\pi \sin \Omega_3 t + 0.75\pi \sin \Omega_2 t \end{aligned} \quad (18)$$

where  $\Omega_1 = 24\pi/T$ ,  $\Omega_2 = 40\pi/T$ ,  $\Omega_3 = 10\pi/T$  and  $T = 1024$  s, the duration of the simulated record. The time step is 0.5 s. The paper wave record was established from Equations 1 and 18; it exhibits substantial groupiness. The computed modulus and unwrapped phase records are included in Figure 1. The result is identical with the target envelope, Equation 18.

**Paper Wave**

$N = 2048$ ;  $dt = 0.5$  sec;  $fp = 0.098$  Hz

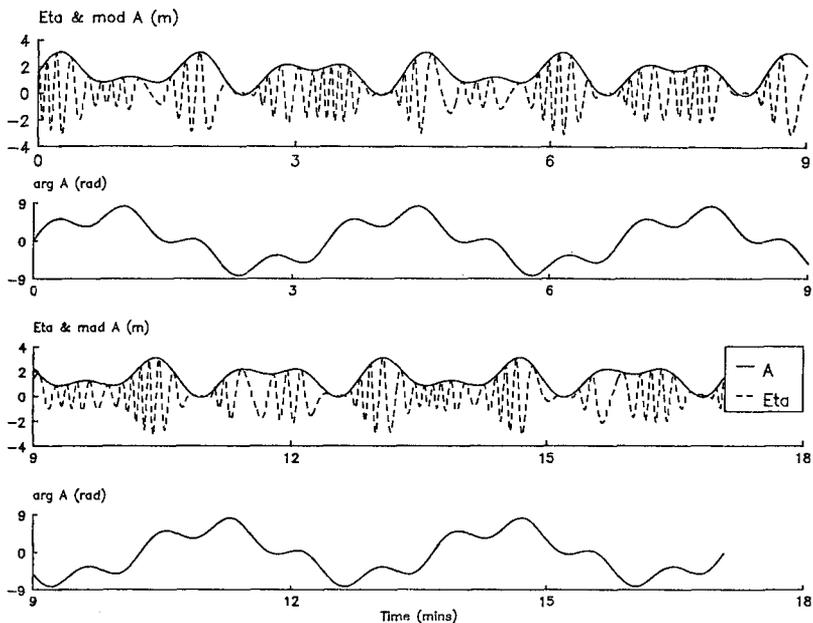


Fig. 1 Complex Envelope for Paper Wave

ALGORITHM EVALUATION

From a purely computational viewpoint, the algorithm described above appeared to be successful. The mod  $A(t)$  trace was a consistently good visual representation of the envelope of the record. Visual confirmation of the arg  $A(t)$  trace does not seem possible, at least not without considerable experience in complex envelope analysis. One measure of success of the phase unwrapping procedure is the difference  $\Delta\ell$  between the real number estimate computed from the right hand side of Equation 14 and the nearest integer  $\ell$  adopted in Equation 15. The differential  $\Delta\ell$  was computed at each time step. The mean of the series is closely zero by definition but the standard deviation

$$s = \langle \Delta\ell^2 \rangle^{1/2} \tag{19}$$

is a reasonable measure of the success of the phase unwrapping. For Figure 1,  $s$  is zero.

Botany Bay 09-Jul-81 11:02 hrs  
 fp= 0.0836Hz fo= 0.1097Hz Variance= 0.97m\*\*2 s= 0.020

