Biperiodic Waves in Shallow Water

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Introduction

The propagation of waves in shallow water is a phenomenon of significant practical importance. The ability to realistically pre-dict the complex wave characteristics occurring in shallow water regions has always been an engineering goal which would make the development of solutions to practical engineering problems a reality. The difficulty in making such predictions stems from the fact that the equations governing the complex three-dimensional flow regime can not be solved without linearizing the problem. The linear equations are solvable; however, their solutions do not reflect the nonlinear features of naturally occurring waves. A recent advance (1984) in nonlinear mathematics has resulted in an explicit solution to a nonlinear equation relevant to water waves in shallow water. This solution possesses features found in observed nonlinear three-dimensional wave fields.

The nonlinear mathematical formulation referred to above has never been compared with actual waves, so that its practical value is unknown. The purpose of the present investigation was to physi-cally generate three-dimensional nonlinear waves and compare these with exact mathematical solutions. The goals were successfully completed by first generating the necessary wave patterns with the new U.S. Army Engineer Waterways Experiment Station, Coastal Engineering Research Center's (CERC) directional spectral wave generation facility. The theoretical solutions were then formed through the determination of a unique correspondence between the free parameters of the solution and the physical characteristics of the generated wave.

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Theoretical Background

One of the first mathematical models of nonlinear waves in shallow water with known solutions was presented by Korteweg and deVries in their famous 1895 paper. Their model, known as the KdV equation can be written in the following nondimensional form

$$f_t + 6ff_x + f_{xxx} = 0 \tag{1}$$

in which f represents the water surface displacement, x is the direction of wave propagation and t is time. This equation admits not only solitary wave solutions but also the periodic solutions commonly known as cnoidal waves. These solutions can be written as

$$f(x,t) = 2\sigma^2 k^2 cn^2(\theta;k) - 2\sigma^2 \left[\frac{E(k)}{K(k)} - 1 + k^2 \right] \tag{2}$$

where each of the terms in the solution are well documented analytic functions which can easily be computed in terms of known wave characteristics such as wave height and wavelength. Unfortunately, cnoidal wave solutions are valid only for long crested waves, e.g., waves which can be described by a single time-dependent one-dimensional surface wave pattern. Natural waves, in contrast, are composed of both long and short crested waves and cannot be adequately described by this theory.

A recent advance in nonlinear mathematics has been reported by Segur and Finkel (1984). They present explicit analytical solutions to a natural three-dimensional extension of the KdV equation proposed by Kadomtsev and Petviashvili (1970), known as the KP equation shown below

$$(f_x + 6ff_x + f_{xxx})_x + 3f_{yy} = 0 \tag{3}$$

where x now represents the primary direction of propagation; however, weak changes in the y-direction are now permitted. When no y-variations occur, the KP equation reverts to the KdV equation.

The KP equation admits an infinitely dimensional family of exact, periodic, solutions (see Dubrovin 1981 and Segur and Finkel 1984) which can be written in the form

$$f(x,y,t) = 2 \frac{\partial^2 \ln\theta}{\partial x^2} \qquad (4)$$

where θ is a Riemann theta function of genus n. Genus 1 solutions are exactly equivalent to cnoidal waves, they are permanent form, singly periodic, two-dimensional (one vertical and one horizontal) nonlinear waves. Genus 2 waves are biperiodic in that they permit the independent specification of two periodicies in both time and space. The solutions are genuinely three-dimensional, nonlinear, and propagate with permanent form at a constant velocity. Genus 3 and higher order solutions are multi-periodic and cannot be characterized as permanent form with respect to any translating coordinate system as the genus 1 and 2 solutions can. This present investigation is limited to the genus 2 solutions developed by Segur and Finkel.

The construction of a genus 2 solution of the KP equation is based on the specification of the appropriate Riemann theta function. This requires the introduction of a two-component phase variable and a 2 X 2 real-valued Riemann matrix. The first of these, the phase variable, is shown below.

$$\phi_1 = \mu_1 x + \nu_1 y + \omega_1 t + \phi_{10}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

$$\phi_2 = \mu_2 x + \nu_2 y + \omega_2 t + \phi_{20}$$

Where the parameters μ_1, μ_2, ν_1, and ν_2 are wave numbers, ω_1, and ω_2 are angular frequencies, and ϕ_{10} and ϕ_{20} are constants with no dynamical significance. The second ingredient involves the specification of a real-valued, negative definite, symmetric 2 X 2 Riemann matrix as shown below.

