A NEW TWO DIMENSIONAL TIDAL MODELLINC SYSTEM

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ABSTRACT

A new method for numerical simulation of tidal currents is presented. Based on a technique involving the splitting of operators, it allows an accurate calculation of momentum advection, and wave propagation for large Courant numbers. It also provides a satisfactory treatment of tidal flat uncovering and flooding, and permits the use of a curvilinear computational grid. The generation method for such grids is presented here, followed by an engineering application on cartesian grid.

1. INTRODUCTION

Mathematical modelling of coastal shallow water areas where currents are influenced by tide and wind became a common engineering practice during the 1970's. Corresponding techniques are still developing and improving not only because faster and bigger computers are available, but, more importantly, because greater emphasis is being given to the environmental aspects of engineering works. To design the facilities themselves free surface elevation and global discharges are usually sufficient, and a fairly coarse modelling is satisfactory. However, the study of the impact that such works have on the environment asks for more refined models, in particular for a reliable simulation of currents.

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In order to satisfy these requirements the Laboratoire National d'Hydraulique (LNH-Electricité de France), Chatou, and Sogreah, Crenoble, France developed jointly a new method of numerical simulation of tidal currents and a new modelling system called CYTHERE ES 1. A schematic diagram of the system, as used at Sogreah, is shown in Fig. 1.

This system was evolved for simulation of currents and consequently great care was taken to avoid three main drawbacks inherent in many programs developed in the 1970's, namely: numerical spurious damping and dispersion, inaccuracies and "polarisation" of velocity fields in ADI (alternating direction implicit method) for greater computational time steps, and inadequate simulation of uncovering tidal flats during ebb flow. Another difficulty is the choice between finite difference and finite element algorithms, the former being more economical, the latter representing the topography more accurately; an intermediate solution was chosen.

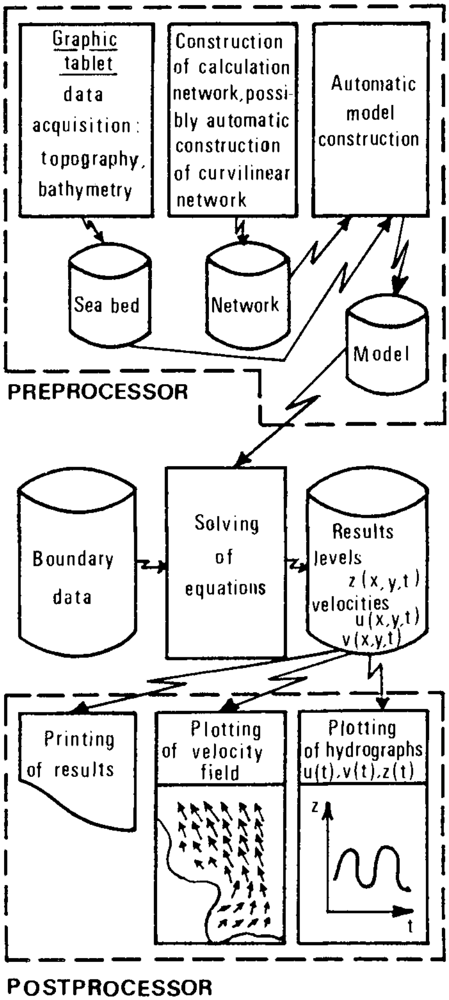
2. EQUATIONS

Nearly all industrially applied modelling systems use finite difference (FDM) discretisation of equations. Although finite elements (FEM) are often mentioned in technical literature, their application to the tidal flow equations is not common because it is expensive.

Indeed, they are no more suitable than FDM where hyperbolic equations are concerned and they lead to much higher computational costs. Their advantage as compared to FDM is that they give a much better fit with topographically complicated boundaries which are represented by stair-like steps in FDM. In order to minimise this disadvantage while using the more economical finite difference method, CYTHERE ES 1 can use an orthogonal curvilinear computational grid which improves boundary and current simulation in narrow estuaries and bays. The choice of an arbitrary (non-orthogonal) curvilinear system of coordinates was discarded because of the inaccuracies which such a transformation of equations would have introduced. Equations of tidal flow are written for an orthogonal curvilinear coordinate system by including projection parameters and metric coefficients. This allows the use of CYTHERE ES 1 in Cartesian coordinates as a special case of orthogonal curvilinear coordinates. In order to render the system efficient a preprocessor which generates a boundary fitted orthogonal curvilinear grid was developed. Its main principles are described later in this paper.

