

CYLINDRICAL LONG WAVE INTO A CYLINDRICAL SHELF

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ABSTRACT

Based on the linear non-dispersive theory, the reflection of a Converging Cylindrical long wave, of wave length L , onto a Cylindrical shelf, of radius $r = a$ and positive or negative height Δh relative to an otherwise flat bottom, is study analitically. It is found that these linear approximation agrees well with the existing non-linear numerical solution when the ratio a/L is large enough. It is also found that these two-dimensional reflection process is the contrary of the corresponding one-dimensional case, since the solution of these problem gives a negative reflected wave for a positive step and a positive reflected wave for a negative step.

1. INTRODUCTION

In this paper we will discuss the reflection of Cylindrical long waves by a submerged Cylindrical shelf of radius $r = a$. Solving, analitically the linear nondispersive Cylindrical wave equation.

It is known that for infinitesimal wave on constant depth, water motion in long waves is essentially horizontal, implying that the vertical variation is weak and the pressure is hydrostatic. If we consider a vertical fluid column of base section $dr r d\theta$ and height $h + \eta$, where h is the water depth and η is the wave amplitude measured from the undisturbed water surface, the rate of change of fluid volume in the column is $\frac{\partial \eta}{\partial t} dr r d\theta$. If, the vertical variation in the horizontal velocity is ignored as is suggested above, the net rate of volume flux into the column is

$$- \nabla \cdot \bar{u} (h + \eta) r dr d\theta,$$

where ∇ denotes the horizontal gradient in polar coordinates

$$\left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right).$$

Mass conservation is satisfied, if the two rates are equal, (incompressible fluid) hence

$$\frac{\partial \eta}{\partial t} + \nabla \cdot \bar{u} (h + \eta) = 0.$$

(1)

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For waves having a vertical axis of symmetry at all times, the above equation can be written as

$$n_t + \frac{1}{r} ((h+n)r u)_r = 0, \quad (2)$$

where u is the radial velocity considered constant in the entire depth. In terms of the velocity potential, inviscid and irrotational flow, we have, that the linear version of equation (2) is

$$n_t + \frac{h}{r} (r \phi_r)_r = 0. \quad (3)$$

In the momentum balance, we only consider horizontal components and neglect convective inertia (non-linear term), therefore

$$\bar{u}_t = -\frac{1}{\rho} \nabla p. \quad (4)$$

Assuming now, that the pressure is hydrostatic,

$$p = \rho (n-z)g,$$

The momentum equation becomes

$$\bar{u}_t = -g \nabla n. \quad (5)$$

Introducing the velocity potential into equation (5), we obtain

$$n = -\frac{1}{g} \frac{\partial \phi}{\partial t}. \quad (6)$$

Substituting equation (6) into equation (3), we have

$$\frac{1}{C^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r}, \quad (7)$$

The above equation is the linear nondispersive Cylindrical wave equation, where $C = \sqrt{gh}$ is the shallow water phase velocity.

Chwang and Wu (1976) investigated the reflection and transmission of a converging cylindrical solitary wave due to a circular step of positive height Δh and radius $r = a$, based on a numerical solution of the three dimensional Boussinesq equation (non-linear theory). Their main conclusion was that after incident wave reaches the circular step, the leading reflected wave takes the form of a negative wave propagating in the positive "r" direction (see figure (1)). Wu (1979) presented a comparative numerical study of the above problem, using different theories of wave propagation. From Wu's work, it is possible to conclude that when a/L is large enough, where L is the wave length, the reflected waves predicted by the Boussinesq theory and the linear nondispersive long wave model are in good agreement (compare figure 2 with figure 1), just as the transmitted waves predicted by the mentioned theories are different, result that can be more appreciated when the waves approach the origin (focusing process). A non-linear analytic solution of this focusing process has been presented by Chwang and Power (1981), based on the inner-outer expansions technique to the Cylindrical Boussinesq equation.

