Water Waves Propagating on Beaches of Arbitrary Slope

by

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ABSTRACT

When a small amplitude wave climbing along an arbitrary sloping beach from deep water toward the shore, the variation of characteristics in the process of wave motion has been described in this paper. From the results of theoretical derivation, it is found out that the variation of water surface and amplitude are function of beach slope(α) *and dimensionless distance* (kx) *from the shore. And under the condition of the beach slope is* $\alpha = 0$ *and* $\alpha = \infty$ *that the solution will become a progressive wave and a standing wave respectively.*

1. *INTRODUCTION*

Concerning the problems of water waves propagating on beaches of arbitrary slope, E.T. Hanson (1926) *assumed the angle of bottom with still water surface to be of the form* $\pi/2q$ *with integral* q, *and constructed a progressive wave derived from two standing waves. Lewy, H.* (1946) *gave a contour integral representation for a progressive wave for all angles between bottom and surface. Then, Stoker* (1947) *derived the exact linear theory and obtained approximate solution for surface waves in variable water depth. Biesel* (1952) *expressed the first-order approximation of the free surface equations and the trajectory of surface*

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particle by Lagrangian form. Carrier & Greenspan (1957) *presented the explict solutions based on the non-linear shallow water theory.*

All the above investigations concentrated on the behavior of wave motion in the region near the coast, however, they didn't include the entire process of wave motion. Therefore, in this paper, the authors pay attention to the theoretical analysis on the variation of the water surface and amplitude when the water waves propagating on beaches of arbitrary slope under the condition without breaking.

2. THEORETICAL CONSIDERATION

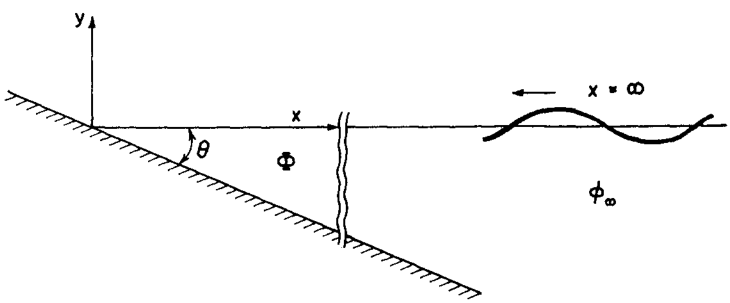

Fig.-1

From the sketch diagram of water wave propagating from deep water toward shallow water area as shown in Fig. 1, *the governing equations of wave motion in two dimensional incompressible fluid are given by the follows:*

$$\Phi_{xx} + \Phi_{yy} = 0 \qquad \eta(x, t) \geq y \geq -h(x) \tag{1}$$

$$\Phi_t + g\eta + \frac{1}{2}(\Phi_x{}^2 + \Phi_y{}^2) = C \qquad y = \eta(x, t) \tag{2}$$

$$\Phi_y = \eta_t + \Phi_x \eta_x \qquad y = \eta(x, t) \tag{3}$$

$$\Phi_y = - h_x \Phi_x \qquad y = - h(x) \tag{4}$$

The subscripts of the above equations denote partial differentiation, and Φ is velocity potential, η is water surface variation, $h(x)$ is water depth which can be expressed as $h(x) = x \cdot \tan\theta = \alpha x$, and C is Bernoulli's constant, g is gravity acceleration.

From Cauchy's integral theorem, there is a constant M existence in the process of wave motion under the conditions without breaking. that

$$|\Phi| + |\Phi^{(n)}| < M \quad \text{for } 0 \le x \le \infty,\ \eta(x,t) \ge y \ge - h(x) \tag{5}$$

where $\Phi^{(n)}$ denotes the n^{th} order's partial derivatives of Φ with respect to x, y or t. Furthermore, since the original source of wave motion comes from deep water, so that

$$\lim_{x \to \infty} \Phi = \phi_\infty \tag{6}$$

According to the above consideration that seems reasonable to coincide with physical grounds and there is not any singularity taken place in the entire process of wave motion from deep water toward the shore. In the following derivation, it is convenient to express the quantities evaluated for $y = \eta(x, y)$ by a bar, "—", over the quantity, and for $y = - h(x)$ by a bar under the quantity. Thus we have