Figure 1

DIAGRAM OF CYTHERE ES1 SYSTEM USED BY SOGREAH



The two-dimensional tidal flow equations written for an orthogonal system of curvilinear coordinates (α,β) are:

$$\underbrace{\frac{\partial \vec{U}}{\partial t}}_{\mathrm{I}} + \underbrace{\vec{U}\,\overrightarrow{\mathrm{grad}}\,\frac{\vec{U}}{h} + \frac{\vec{U}}{h}\,\mathrm{div}\vec{U}}_{\mathrm{II}} + \underbrace{gh\,\overrightarrow{\mathrm{grad}}Z}_{\mathrm{III}} + \underbrace{\frac{\vec{\tau}_B - \vec{\tau}_S}{\rho}}_{\mathrm{IV}} \underbrace{-\,2\vec{\Omega}\wedge\vec{U}}_{\mathrm{V}} \underbrace{-\,\mathrm{div}(K\,\overrightarrow{\mathrm{grad}}\vec{U})}_{\mathrm{VI}} = 0 \quad (1)$$

$$\underbrace{\frac{\partial Z}{\partial t} + \mathrm{div}\vec{U}}_{\mathrm{III}} = 0 \quad (2)$$

Where:

$\vec{U}$ = (U,V) = unit width discharge vector through depth h;
Z = free surface elevation;
$\vec{\tau}_B, \vec{\tau}_S$ = bottom friction and wind stresses, respectively;
$2\vec{\Omega}\wedge\vec{U}$ = Coriolis term, $\vec{\Omega}$ being the earth's rotation speed;
K, ρ = horizontal momentum diffusivity, water density.

As mentioned in previous publication [1], it is useful to give a physical interpretation to the six groups of terms of Eqs. (1) and (2) as follows:

I. Local flow acceleration,

II. Momentum transport by advection,

III. Mass conservation and momentum transfer by propagation,

IV,V. Momentum sources or sinks due to Coriolis force, surface wind stress and bed friction,

VI. Horizontal diffusion of momentum.

The three dependent variables of Eqs. (1),(2) are: $U(\alpha,\beta,t)$, $V(\alpha,\beta,t)$ and $Z(\alpha,\beta,t)$.

3. FRACTIONAL STEP ALGORITHM

A qualitative examination of Eqs. (1) and (2) shows that they represent three physical processes with distinctive features and for which numerical approximations present different kinds of difficulties:

(i) momentum advection, groups I and II,
(ii) momentum diffusion, groups I, V and VI,
(iii) propagation, groups I,III and IV.

The algorithm used in CYTHERE ES 1 is based on the recognition of three physical phenomena and proceeds as follows. Suppose that the state of the model is known at time $n\Delta t$ (Z^n, U^n, V^n) known at all points), the new state at time $(n+1)\Delta t$ is computed by the successive resolutions of the advection, diffusion and propagation operators. Each step is solved by a different numerical method so as to increase the accuracy and economy of the solution.

3.1 First step - Advection

The finite difference discretisation of advective terms in flow equations is the most important source of spurious damping and dispersion, both due to discretisation errors. To avoid this, space-centered finite differences are used such as:

$$\frac{\partial u}{\partial x} \simeq \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \quad ; \quad \frac{\partial u}{\partial y} \simeq \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \tag{3}$$

Where:

- (x,y) = horizontal coordinate axes,
- $u(x,y,t)$ = flow velocity in x-direction,
- (i,j) = indices of computational points in x- and y- directions respectively,
- $(\Delta x, \Delta y)$ = space computational steps in x- and y- directions respectively.