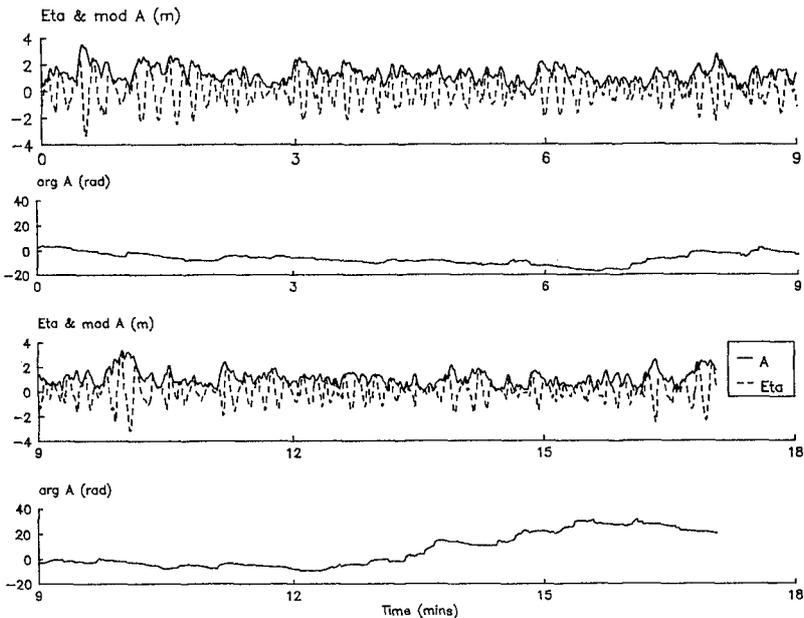


Fig. 2a Complex Envelope for Botany Bay, 9 July 1981, 11:02 hrs.

Figures 2 and 3 are analysis results from field records. The Figure 2 record was measured off Botany Bay on the east coast of Australia in 75 m of water. The record variance is  $0.97 m^2$ , the peak frequency is 0.0836 Hz, the computed dominant frequency is 0.1097 Hz and the standard deviation  $s$  is 0.020. The Figure 3 record was measured off Wilson Bluff on the south coast of Australia, the record variance and peak frequency being  $1.61 m^2$  and 0.0550 Hz respectively. This record is predominantly long period Southern Ocean swell. The computed dominant frequency is 0.0719 Hz and the standard deviation is 0.025. The computed dominant frequencies were typically about thirty percent greater

than the record peak frequencies, but there did not seem to be any apparent physical significance that could be attached to the dominant frequency. The significance of the computed standard deviations might be established by comparison of typical values with expected values if  $\Delta\ell$  were a random variable, uniformly distributed in the interval  $-0.5$  to  $+0.5$ . The theoretical mean and standard deviation of such a distribution would be  $0.0$  and  $12^{-1/2} = 0.289$ , respectively (Ang and Tang 1975). The Figure 2 and 3 results are indeed small with respect to  $0.289$ . These are typical values and clearly demonstrate the relative success of the algorithm.

Botany Bay 09-Jul-81 11:02 hrs

$f_p = 0.0836\text{Hz}$   $f_o = 0.1097\text{Hz}$  Variance =  $0.97\text{m}^2$   $s = 0.020$

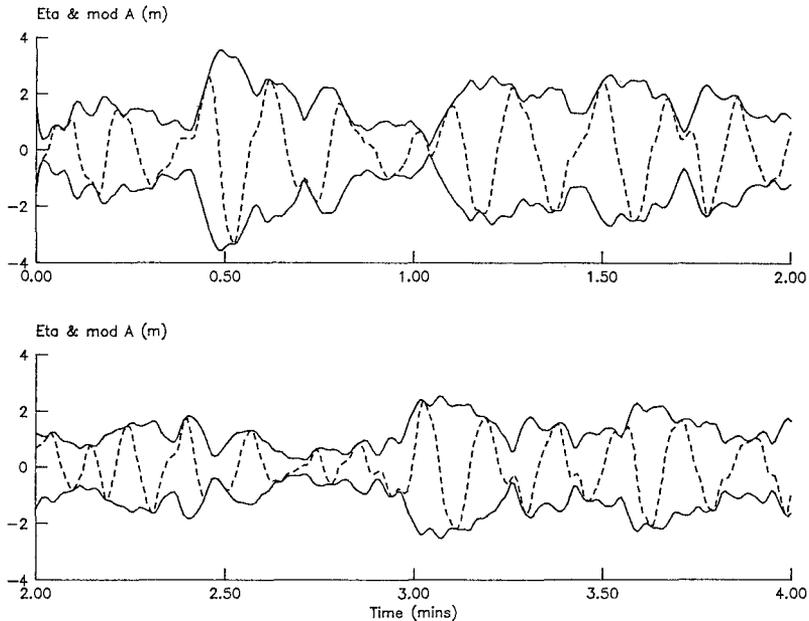


Fig. 2b Expanded Complex Envelope for Botany Bay

An occasional problem was identified in the phase unwrapping procedure where  $\text{mod } A(t)$  was close to zero. The  $d\Phi/dt$  estimate from Equation 13 became very large, resulting in a sudden step change in the  $d\Phi/dt$  trace and correspondingly in the  $\Phi$  trace. This discontinuity could be accommodated by a L'Hospital's rule procedure, as adopted by Read and Sobej (1985) in unwrapping the phase spectrum of the original record, but near zero magnitude points were found to be much less of a problem for the complex envelope function. A technique suggested by Brillinger (1962) proved to be satisfactory in the vicinity of near-zero  $A(t)$  points. This is based on the trigonometrical identity