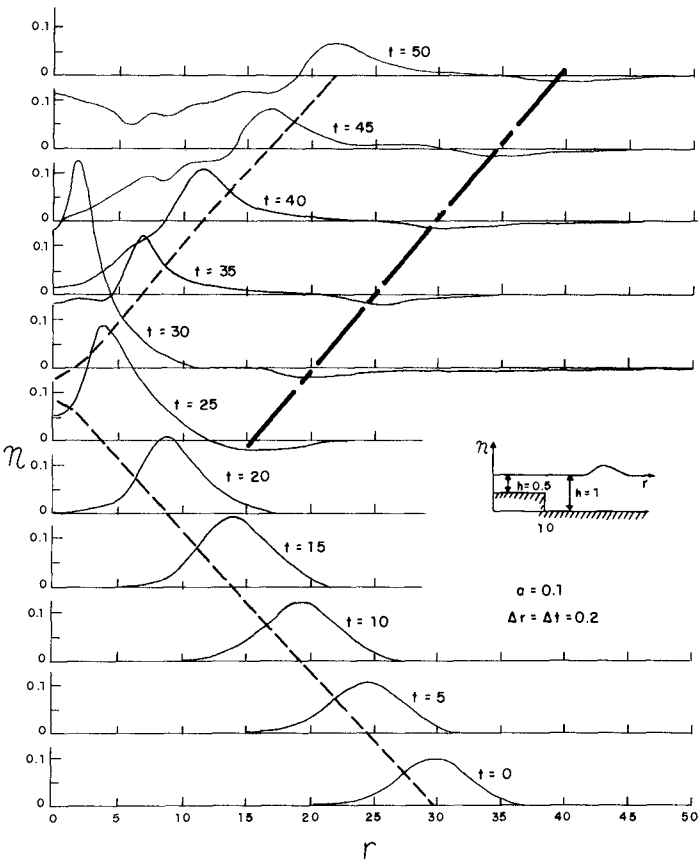


FIG. # 1

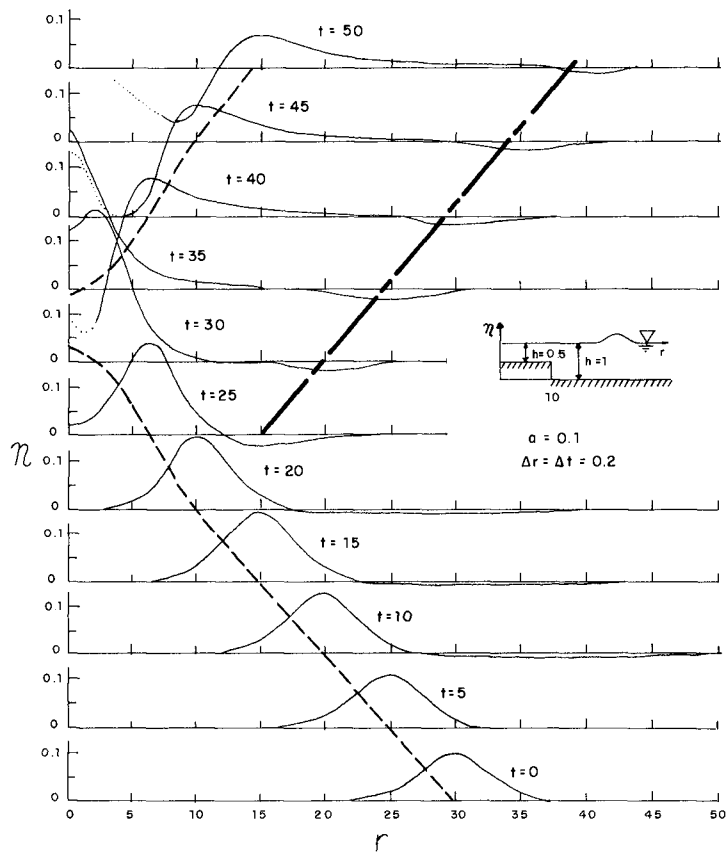


FIG. # 2

The solution of such a problem in one dimension (reflexion of a planar wave due to abrupt change in the water depth) is extremely easy, owing to the simple general integral of the one-dimensional wave equation. As it is known in one dimension we have the deviations from the mean water level as

$$\eta_I = f\left(t + x - \frac{x_0}{c_1}\right) \quad (8)$$

for the incident wave,

$$\eta_r = \frac{c_1 - c_2}{c_1 + c_2} f\left(t - \frac{x + x_0}{c_1}\right) \quad (9)$$