Approximation equation (3) is of 2^{nd} order if Δx = const, Δy = const. Numerical damping due to this approximation is negligible but dispersion is important and hampers the results so much that higher order approximations (4th order) are sought. The results are apparently more satisfactory, but this approach was not followed by the authors, who think that the high accuracy may often be misleading. Indeed, if the computational grid is not uniform (Δx,Δy variable) the order of approximation decreases. More importantly a higher order of approximation is theoretically better when assuming that Δx, Δy are small. Typically these intervals vary between 100 m and 1000 m. For a computational grid with Δx = 200 m Eq. (3) approximates the derivative with a difference of two velocities at points which are 400 meters apart! Fourth order approximation would involve 5 points distributed over a distance of 1000 m. In such a situation, the higher order approximation has the effect of smoothing and spreading out the differences through interpolation polynomials and local variations of advective currents cannot be accurately computed.

For the above reasons advective terms in the modelling system presented here were treated by the method of characteristics based on the principle derived by Holly and Preissmann [4]. During the advection step the following system of equations is solved between the time $n\,\Delta t$ and an intermediate time notation n+1/3:

$$\frac{u^{n+1/3} - u^{n}}{\Delta t} + \frac{u_\alpha}{e_\alpha}\frac{\partial u}{\partial \alpha} + \frac{u_\beta}{e_\beta}\frac{\partial u}{\partial \beta} = 0 \; ; \; \frac{v^{n+1/3} - v^{n}}{\Delta t} + \frac{u_\alpha}{e_\alpha}\frac{\partial v}{\partial \alpha} + \frac{u_\beta}{e_\beta}\frac{\partial v}{\partial \beta} = 0 \tag{4}$$

Where:

(u,v) = velocity components along x- and y- axes of cartesian coordinate system,

u_α, u_β = projections of the velocity vector on the α- and β- axes of curvilinear coordinate system,

e_α, e_β = metric coefficients such that:

$$e_\alpha = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2\right]^{1/2} \qquad e_\beta = \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right]^{1/2} \tag{5}$$

The solution $u^{n+1/3}$, $v^{n+1/3}$ of Eq(4) is obtained by the method of characteristics in two dimensions as described in [1]. Although the computations are made at specified intervals, the method described in [4] enables numerical damping and dispersion to be almost completely avoided. Once intermediate velocities $u^{n+1/3}$, $v^{n+1/3}$ are found for all computational points, unit discharges $U^{n+1/3} = h^n\, u^{n+1/3}$, $V^{n+1/3} = h^n\, v^{n+1/3}$ can be computed.

3.2 Second step - Diffusion

Diffusion step equations are solved between two intermediate time notations n+1/3 and n+2/3 in terms of unit discharges.

$$\frac{U^{n+2/3} - U^{n+1/3}}{\Delta t} = \frac{1}{e_\alpha e_\beta}\left[\frac{\partial}{\partial \alpha}\left(K \frac{\partial U}{\partial \alpha}\frac{e_\beta}{e_\alpha}\right) + \frac{\partial}{\partial \beta}\left(K \frac{\partial U}{\partial \beta}\frac{e_\alpha}{e_\beta}\right)\right] + FV \tag{6}$$

$$\frac{V^{n+2/3} - V^{n+1/3}}{\Delta t} = \frac{1}{e_\alpha e_\beta}\left[\frac{\partial}{\partial \alpha}\left(K \frac{\partial V}{\partial \alpha}\frac{e_\beta}{e_\alpha}\right) + \frac{\partial}{\partial \beta}\left(K \frac{\partial V}{\partial \beta}\frac{e_\alpha}{e_\beta}\right)\right] - FU \tag{7}$$

Where F is the Coriolis acceleration parameter. Eqs.(6) and (7) are well-known parabolic equations. Their numerical solutions are accurate even with crude methods, so an ADI approach based on a fully implicit finite difference scheme is used for this step in CYTHERE ES 1. It gives new values $(U,V)^{n+2/3}$ at all computational points.

3.3 Third step - Propagation

This step is a crucial one because of the computer time it requires and also because of the numerical inaccuracies generated when economy in computer time is sought by using inadequate methods. "Polarisation" of velocity fields when using Alternating Directions (ADI) methods was described firstly in [3] then in [1]. Let us define Courant number Cr as:

$$Cr = \Delta t \sqrt{gh} \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}} \tag{8}$$

where:

Δt = computational time step,
h = local flow depth,
g = acceleration due to gravity.