$$\frac{d\Phi}{dt} = \cos \Phi \frac{d}{dt} (\sin \Phi) - \sin \Phi \frac{d}{dt} (\cos \Phi) \quad (20)$$

A forward difference approximation to the right hand side gave an alternate estimate of

$d\Phi/dt$  that avoided any sudden steps in the  $d\Phi/dt$  trace. This procedure was not suitable for general application however as the discrete approximation restricted the estimate of the phase derivative to the range - 1 to + 1.

Wilson Bluff 25-April-80 05:29 hrs  
 fp= 0.0550Hz fo= 0.0719Hz Variance= 1.61m\*\*2 s= 0.025

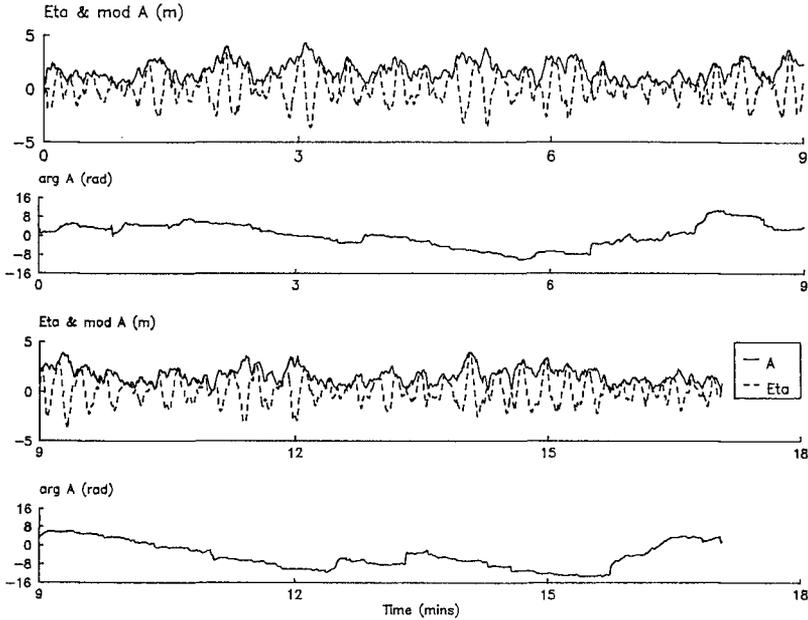


Fig. 3 Complex Envelope for Wilson Bluff, 25 April 1980, 05:29 hrs.

In practice, it was necessary to adopt a rational definition of a near-zero magnitude point. If the envelope amplitudes followed the Rayleigh distribution (Longuet-Higgins 1952), then a characteristic amplitude would be the mean amplitude  $(2\sigma^2)^{1/2}$ , where  $\sigma^2$  is the variance of the record. The Equation 20 algorithm was used when  $mod A(t)$  was smaller than one-tenth of this value. This tolerance level was not found to be critical.

The value of the  $mod A(t)$  traces is clear from Figures 2 and 3, and particularly from Figure 2b which shows only the first four minutes of the Figure 2a results. The removal of the dominant frequency focusses attention quite sharply on the envelope modulations in a very natural manner. Wave grouping is clearly identified, conveniently in a manner that (unlike low pass filtering of  $\eta^2(t)$ ) does not remove potentially valuable information from the record. With only casual experience, the identification of wave grouping in the phase trace alone does not seem possible but some assistance may be forthcoming from the cross-correlation and the cross-spectra between the modulus and argument traces. It is apparent from Figure 2b in particular that there remains potentially significant detail imbedded in the complex envelope amplitude and phase traces.