$$B = \begin{pmatrix} b & b\lambda \\ b\lambda & b\lambda^2 + d \end{pmatrix} \qquad (6)$$

The parameters b, d, and λ represent solution non-linearity. The genus 2 theta function can now be defined in terms of the above components by the following double Fourier series:

$$\theta(\phi_1,\phi_2,B) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \exp(\tfrac{1}{2} m \cdot B \cdot m + i m \cdot \phi) \qquad (7)$$

The calculation of a general case genus 2 KP solution requires the specification of the 11 parameters shown in Equations 5 and 6. Two of these parameters (ϕ_{10} and ϕ_{20}) have no dynamical significance, their only effect is to shift the origin of the resulting solution. Dubrovin (1981) proved that a genus 2 theta function in

the form of Equation 7 was a solution to the KP equation if, and
only if, the solution parameters were related by four additional
equations. One of these equations contains a constant of
integration. Use of this additional criteria reduces the number of
free parameters to 8, representing the minimum number of free param-
eters required to specify a general case genus 2 solution.

Genus 2 solutions of the KP equation describe a complex two-
dimensional surface wave pattern. Similar features were observed by
Hammack (1980) to result from the nonlinear interaction of two
intersecting waves. The theoretical development by Segur and Finkel
was partially prompted, in fact, by these reported waves. The devel-
opment of an experimental program which would result in the genera-
tion of surface wave patterns qualitatively similar to genus 2 solu-
tions was achieved by attempting to experimentally reproduce the
conditions reported by Hammack, i.e. intersecting waves. This
generation technique can best be described by presenting the analogy
of interacting waves. Consider, for example, two periodic waves
which intersect and pass through each other as shown in Figure 1.
The angles α_1 and α_2 represent the angle of the crest of each
wave front with respect to some reference line. The resulting sur-
face wave pattern, according to linear wave theory, would simply be
a superposition of the two individual waves. This would produce a
diamond shaped surface pattern as indicated in Figure 1. It can be
seen that certain of the basic characteristics of the individual
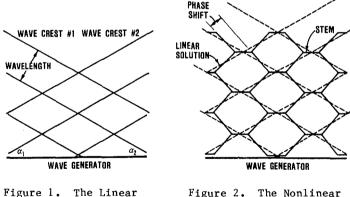waves, wavelength and angle of propagation for example, have been
preserved.

Figure 1. The Linear   Figure 2. The Nonlinear
          Intersection   Intersection
          of Waves   of Waves

Now, consider the analagous case in which similarly intersecting
waves interact nonlinearly with each other. This scenario is shown
schematically in Figure 2. The resulting wave pattern shows that a
"stem of interaction" is formed at the point where the two waves

cross each other. The formation of this stem region is a result of
a phase shift in the crest line angles of the original waves. This
phenomenon is shown in Figure 2 superimposed on the corresponding
linear wave solution. The resulting surface wave pattern now
assumes a hexagonal pattern in which a third wave crest, separate of
the original two, is formed. This phase shift and stem formation
are indicative of the nonlinear interaction of the two waves since
the exact linear solution does not predict either the phase shift or
the new wave crest. Genus 2 solutions of the KP equation predict
these features and was tested as a possible model for their
description.

Laboratory Facilities and Experimental Procedures

    A project was initiated at CERC to generate three-dimensional
nonlinear wave fields in the laboratory and then to apply KP theory
to the resulting waves in order to determine whether or not the KP
equation was a model for these waves and, if so, what was the range
of its applicability. This required the use of the CERC directional
spectral wave generation facility. This unique wave generator,
shown in Figure 3, was designed and constructed for CERC by MTS
Systems Corporation of Minneapolis, Minnesota, based on design
specifications provided by CERC. The generator is comprised of 60
individually programmable electromechanical wave paddles. Each wave
paddle is 1.5 ft wide making the generator a total of 90.0 ft in
width. The generator is located in a 98.0 by 184.0-ft wave basin
with 2.5 ft high side walls. Computer control of the system is
provided by a Digital Equipment Corporation (DEC) VAX 11/750 central
processing unit. The above facilities were utilized to generate
genus 2 candidate waves in a comprehensive experimental program.