Nearly all industrial programs use implicit finite difference schemes which enable Cr values of greater than 1 to be used, whereas explicit schemes are numerically unstable in these conditions. Since the fully implicit methods lead to excessive computational times, the ADI implicit methods are widely used. These methods, however, may give completely wrong results for Courant numbers greater than 5 to 10. Indeed the velocity fields become "polarized" along either x- and y- axes when the time step Δt corresponds to Cr greater than 5. This phenomenon is described in reference [1] and examples of obviously absurd results obtained with the ADI method for higher Cr values are given. Moreover the ADI method cannot correctly compute velocity fields along and across channels which are not parallel to the x- or y- axis, as has been shown in [6]. For these reasons a more efficient and accurate algorithm was used by the authors.

Working equations for the propagation step are:

$$\frac{Z^{n+1} - Z^n}{\Delta t} = - \frac{1}{e_\alpha e_\beta} \left[\frac{\partial}{\partial \alpha} (U_\alpha e_\beta) + \frac{\partial}{\partial \beta} (U_\beta e_\alpha) \right] \tag{9}$$

$$\frac{U_\alpha^{n+1} - U_\alpha^{n+2/3}}{\Delta t} = - gh \frac{1}{e_\alpha} \frac{\partial Z}{\partial \alpha} + \frac{U_\alpha}{h} \frac{\partial Z}{\partial t} - g \frac{U_\alpha \|\vec{U}\|}{C^2 h^2} + \frac{1}{\rho} \tau_{S\alpha} \tag{10}$$

$$\frac{U_\beta^{n+1} - U_\beta^{n+2/3}}{\Delta t} = - gh \frac{1}{e_\beta} \frac{\partial Z}{\partial \beta} + \frac{U_\beta}{h} \frac{\partial Z}{\partial t} - g \frac{U_\beta \|\vec{U}\|}{C^2 h^2} + \frac{1}{\rho} \tau_{S\beta} \tag{11}$$

Where:

C = Chezy coefficient,
$\|\vec{U}\| = (U_\alpha^2 + U_\beta^2)^{1/2}$.

Lack of space does not allow the authors to give details of the discretisation, the main principles of which can be found in [1]. Essentially, right hand terms of Eqs (9) to (11) are written under implicit formulation such as:

$$\frac{gh}{e_\alpha}\frac{\partial Z}{\partial \alpha} = \frac{g}{e_\alpha}\left[\theta h^{n+1}\left(\frac{\partial Z}{\partial \alpha}\right)^{n+1} + (1-\theta)h^{n}\left(\frac{\partial Z}{\partial \alpha}\right)^{n}\right] \qquad (12)$$

Then, from Eqs. (10) and (11) expressions for increments ΔU, ΔV during one time step Δt, as functions of known values and Z^{n+1} are extracted, (Note: $\Delta U = U^{n+1} - U^{n+2/3}$, etc). These increments are substituted into Eq (9) which, discretised, becomes an algebraic system of equations for the Z_k^{n+1} , (k = 1, 2, ..., m), where m is the number of computational points in the model.

This final system of equations is solved by an iterative method which is based upon the conjugate gradient method and is, with the method of characteristics used for the advection step, of crucial importance in the algorithm. Indeed, nowadays models of tidal areas can contain anywhere between 2000 and 8000 computational points; this effectively prohibits matrix inversion. Explicit formulation would lead to excessively small time steps. The ADI methods may give, as mentioned above, misleading results even when the time step is physically reasonable, hence the use of a special development of the conjugate gradient method (iteration through alternating direction operator with coordinator) described in [5].

The iteration procedure permits the use of high Courant numbers (20 and more) without loss of accuracy even when simulating channels angled at 45° to the coordinate axes. It should be stressed that with efficient programming one iteration needs less computer time than resolution of the propagation step by the explicit method.

4. ORTHOGONAL CURVILINEAR GRID GENERATION

The generation of curvilinear computational grid is semi-automatic in the CYTHERE ES 1 system at the preprocessor level (see Fig. 1). The details of the method, developed by Sogreah, are given elsewhere [2], only the main features are followed here.

The mathematical tool used to generate the orthogonal curvilinear grid is the conformal transformation. It is assumed that for a given domain D, D being a coastal zone or an estuary for example, which satisfies a number of conditions, there is a conformal transformation f which associates a rectilinear domain D' to D (a rectilinear domain is defined as a domain bounded by straight lines intersecting at right angles).