**CORRELATION AND CROSS-CORRELATION**

In the time or lag domain, the appropriate statistical summary is the cross-correlogram, plotting the normalized cross-correlation

$$\rho_{R\Phi}(\tau) = C_{R\Phi}(\tau) / [C_{RR}(0)C_{\Phi\Phi}(0)]^{1/2} \tag{21}$$

against the time lag  $\tau$ . The cross-correlation  $C_{R\Phi}(\tau)$  is the expectation  $E[(R(t) - \bar{R})\Phi(t+\tau)]$ , where  $\bar{R}$  is the average modulus of the envelope. The  $C_{RR}(\tau)$  and  $C_{\Phi\Phi}(\tau)$  are similarly defined. In practice, the correlograms were computed from the inverse Fourier transform of the variance spectra and cross-spectrum of  $R$  and  $\Phi$ , to take advantage of the FFT algorithm. It is reasonably common in complex demodulation (Hasan 1983) to consider  $\ell n R(t)$  rather than  $R(t)$ , as the probability distribution of  $\ell n R(t)$  is more nearly Gaussian. This practice was considered but not adopted as the range of  $R$  in the present problem is relatively small.

Figure 4 shows the correlograms corresponding to the Botany Bay record in Figure 2. The normalized form of the cross-correlation  $C_{R\Phi}$  is presented together with the normalized  $C_{RR}$  and  $C_{\Phi\Phi}$  autocorrelations;  $C_{R\Phi}$  is not an odd or even function and there are different traces for positive and negative lags. The correlograms are truncated at  $20/f_p$ , a typical record length being  $100/f_p$ .

Botany Bay 09-Jul-81 11:02 hrs  
Short dashed lines are 95% confidence limits

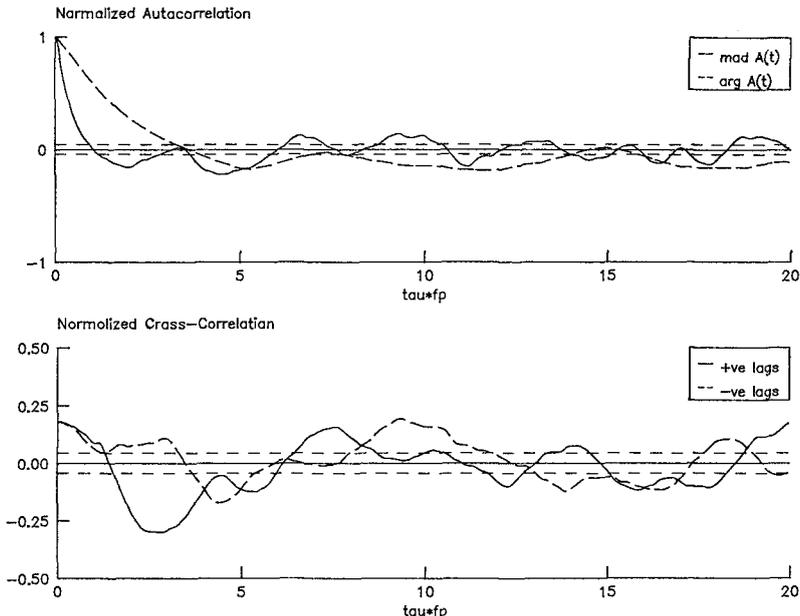


Fig. 4 Correlograms for Botany Bay, 9 July 1981, 11:02 hrs.

The initial part of the  $C_{RR}$  trace is typical of random noise. The tail however does not decay but evolves into an irregular long period oscillation with a period of order  $4/f_p$ . Referring back to the envelope trace in Figure 2, this is the anticipated result, the correlogram oscillation corresponding roughly to the envelope modulations. Similar, but longer period, oscillations are observed in the  $C_{\phi\phi}$  traces, indicating that the envelope modulations are perhaps a feature of the phase trace as well as the amplitude trace. Periodicity of order  $4/f_p$  remains the dominant feature of the cross-correlogram  $C_{R\phi}$ , implying some residual coupling between the envelope amplitude and phase. This would appear to impose some qualification on the Gaussian random wave model, which explicitly assumes that the phase trace is completely random. The magnitude and residual periodicity of the cross-correlations imply that the record involves a little order against a lot of background disorder.