$$\int_{-h(x)}^{\eta(x,t)} \Phi_{yy}\, dy = \overline{\Phi_y} - \underline{\Phi_y} = \overline{\Phi_y} + h_x \underline{\Phi_x} \tag{7}$$

and

$$\int_{-h(x)}^{\eta(x,t)} \Phi_{yy}\, dy = - \int_{-h(x)}^{\eta(x,t)} \Phi_{xx}\, dy$$

$$= - \frac{\partial}{\partial x} \int_{-h(x)}^{\eta(x,t)} \Phi_x\, dy + \overline{\Phi_x}\, \eta_x + \underline{\Phi_x}\, h_x \tag{8}$$

From eq. (7) *and eq.* (8) , *we get*

$$\overline{\Phi}_y = -\frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} \Phi_x dy + \overline{\Phi}_x \eta_x \tag{9}$$

Then the integral term of eq. (9) *will be the following relation through integration by parts*

$$\int_{-h(x)}^{\eta(x,t)} \Phi_x dy = \eta\,\overline{\Phi}_x + h\,\underline{\Phi}_x - \int_{-h(x)}^{\eta(x,t)} y\,\Phi_{xy} dy \tag{10}$$

and also the relation will be as

$$\int_{-h(x)}^{\eta(x,t)} h(x)\Phi_{xy}\,dy = h(x)\overline{\Phi}_x - h(x)\underline{\Phi}_x \tag{11}$$

Eliminating $\underline{\Phi}_x$ *from eq.* (10) *and eq.* (11) , *and this in turn lead through use of eq.* (9) *to*

$$\begin{aligned}\overline{\Phi}_y &= -\frac{\partial}{\partial x}\left[h(x)\overline{\Phi}_x - \int_{-h(x)}^{\eta(x,t)} (y+h)\Phi_{xy}dy \right] + \eta_x\Phi_x - \frac{\partial}{\partial x}(\eta\overline{\Phi}_x) \\ &= -(h\overline{\Phi}_x)_x + \frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} (y+h)\Phi_{xy}dy - \eta\,\overline{\Phi}_{xx}\end{aligned} \tag{12}$$

From the kinematic condition at free surface of eq. (2) , *taking the partial derivative with respect to t, so that*

$$\eta_t = -\frac{1}{g}\left[\overline{\Phi}_{tt} + (\overline{\Phi}_x\overline{\Phi}_{xt} + \overline{\Phi}_y\overline{\Phi}_{yt}) \right] \tag{13}$$

Then substituting eq. (13) *into eq.* (3) , *we get*

$$\overline{\Phi}_y = -\frac{1}{g}\left[\overline{\Phi}_{tt} + (\overline{\Phi}_x\overline{\Phi}_{xt} + \overline{\Phi}_y\overline{\Phi}_{yt}) \right] + \overline{\Phi}_x\eta_x \tag{14}$$

Thus, from eq. (12) *and eq.* (14) , *we have*

$$\overline{\Phi}_{tt} = (gh\overline{\Phi}_x)_x - (\overline{\Phi}_x\overline{\Phi}_{xt} + \overline{\Phi}_y\overline{\Phi}_{yt}) - g [\frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} (y+h)\Phi_{xy}dy] + g (\eta\Phi_x)_x \tag{15}$$

Up to now we have made no assumptions in addition to those made in deriving the non-linear theory. In other words, water waves propagating along an arbitrary sloping beach will be described completely in eq. (15) *. Unfortunately, since eq.* (15) *is a high order non-linear partial differential equation, so that analytical solution is not able to be obtained and the approximate solution will be presented in this paper.*

From eq. (6)*, we know that wave motion propagating along an arbitrary sloping beach comes from deep water. Accordingly, it is reasonable to imagine that the velocity potential in deep water,* ϕ_∞ *, is part of wave motion in the propagating process. In this case, the velocity potential would be proposed by the following form,*

$$\Phi = \phi + \phi_\infty \tag{16}$$

The above equation makes a brief statement that the velocity potential, Φ *, existed in the propagating process is consisted of the velocity potential in deep water,* ϕ_∞ *, and the velocity potential due to shoaling and reflection,* ϕ *, .*