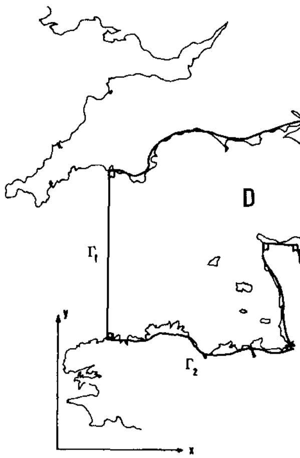


Fig. 2a Simplified domain D for the English Channel

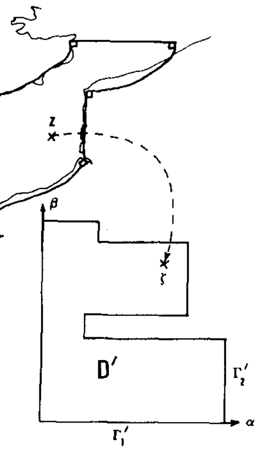


Fig. 2b A priori shape of domain D'

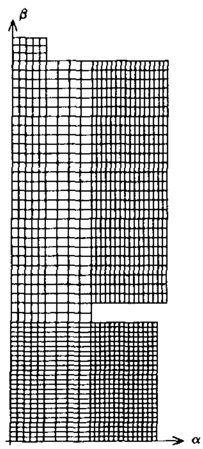


Fig. 3a Final shape of D' with cartesian grid

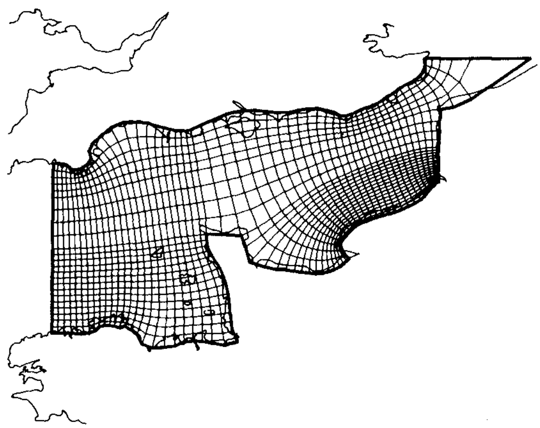


Fig. 3b Orthogonal Curvilinear grid in D

Using complex notations, to any point $Z = x + iy$ in D, f associates a point $\zeta = f(Z) = \alpha + i\beta$ in D'. The function f is analytic, hence α and β form a conjugate harmonic pair $\Delta\alpha = 0, \Delta\beta = 0$ and they satisfy the Cauchy-Riemann equations:

$$\frac{\partial\alpha}{\partial x} = \frac{\partial\beta}{\partial y} \qquad \frac{\partial\alpha}{\partial y} = -\frac{\partial\beta}{\partial x} \tag{13}$$

which implies that lines of constant α are orthogonal to lines of constant β . These conditions must hold true on the boundaries which are formed of segments on which either α or β is constant.

This method is applied to jagged coastlines or estuaries. In a first approach, the real topography of the domain is approximated by a simplified contour which represents only the most characteristic features. The size of the features to be taken into account for the model depends, of course, on the scale of the phenomenon to be modelled. The physical domain is thereby schematised into a simplified domain D (e.g. Fig. 2a).

The domain D must satisfy a number of conditions. It must be simply connected in the (x,y) plane, i.e. its boundary is a continuous line and D does not contain holes. Next, its boundary Γ has to be composed of an even number of curved segments Γ_i intersecting at right angles, $\Gamma = \cup \Gamma_i$. Hence, the transformed domain D' will be delimited by a contour $\Gamma' = \cup\Gamma'_i$, Γ'_i being the image of Γ_i, so that all straight segments Γ'_i and Γ'_{i+1} intersect at right angles, since the conformal transformation keeps the values of angles. Using this property, the contour Γ' can be drawn a priori, simply by turning at the end of each Γ'_i of an angle equal to the angle between Γ_i and Γ_{i+1}.

Thus, the domain D' is defined in a (α,β) plane where the α- axis is parallel to horizontal segments and the β- axis parallel to vertical ones. The direction of axes is chosen according to the contour orientation (see Fig. 2b).