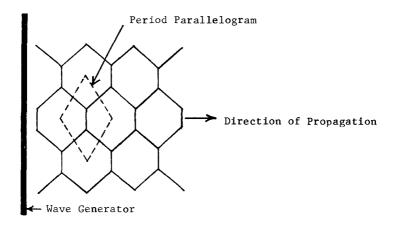Figure 3. The Directional Spectral Wave Generator

The wave generator was programmed to simultaneously generate intersecting cnoidal wave trains. A variety of wave fields were generated by varying both the wavelength of the individual waves and their angle of intersection. Twelve wave fields, generated in this manner, were used to test the KP equation. The wave fields selected for the experimental program are presented in Table 1. Waves characterized by three wavelengths (7, 11, and 15 ft) were combined with phase shifts between adjacent wavemaker paddles. These phase shifts were approximately equivalent to the angle of the wavecrest with respect to the axis of the wave generator. The angle in the tahle shows the approximate correspondence between the phase lag and the angle of propagation.

<div align="center">Table 1
The Experimental Waves</div>

Test Number	Wavelength (ft)	Phase Shift (deg)	Angle (deg)	Period (sec)
CN1007	7.0	10.0	7.45	1.378
CN1507	7.0	15.0	11.21	1.378
CN2007	7.0	20.0	15.03	1.378
CN3007	7.0	30.0	22.89	1.378
CN4007	7.0	40.0	31.23	1.378
CN1011	11.0	10.0	11.75	1.947
CN1511	11.0	15.0	17.79	1.947
CN2011	11.0	20.0	24.04	1.947
CN3011	11.0	30.0	37.67	1.947
CN1015	15.0	10.0	16.12	2.553
CN1515	15.0	15.0	24.62	2.553
CN2015	15.0	20.0	33.75	2.553

Genus 2 solutions can be visualized as a series of repeating two-dimensional permanent form surface patterns, referred to as period parallelograms. These patterns translate at a constant velocity in a constant direction. The global wave field is represented by a tiling of these basic patterns; therefore, the entire wave pattern can be exactly specified by quantifying just one period parallelogram. The location of a basic period parallelogram within the hexagonal wave field of Figure 2 is shown in Figure 4. The phase variables of Equation 5 define the horizontal limits of these patterns such that each side is uniquely defined by ϕ_1 = constant and ϕ_2 = constant. The components of the Riemann matrix define the vertical and horizontal distribution within the period parallelogram.
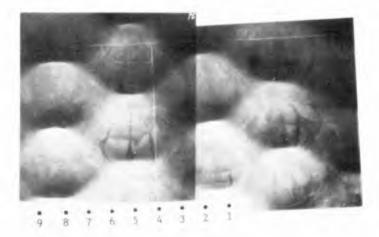
Figure 4. The Period Parallelogram

    Detailed measurements of each of the generated wave fields shown
in Table 1 were required in order to relate the physical charac-
teristics of the waves to the parameters of the corresponding period
parallelogram of the exact solution. This quantification was accom-
plished by first using overhead photography to determine the dimen-
sions of the period parallelogram and to provide an estimate of the
internal features, such as the phase shift and stem length.
Knowledge of these horizontal features and their location within the
wave tank were then used to locate a linear array of 9 recording
wave gages in the wave basin. This approach provided a vertical
wave record which could be identified with a known location within
the parallelogram.
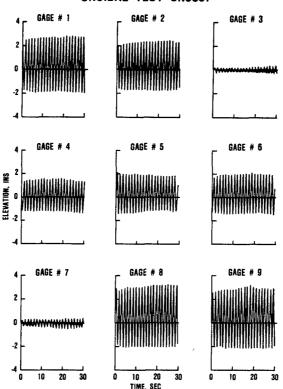
Comparing Theoretical Solutions To Observed Waves

    The experimental program described above generates symmetric
cnoidal waves ($\alpha_1 = \alpha_2$ in Figure 1) resulting in a symmetric period
parallelogram. This simplification was adopted so that the
generated wave patterns would propagate perpendicularly off the face
of the wave generator, making it possible to measure all wave forms
with a single stationary wave gage array. Symmetry also reduces the
number of free parameters which need to be specified, for example,
$\mu_1 = \mu_2$, $\nu_1 = -\nu_2$, and $\omega_1 = \omega_2$ from Equation 5. This simpli-
fication results in the requirement of only three dynamical parame-
ters and two nondynamical parameters. The parameters chosen were
b , μ , and λ along with the phase shift parameters ϕ_{10} and ϕ_{20}
The following sequence of events was used for optimizing these
coefficients. Experiment CN3007 will be used to demonstrate the
verification process.