The detailed interpretation of correlograms however must acknowledge a number of uncertainties. It is well known (Kendall & Stuart 1966, Jenkins & Watts 1968) that correlograms for short series can be unreliable and may not decay as rapidly as expected; as a rule of thumb, only the first twenty percent might be given any credence. Wave records are moderately long series however, typically 2048 points over seventeen minutes; the maximum lag presented in Figure 4 is  $20/f_p$ , corresponding typically to less than twenty percent of the record length. These potential uncertainties may be addressed in two ways, the first being the estimation of confidence limits on the correlogram tail. Kendall and Stuart (1965) show that the variance of a normalized auto-correlation estimate is dependent on all autocorrelations in the series, even for large samples with the simplifying assumption of normality. However for a random series where all parent autocorrelations are zero, the expectation of the mean approaches  $-1/N$  and the variance  $1/N$ . The 95 percent confidence limits are accordingly  $-1/N \pm 2/(N)^{1/2}$ . The width of this confidence band decreases with the length of the series, but is already moderately small at  $\pm 0.044$  for typical 2048 point records. These 95 percent confidence limits are included as the short-dashed lines on Figure 4. Note that rather more than five percent of the autocorrelogram and cross-correlogram tails falls outside this band.

A second approach to these potential uncertainties for moderately small lags is to compute the correlograms from a longer sample, significantly in excess of the typical  $100/f_p$ . The nominal twenty minute record has become a field measurement standard and longer records are not commonly available. Long records also introduce another problem, in that the record may no longer be statistically stationary. Sea states typically develop at time scales of order hours (Sobey 1986). A record of twenty minutes could reasonably be expected to be stationary, a record of one hour may or may not be stationary according to circumstance, but a record of several hours is unlikely to be stationary. An attempt to solve one problem unfortunately introduces another. Non-stationarity appears as very low frequency trend in the Fourier transform. There is a sharp spectral spike at near-zero frequency, whose magnitude increased substantially with longer records and accordingly finer frequency resolution. This was a particular problem in the phase trace. Jenkins and Watts (1967) recommend removal of such very long period trend prior to analysis by high-pass filtering of the record; this approach was adopted with a cutoff frequency of  $0.05 f_p$ . The phase trace was first low-pass filtered by a moving average filter of width equivalent to a  $0.05 f_p$  cutoff. This filtered record was then subtracted from the original phase trace to yield the detrended phase trace. Without detrending, the very low frequency trend completely dominated the Fourier transform and just as completely negated any advantage from a longer record. Detrending was adopted with considerable reservation however, as a developing sea state will likely be accompanied by physical developments in the wave grouping structure which will also negate any advantage from a longer record. An excessively long record (say several hours) is clearly inappropriate. A one hour record might be a reasonable compromise, provided the record is reasonably stationary and due

attention is given to the analysis uncertainties.

The record presented in Figure 2 and again in Figure 4 is the central 2048 points (nominal twenty minutes) of a 8000 point (nominal one hour) record. The variance of this extended record was computed in 1000 point blocks, being 1.40, 0.98, 0.77, 0.99, 0.75, 0.94, 1.06 and 1.18  $m^2$  for the Botany Bay record. This variation is typical of visually-stationary wave records. Figure 5 is a repetition of the correlogram analyses for the 8000 point record. Record length is of order  $400/f_p$  and lags up to  $20/f_p$  now represent only five percent of the record length. The confidence band is also somewhat narrower, reflecting the larger sample size. The general trend of the correlogram however remains largely unchanged.