The conformal mapping f is then computed numerically, with the help of a finite element method by solving a Laplacian equation with a Neumann boundary condition given on the contour Γ, and the final shape of D', i.e. the length of each Γ'_i is determined, as explained in [2].

In the second stage a Cartesian grid is built inside the transformed rectilinear domain D' parallel to the α and β axes, the spacing between vertical and horizontal lines need not to be constant (Fig. 3a). Using the inverse conformal transformation f^{-1}, this Cartesian grid will be mapped into an orthogonal curvilinear grid inside D (see Fig. 3b) since the property of orthogonality is conserved.

Numerical computation of f^{-1} uses the same finite element program as for f, for the simultanous resolution of four Laplacian equations with Dirichlet boundary conditions. Finally projection parameters and metric coefficients e_α, e_β of Eq. (5) are computed automatically.

The spacing between Cartesian grid lines in D' has been chosen to suit the desired spacing in the curvilinear grid, however the packing of the curvilinear grid lines obtained may be slightly different from that expected. It is not possible to add lines after the resolution of the second stage computations unless the process is reiterated from a new Cartesian grid inside D'. On the other hand, some of the lines can be eliminated provided that the new spacing satisfies a regular increase or decrease acceptable by the finite difference program. According to user's choice, one final program "erases" lines of the curvilinear grid, checks the regularity of the spacing and establishes a correspondence between the lines that have been kept and integer numbering of the finite difference grid (I = 1, 2,... ; J = 1, 2,...). The example of the orthogonal grid generated by the preprocessor shown in Fig. 3b illustrates the method. All points are determined by the intersection of two orthogonal lines, but since they are joined on the drawing by straight segments which do not exactly coïncide with the lines, some drafted angles are different from 90°, as in the top-right corner.

5. EXAMPLE OF AN APPLICATION TO AN ENGINEERING STUDY

The CYTHERE ES 1 modelling system has recently been used by Sogreah for a complex engineering study of development in South Korea: Kwang Yang Bay. The project consists of reclaiming a vast area in the bay located on river delta soils, and of dredging from a nearby area. A steel mill will be built on the reclaimed site together with a harbour for raw materials and finished products, for up to 250,000 dwt ships. The study involved:

- in-situ measurements;
- two-dimensional mathematical modelling of tidal currents in the bay with the projected steel mill, two options of river discharge: average or flood, and three tide conditions;
- one-dimensional mathematical modelling of the tide influenced section of the major river flowing into the bay;
- mathematical modelling of large ship navigation in the entrance channel in the bay, based on currents predicted for the new conditions (reclaimed and dredged area);