Each of the waves of Table 1 were generated in the wave basin.
Two overlapping photographs were taken with duel Hasselblad model
500EL/M 70mm cameras equipped with 50mm lenses mounted 23 ft above
the floor of the basin. The resulting mosaic photograph, shown in
Figure 5, was used to estimate the length and width of the period
parallelogram. This resulted in estimates for $\mu_1 = \mu_2$ and
$\nu_1 = -\nu_2$. An estimate for the phase shift parameter λ was also
determined from the photograph. The accuracy of μ , ν , and λ is a
function of the distortions in the photograph. Because of this
distortion, their values were considered to be initial estimates.
Following the photographing of all waves, a wave gage spacing of
2.5 ft apart and 40.0 ft from and parallel to the generator was
selected for use in all tests. The location of each of the gages
with respect to wave CN3007 is shown in Figure 5. It can be seen
that each gage can be uniquely referenced according to a distance
from the center of the parallelogram. Since all parallelograms are
identical, wave gages located in an adjacent parallelogram can be
referenced to the common center point.

Figure 5.   Overhead Mosaic Photograph of Test Wave CN3007

Wave gages were located in the basin and each of the waves of
Table 1 were regenerated. Data was sampled for each of the gages at
a rate of 50 samples per second for a total of 30.0 seconds.
Figure 6 shows the wave traces for one period of wave CN3007. The
correspondence between the wave traces and their location within the
parallelogram can easily be seen. For example, gage 5 is located on
a stem where only one peak per passing of the parallelogram is
experienced. Gage 3 is located in the saddle region where two
smaller peaks per period are seen. This comparison demonstrates the
usefulness of the photographs in interpreting the data since three-
dimensional effects are difficult to deduce from two-dimensional
data.

CNOIDAL TEST CN3007

Figure 6.   Wave Gage Traces for Test Wave CN3007

The determination of the free coefficients can now be made. Known or estimated data are the period of the wave (determined from the recording wave gages), the length and width of the period parallelogram and an estimate of the phase shift parameter λ determined from the photographs, and a maximum wave height selected from the wave gage data. The following iteration procedure was used to optimize the coefficients:

a. The estimated values for $\mu_1 = \mu_2$ and λ were specified. The nondynamical parameters ϕ_{10} and ϕ_{20} were accounted for by specifying solutions to be computed at location within the period parallelogram corresponding to the location of the wave gages. A value of b was then selected such that the dimensionalized maximum KP solution was within 5.0 percent of the measured value.

   b. The value of $\mu_1 = \mu_2$ was adjusted, if necessary, until the
dimensionalized period was within 3.0 percent of the measured
period.

   c. The value of λ was adjusted, if necessary, until the dimen-
sionalized value of $\nu_1 = -\nu_2$ was within 10.0 percent of the esti-
mated value. A 10-percent criteria was used for this iteration
since the length of the parallelogram was difficult to determined
from the photographs.

   d. Because of the nonlinear coupling of the solution coeffi-
cients, each adjustment affected all parameters to some extent. If
corrections were found to be necessary, steps (a.) through (c.) were
repeated until all of the specified tolerances were

CNOIDAL TEST CN3007

ZETA/(MAX ZETA)
— COMPUTED
* MEASURED

MAX ZETA (CMP) = 0.394
DEPTH (FT) = 1.000
2.*PI/NU (FT) = 17.028
2.*PI/MU (FT) = 7.854
PERIOD (SEC) = 1.378
MAX ZETA (OB-IN) = 3.239

GAGE # 9, y = 1.50 FEET
RMS ERROR = 0.077

GAGE # 8, y = 1.00 FEET
RMS ERROR = 0.089

GAGE # 7, y = 3.50 FEET
RMS ERROR = 0.209

GAGE # 6, y = 6.00 FEET
RMS ERROR = 0.095

GAGE # 5, y = 6.50 FEET
RMS ERROR = 0.263

GAGE # 4, y = 6.00 FEET
RMS ERROR = 0.112

GAGE # 3, y = 3.50 FEET
RM8 ERROR = 0.163

GAGE # 2, y = 1.00 FEET
RMS ERROR = 0.093

GAGE # 1, y = 1.50 FEET
RMS ERROR = 0.187

AVERAGE RMS ERROR = 0.143

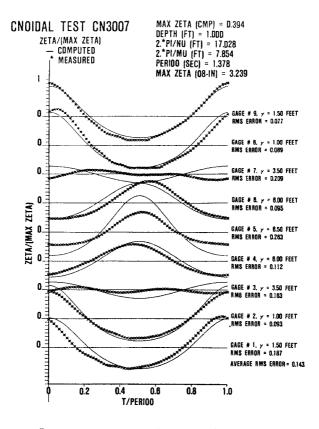ZETA/(MAX ZETA)