Botany Bay 09-Jul-81 11:02 hrs  
Short dashed lines are 95% confidence limits

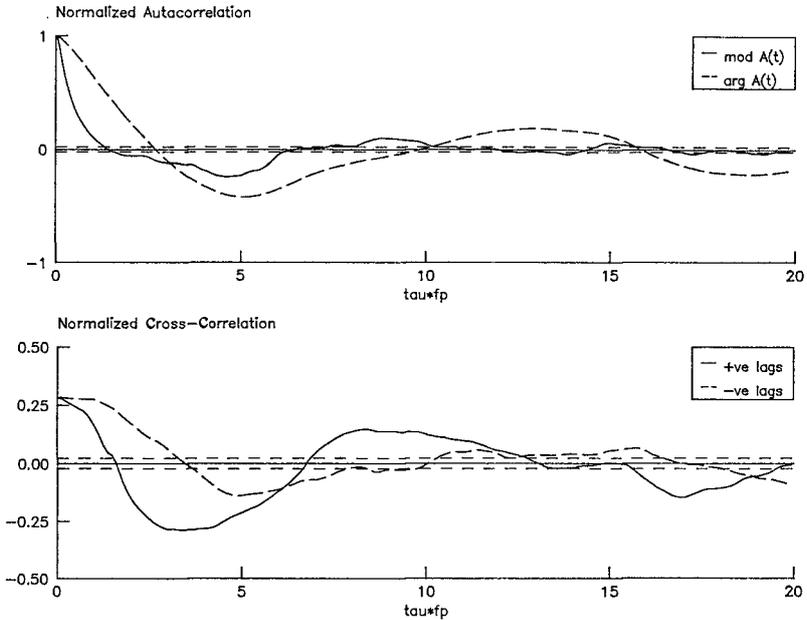


Fig. 5 Correlograms for Botany Bay, 8000 point record.

The Figure 4 correlogram is quite a typical result. It provides a useful perspective on the nature of wave groups in surface gravity waves. A large part of a typical wave record can reasonably be described as random; of order ninety percent of the information can be so described. The residual ten percent or so would appear to be nonrandom, involving long period modulations of the complex wave envelope. The implied low-level correlation between the envelope amplitude and envelope phase traces appears to be an intrinsic property of wave groups.

## SPECTRA AND COHERENCE

In the frequency domain, the statistical summary is the cross spectrum

$$E_{R\Phi}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{R\Phi}(\tau) \exp(-i\omega\tau) d\omega \quad (22)$$

$E_{R\Phi}(\omega)$  is a complex function with a real part (the cospectrum) and an imaginary part (the quadrspectrum). A useful real function is the coherence

$$\gamma_{R\Phi}(\omega) = \text{mod } E_{R\Phi}(\omega) / [E_{RR}(\omega)E_{\Phi\Phi}(\omega)]^{1/2} \quad (23)$$

or the squared coherence, the square of Equation 22. The squared coherence approaches unity when the two traces are strongly correlated and it is zero when the traces are uncorrelated. High squared coherence levels suggest some relationship between the phase and amplitude traces, a relationship that is assumed by the Gaussian random wave model to be very small.

The variance spectra  $E_{RR}(\omega)$  and  $E_{\Phi\Phi}(\omega)$  and the cross-spectrum were estimated directly from the Fourier transforms of the envelope amplitude and phase traces, taking advantage of the FFT algorithm. Raw spectral estimates resulting from the FFT algorithm have only two degrees of freedom and the raw estimate of squared coherence is identically one for all frequencies (Bloomfield 1976). Statistically reliable spectral estimates require a significant increase in the degrees of freedom, achieved by frequency domain smoothing of  $E_{RR}(\omega)$ ,  $E_{\Phi\Phi}(\omega)$  and  $E_{R\Phi}(\omega)$  prior to estimation of the squared coherence. Frequency domain smoothing (specifically here a moving average filter of half-width  $L$  points) increases the degrees of freedom from 2 for the raw FFT estimates to  $4L + 2$ . The widths of 95 percent confidence bands on variance spectral estimates and on squared coherence estimates are accordingly reduced (Jenkins and Watts 1968). A filter half-width  $L$  of the nearest integer to  $1/(5f_p\Delta t)$  proved to be a suitable compromise between statistical confidence and resolution of detail.

The complex demodulation procedure introduces an immediate problem of frequency resolution in spectral analysis of the complex envelope. A typical 1024 s wave record imposes a frequency resolution of 0.000977 Hz, more than adequate to accommodate peak frequencies of order 0.1 Hz. Complex demodulation however removes these frequencies and shifts the spectral peak to frequencies of order 0.01 Hz, where the frequency resolution is distinctly marginal. Frequency resolution is enhanced by a longer record, specifically 8000 points where the frequency resolution is 0.000150 Hz. While this resolution remains marginal, longer record lengths will again introduce problems of non-stationarity.