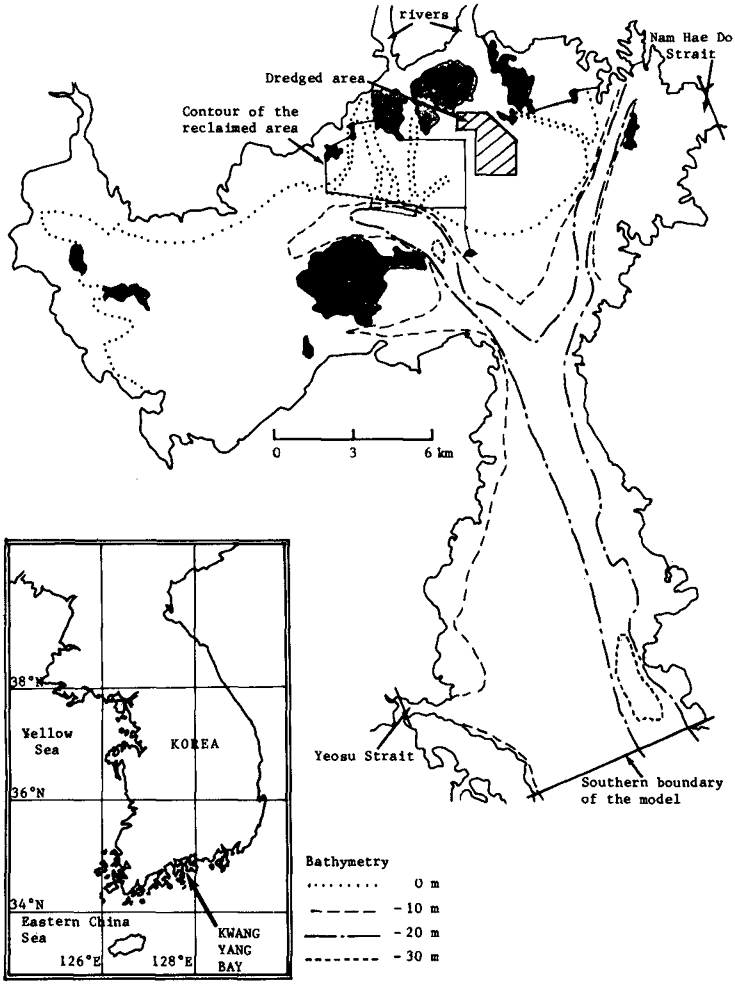


Fig. 4 Map of Kwang Yang Bay
(islands in dark)

- scale model tests (1:25), in a specially equipped lake, of ship manoeuvres in the harbour, conducted by ship's pilots. The lake dimensions are 255 m by 190 m and the test crafts were scaled to represent up to 250,000 dwt ships;

- sediment transport evaluation.

Figure 4 shows a map of the bay, which has five open boundaries; two rivers to the north with given constant discharge, and three boundaries with time dependent water conditions obtained from in-situ measurements: the southern boundary directly connected to the Eastern China Sea, the Nam Hae Do Strait to the north east and Yeosu Strait to the south west, both connected to adjacent bays.

The grid was cartesian with irregular intervals in the x and y directions, and varying Strickler friction coefficients were used to account for sea weed culture on the tidal flats.

A most important feature of the model was its rapidly varying bathymetry: within a distance of less than 600 meters (three computational points) the bottom depth goes from a positive value (above lowest low water datum) to -20 meters. The original tidal flat computation procedure used in the propagation step (see [1]) and its link with deep water computation proved to be the critical point of the simulation.

Once the calibration of the model was completed, computed levels and currents compared very well with measured values (fig. 5). The current phase agreement was quite remarkable, and of particular importance as currents computed with the proposed facilities were to be analysed to define the time period of ship entry into the harbour. Current fields were drawn at different times during the tidal cycles. Figure 6 shows the currents in the vicinity of the steel mill, they are parts of the current charts drawn for spring tide at high water and at 4 hours before high water.

6. CONCLUSION

The original mathematical method used in the CYTHERE ES 1 method enables very good accuracy to be obtained at a reasonable computer cost. It has been applied to many engineering studies including modelling of the thermal impact of 30 coastal nuclear power plants in France by the LNH [7], investigation of ocean eroding action (river Canche estuary, France, LNH), a pollution study (bay of Saint-Brieuc, France, Sogreah), ... New developments include modelling of wind-induced currents and this is being applied to the study of a lake.