T/PERIOD

Figure 7. Theoretical and Measured Wave Profiles
            for Test Wave CN3007

met or exceeded. Possible phasing problems regarding the gage
locations within the parallelogram were rectified by adjusting the
nondynamical phase parameters.

   e. A KP solution corresponding to the location of each of the
wave gages was calculated. A normalized plot comparing theory to
measurements was made, as shown in Figure 7 for the present
example. Included in each plot is the Root Mean Square (RMS) error
for each comparison.

   f. A normalized contour plot (Figure 8) and a three-dimensional
plot (Figure 9) for each wave field was finally prepared as a visual
example of the KP solution.

9 CONTOURS
CONTOUR LEVELS FROM -.600 TO 1.00
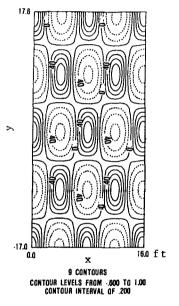CONTOUR INTERVAL OF .200

Figure 8. Normalized Contour Plot for Test Wave CN3007

Figure 9. Three-Dimensional Plot of Test Wave CN3007

The above procedures were followed for each of the test wave
fields of Table 1. A minimum tolerance of 5.0 percent for
waveheight, 3.0 percent for period, and 10.0 percent for the
Y-direction wavelength was maintained for all experiments. Table 2
presents those computed results. For each case, an average RMS
error is provided which represents a simple average of the 9 RMS
values computed for each gage. In no case did this error exceed
20 percent even though variations in the elevation of the basin
floor of 10 percent were known to exist. Additionally, the
experimental wave fields were generated almost to the point of
breaking in order to span the range of solution parameters and
investigate the limits of applicability of the genus 2 solutions.
In view of these introduced and existing sources of potential error,
the degree of fit between the generated wave fields and the exact
solutions were found to be very good.

Table 2
Computed wave parameters

Test Number	Max. Height (in)	X-Wavelength (ft)	Y-Wavelength (ft)	Ave. RMS Error
CN1007	2.44	7.0	46.5	0.141
CN1507	3.59	7.2	35.1	0.188
CN2007	3.06	7.5	27.3	0.150
CN3007	3.24	7.9	17.0	0.143
CN4007	3.30	8.7	13.6	0.184
CN1011	2.23	10.7	48.0	0.174
CN1511	2.87	11.1	40.3	0.122
CN2011	3.10	11.6	27.6	0.126
CN3011	2.48	12.6	20.7	0.172
CN1015	2.65	15.0	59.3	0.120
CN1515	2.84	16.1	32.6	0.094
CN2015	2.86	17.1	29.0	0.098

Conclusions

Twelve separate nonlinear wave fields were generated for the
purpose of verifying the KP equation to be an accurate model for
three-dimensional nonlinear waves. Criteria were developed which
provided a unique correspondence between the solution parameters of
the KP equation and the physical characteristics of the laboratory
generated waves. Results of these experiments showed that both the
generated waves and the genus 2 solutions are remarkable robust in
that both were stable over a wide range of parameters, including the
near breaking of waves. The excellent degree of fit between the
observed and computed solutions shows that genus 2 solutions of the
KP equation represent a viable model for three-dimensional,
nonlinear, shallow water waves.

References

Dubrovin, B. A. (1981), "Theta Functions and Non-Linear Equations", Russian Math. Surveys, Vol. 36:2, pp 11-92.

Hammack, J. L. (1980), Unpublished Experiments.

Kadomtsev, B. B. and Petviashvili, V. I. (1970), "On the Stability of Solitary Waves in Weakly Dispersive Media", Soviet Physics — Doklady, Vol. 15, No. 6, pp 539-541.

Korteweg, D. J. and deVries, G. (1895), "On the Change of Form of Long Waves Advancing in a Rectangular Channel, and on a New Type of Long Stationary Waves, Phil. Mag., Ser. 5, Vol. 39, pp 422-443.

Segur, H.. and Finkel, A. (1984), "An Analytical Model of Periodic Waves in Shallow Water", Aeronautical Research Associates of Princeton, Inc., Tech. Memo. 84-12.