The frequency domain description for the Botany Bay record is shown in Figure 6, which includes four spectra. Part (a) is the variance spectrum for  $R - R$ , the solid line being the raw estimate and the dash-dot line the smoothed spectral estimate, 95 percent confidence bars being shown at selected frequencies for the smoothed spectral estimates. Part (b) is an analogous presentation of the variance spectrum for the envelope phase trace. Part (c) is the smoothed squared coherence estimate, the dotted line being the 95 percent confidence limit on zero coherence between the amplitude and phase traces. Part (d) is  $\arg E_{R\Phi}(\omega)$ , the smoothed phase of the cross spectrum between  $R$  and  $\Phi$ . Variance levels at frequencies above the spectral peak are very small and these presentations are terminated at  $f_p$ .

Botany Bay 09—Jul—81 11:02 hrs

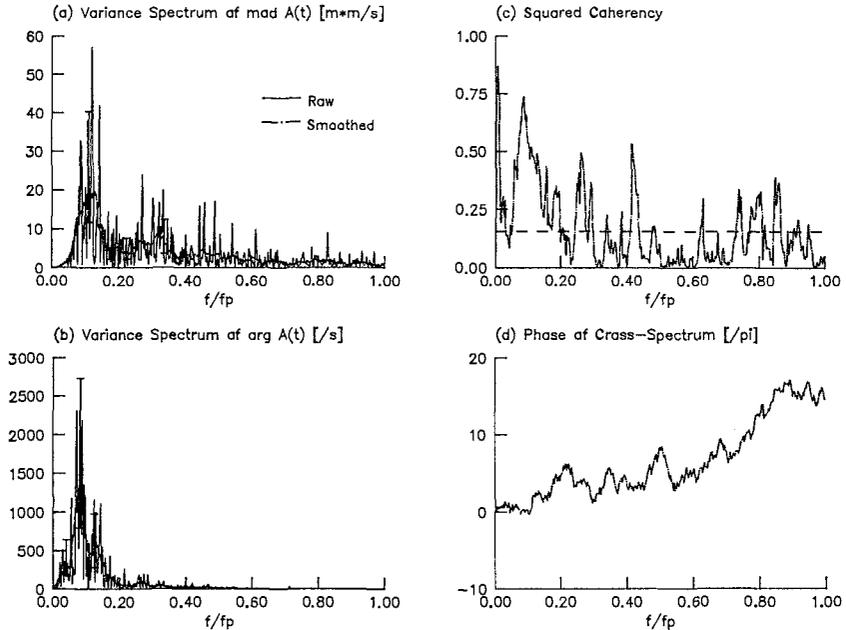


Fig. 6 Spectra for Botany Bay, 8000 point record.

The  $E_{RR}$  spectrum for Botany Bay is the result that is anticipated from the correlograms in Figure 4 and 5; it is dominated by short frequencies about an order of magnitude less than the peak frequency. Short frequencies are also clearly dominant in the phase trace in Figure 2 and this is confirmed in Figure 6. The squared coherence spectrum shows moderate coherence in excess of the 95 percent confidence limit on zero coherence at frequencies of order  $0.2 f_p$  to and also  $0.8 f_p$ . The Gaussian random wave model would suggest negligible coherence throughout. Coherence levels are small but not entirely negligible. Analysis of the cross-phase spectrum was not pursued, as marginal frequency resolution and low coherence levels indicated that results would most likely be inconclusive. The higher frequency peak might perhaps be the result of non-linear interactions near the spectral peak. Moderate coherence levels around  $0.2 f_p$  seemed to be a reasonably typical result, at least for waves off Botany Bay.

## CONCLUSIONS

The complex envelope function is a very useful analysis technique for wave records where the identification of wave grouping is a relevant interest. The removal of the dominant frequency concentrates attention directly on the envelope modulations without removing potentially useful information from the record. Wave groups are clearly identified against a predominantly random background. An algorithm has been presented for extraction of the complex envelope function, identification of the dominant frequency and unwrapping of the phase trace.

Cross-correlogram and cross-spectral analyses of the simultaneous envelope amplitude and unwrapped phase traces has associated wave grouping with low level phase locking between the amplitude and phase traces. The ordered structure comprises about ten percent of the information contained in a wave record. The balance of the record can reasonably be described as random, as assumed by the Gaussian random wave model. This dominant random component is clearly responsible for the relative success of the Gaussian random wave model.

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## APPENDIX I - REFERENCES

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