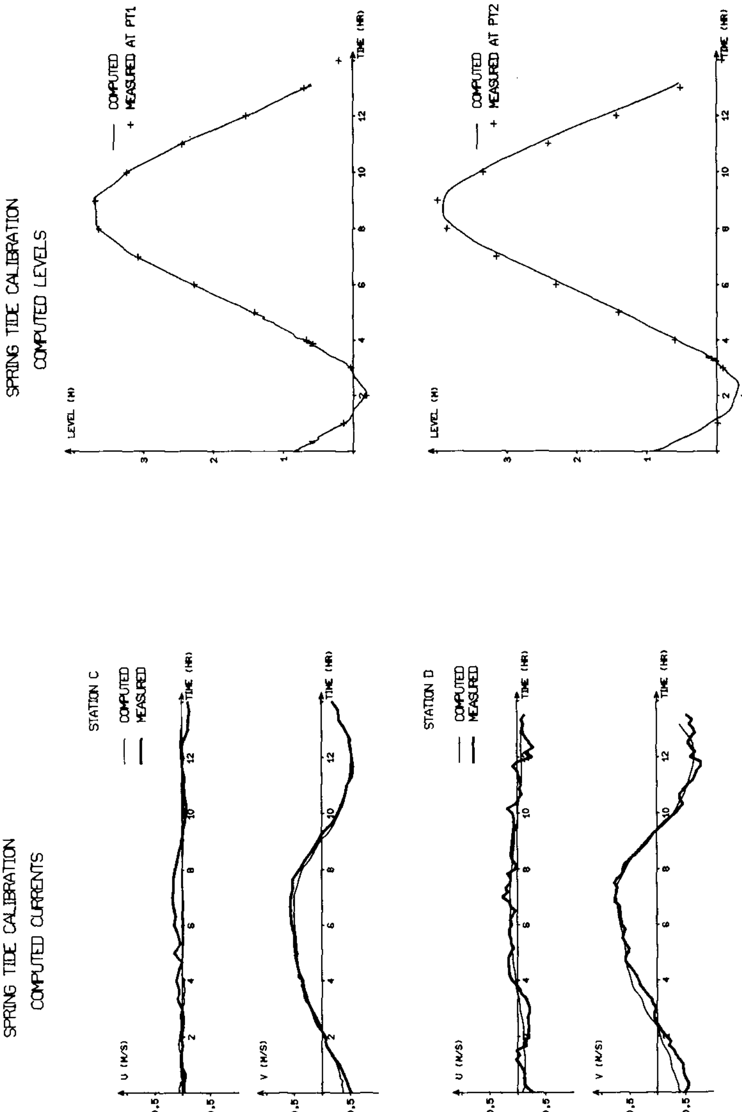


Fig. 5 Comparison of currents and levels for calibration of the model

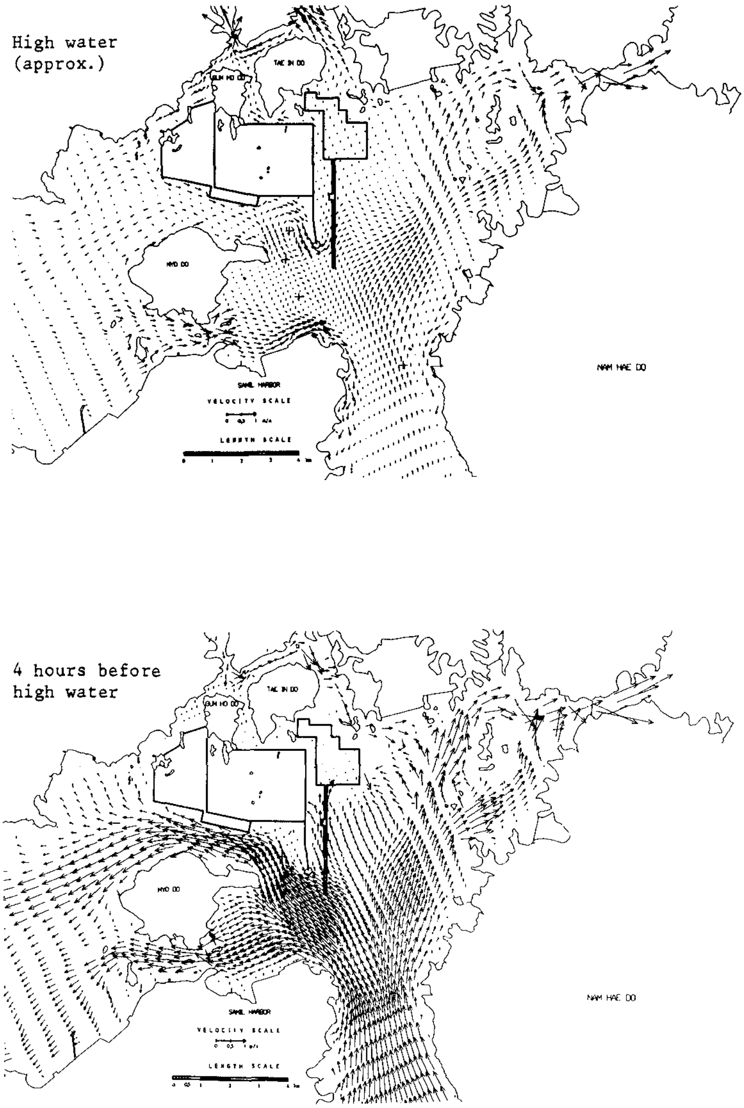


Fig. 6 Current fields for spring tide with simulated reclaimed and dredged areas

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