After susbstituting ϕ_∞ *into the last three terms of the right side of eq.* (15)*, then the first-order approximation of wave motion will be given as*

$$\overline{\Phi}_{tt} = (gh\overline{\Phi}_x)_x - [(\overline{\phi}_\infty)_x (\overline{\phi}_\infty)_{xt} + (\overline{\phi}_\infty)_y (\overline{\phi}_\infty)_{yt}] - g [\frac{\partial}{\partial x}\int_{-h(x)}^{\eta(x,t)} (y+h) (\phi_\infty)_{xy} dy] + g [\eta (\overline{\phi}_\infty)_x]_x \tag{17}$$

Furthermore, expanding the integral term of eq. (17) *and from the* *condition that* ϕ_∞ *substituted in eq.* (14), *we have*

$$\overline{\Phi}_{tt} = (gh\overline{\Phi}_x)_x - [gh(\overline{\phi}_\infty)_x]_x + (\overline{\phi}_\infty)_{tt} \tag{18}$$

Because ϕ_∞ *is part of* Φ, *we know that* ϕ_∞ *is a particular integral of wave motion. In other wordes, eq.* (18) *involves a particular solution and a complementary function which will satisfy the following homogeneous equation.*

$$\overline{\Phi}_{tt} = (gh\overline{\Phi}_x)_x \tag{19}$$

Since the complementary function is due to shoaling and reflection of sloping bottom boundary, that the velocity potential at water surface is well to be proposed as the following form.

$$\phi = e^{i(kx-\sigma t+\epsilon)} Z(x) \tag{20}$$

where the exponential in eq. (20) *represents the factor due to reflection,* k *and* σ *are wave number and angular frequency in deep water respectively,* ϵ *is the change in phasse as wave climbing along beach.* $Z(x)$ *is the function of water surface elevation resulted from shoaling and reflection.*

After substituting eq. (20) *into eq.* (19), *and taking real part, that it is easy to transform the result into zero order of Bessel function. And the solution will be given as*

$$Z(x) = A J_0\left(2\sqrt{\frac{kx}{\alpha}}\right) \tag{21}$$

where J_0 *is the Bessel function of the first kind of order zero, and* A *is a function of* α, *which has to be determined by an appropriate condition. Therefore, the velocity potential due to shoaling and reflection will be*

$$\overline{\phi} = A J_0\left(2\sqrt{\frac{kx}{\alpha}}\right) \cos(kx - \sigma t + \epsilon) \tag{22}$$

Then the solution of wave motion can be expressed as

$$\overline{\Phi} = \overline{\phi} + \overline{\phi}_{\infty} = -\frac{gA_o}{\sigma} J_o\left(2\sqrt{\frac{kx}{\sigma}}\right)\cos(kx - \sigma t + \epsilon)$$

$$+\frac{ag}{\sigma}\cos(kx + \sigma t) \tag{23}$$

The last term of the above equation is the solution of progressive small amplitude wave at water surface, and a is the amplitude, A is to be of $A_o = -\frac{\sigma}{g}A$ 。

From the theory of reflection as light —wave, the change of phase, ϵ *, is function of bottom slope,* α *, will be found out, as*

$$\epsilon = \epsilon(\alpha) = 2\tan^{-1}\alpha \tag{24}$$

Combining eq.(23) *and eq.*(24) *then substituting in eq.*(2)*, and take the first — order approximation that we have the water surface elevation*

$$\eta(x, t) = A_o J_o\left(2\sqrt{\frac{kx}{\sigma}}\right)\sin(kx - \sigma t + 2\tan^{-1}\alpha) + a\sin(kx + \sigma t) \tag{25}$$

Based on the energy conservation at the intersecting point of mean water level and bottom boundary, where the potential energy exists only, that we find

$$A_o(\alpha) = a\left[\cos(2\tan^{-1}\alpha) + \sqrt{\cos^2(2\tan^{-1}\alpha) + 3}\right] \tag{26}$$

The relationship between A_o *and* α *has been shown in Fig. — 2 From this figure, it illustrates that* A_o *decreases with* α *increases, and as* $\alpha = \infty$ *, it becomes standing wave,* $\alpha = 0$ *that it will be a progressive wave.*

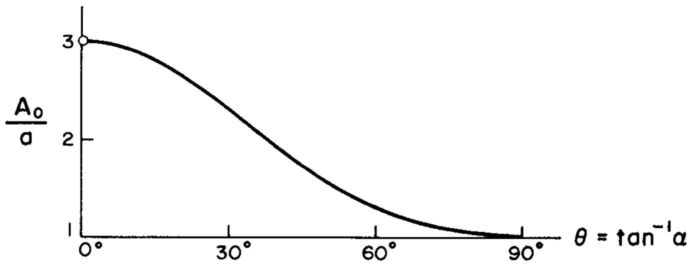

Fig. — 2

After substituting eq. (26) into eq. (25), the water surface elevation, $\eta(x, t)$, will be abtained. There are three examples presented through Fig. — 3 to Fig. — 5

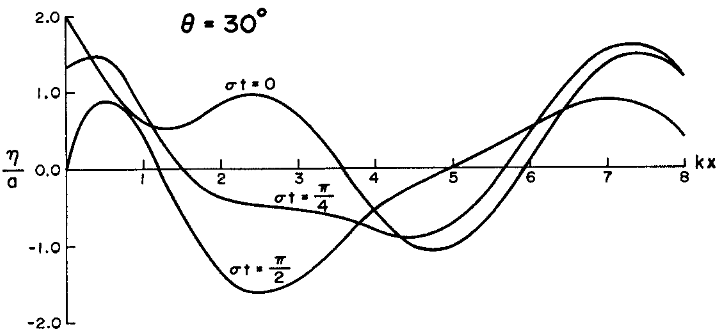

Fig. — 3

Furthermore, arrange the result of the expansion of eq. (25), that the relationship among water surface elevation, wave amplitude, wave number, bottom slope and dimensionless distance can also be expressed as the following forms,

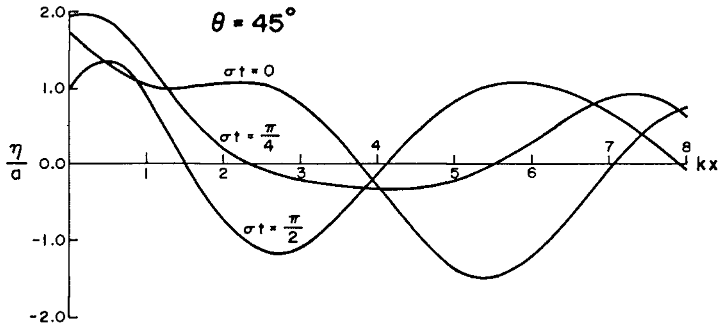

Fig. – 4

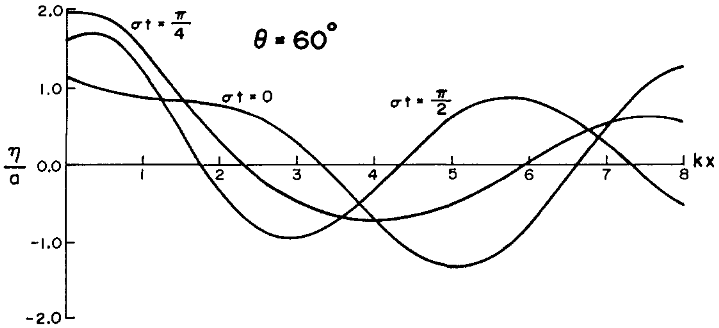

Fig. – 5

$$\eta(x,t) = \left[A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \sin(kx+\epsilon) + a \sin kx \right] \cos \sigma t$$

$$+ \left[a \cos kx - A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \cos(kx+\epsilon) \right] \sin \sigma t$$

$$= B(x,\alpha) \sin\left[m(x,\alpha) \cdot x + \sigma t \right] \tag{27}$$

and

$$B(x,\alpha) = \left\{ \left[a \sin kx + A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \sin(kx+\epsilon) \right]^2 \right.$$

$$+ \left[a \cos kx - A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \cos (kx+\epsilon) \right]^2 \}^{1/2} \tag{28}$$

$$m(x, \alpha) = \left[\tan^{-1} \left\{ \frac{a \sin kx + A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \sin (kx+\epsilon)}{a \cos kx - A_0 J_0 \left(2\sqrt{\frac{kx}{\alpha}} \right) \cos (kx+\epsilon)} \right\} + n\pi \right] / x \tag{29}$$

where n is a positive integer.

From the expression of eq. (27), the amplitude of wave motion climbing along an arbitrary sloping beach has been abtained as eq. (28) and it is function of the dimensionless distance kx and bottom slope α. That is to say, the amplitude varies along the horizontal distance due to shoaling and reflection as water waves propagate from deep water toward the shore. Several illustrations have been presented in Fig. – 6 for $\alpha = 0$, $1/\sqrt{3}$, 1, $\sqrt{3}$ *and* ∞

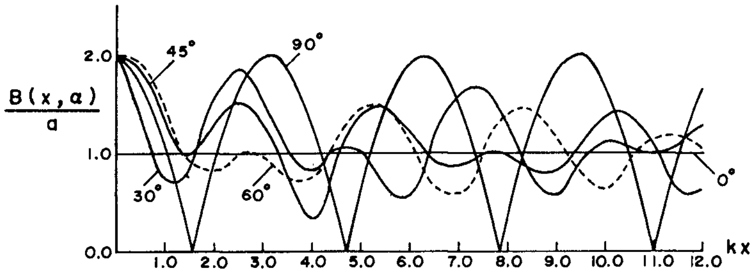

Fig.-6

Figure – 6 illustrates the solutions for the bottom slope 0°, 30°, 45°, 60° *and* 90° *according to eq. (28). And we can see that all the variation of water surface will gradually decrease and become small amplitude wave in deep water except for the case of,* $\theta = 90°$, *under that condition, it becomes a standing* *wave.*

3. *DINCUSSION AND CONCLUSION*

From the theoretical results as above metioned that the variation of water surface and amplitute are function of beach slope and dimensionless distance from the shore. And the results will be more reasonable than Stokers' (1947) *which is expressed in eq.* (30)

$$\bar{\Phi}(x,t) = A_1 [\cos(\sigma t - \epsilon_1) Y_0 (2\sqrt{\frac{kx}{\alpha}}) + \sin(\sigma t - \epsilon_1) J_0 (2\sqrt{\frac{kx}{\alpha}})] \quad (30)$$

where A_1, ϵ_1 *are the same significance as in this paper. But Stoker's solution, because contains only the linear terms so that the solution for the deep water becomes a straight line. Our solution, on the other hand, give a more reasonable solution which is a small amplitude wave when we include the potential function in deep water.*

4. *REFERENCE*

1 *Hanson, E.T., The theory of ship waves. Proc. Roy. Soc. Lond. A.* 111, 1926, 491-529.

2 *Lewy, H., Water waves on sloping beaches. Bulletin of the American Mathematical Society,* 52, 1946. 737-775

3 *Stoker, J.J., Surface waves in water of variable depth. Quarterly of Applied Mathematics,* 5, 1947, 1-54

4 *Stoker, J.J. Water waves. Institute of Mathematical Sciences New York University, New York,* 1968.

5 *Biesel, F., Study of wave propagation in water of gradually varying depth. U.S. National Bureau of Standards. Gravity wave NBS Circular* 521, 1952

6 *Carrier, G.F. and Greenspan, H.P., Water waves of finite amplitude on a sloping beach.J. Fluid Mech.*4, 1958. 97-109.

7 *Greenspan, H.P., On the breaking of water waves of finite amplitude on a sloping beach. J.Fluid Mech.* 4, 1958. 330-334.

8 *Stokes, G.G., On the theory of oscillating waves Trans. Camb. Phil. Soc. Vol.*8, 1847, *and Supplement. Scientific Paper, Vol.*1

9 *Chen Y.Y., Waves reflection and diffraction in Kaohsiung Harbor, Oceanogrophy,*6. *National Taiwan College of Marine Science and Technology,* 1978.

10 *Chen, Y.Y., Extension – poisson's Integral Formula, (in preparation),* 1981.