for the reflected wave, and

$$\eta_t = \frac{2c_1}{c_1 + c_2} f\left(t + \frac{x - \frac{c_2}{c_1} x_0}{c_2}\right) \quad (10)$$

for the transmitted wave, where the incident, transmitted and reflected waves have the same shapes, and for $(C_1 = \sqrt{gh_1}) > (C_2 = \sqrt{gh_2})$ the reflected wave takes the form of a positive wave propagating through in finity. The above conclusion for one dimensional wave is totally different from the one found by Chwang and Wu numerical solution for a two dimensional wave.

2. CYLINDRICAL LONG WAVES IN WATER OF CONSTANT DEPTH

In these section we are interested in studying the propagation of cylindrical incoming long waves in water of constant depth, whose maximum wave amplitude is located at $r = r_0$, sufficiently far from the origin when $t = 0$. To do it, we will follow Lamb(1902) original paper, "on wave propagation in two dimensions", and Whitham (1974).

To find a general solution for incoming waves of equation 7, we can use two different approaches, one consist on a Fourier superposition of the periodic incoming solution of equation 7 and the other technic consists on a uniform line distribution on the z axis, of three-dimensional point wave sources, the total disturbance from this line distribution is clearly a function only of the distance r from the z axis and the time t .

The wave equation in spherical coordinates reduces to

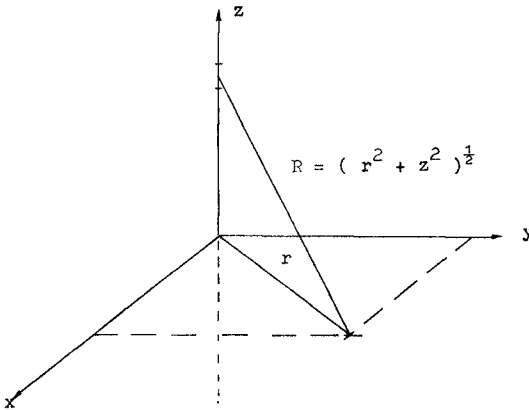
$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial R^2} + \frac{2}{R} \frac{\partial \phi}{\partial R}, \quad (11)$$

when $R = \sqrt{r^2 + z^2}$. The source solution of the above equation producing incoming wave whose maximum wave amplitude is located at $R = r_0$, is

$$\phi = -\frac{1}{4\pi R} f\left(t + \frac{R - r_0}{c}\right). \quad (12)$$

The total potential produced by a line distribution of the above source is

$$\phi\left(\frac{r-r_0}{c}, t\right) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{R} f\left(\frac{t+R-r_0}{c}\right) dz. \quad (13)$$



Let $z = r \sinh u$, therefore $R = r \cosh u$, and substituting this change of variable in equation (13), we obtain

$$\phi = -\frac{1}{2\pi} \int_0^{\infty} f\left(t - \frac{r_0}{c} + \frac{r}{c} \cosh u\right) du. \quad (14)$$

To verify that the above equation is a solution of equation (7), subject to certain condition, let us substitute equation (14) into equation (7). Then

$$\begin{aligned} 2\pi\left(c^2\left(\phi_{rr} + \frac{1}{r}\phi_r\right) - \phi_{tt}\right) &= \int_0^{\infty} \left(\cosh^2 u f''(\xi) + \frac{c}{r} \cosh u f'(\xi) - f''(\xi)\right) du \\ &= \int_0^{\infty} \frac{c^2}{r^2} \frac{\partial^2}{\partial u^2} f(\xi) du \\ &= \frac{c}{r} \left(\sinh u f'(\xi)\right)_{u=0}^{u=\infty}, \end{aligned} \quad (15)$$

where $\xi = (t - r_0/c) + (r/c) \cosh u$. Equation (15) is identically equal to zero, therefore (14) is a solution of (7), if and only if f' tends to zero faster than $\sinh u$ tends to infinity when u tends to infinity. Then under the above condition, equation (14) is a two-dimensional source of strength $f(t - r_0/c)$, producing cylindrical incoming waves, whose maximum wave amplitude is located at $r = r_0$ when $t = 0$. Since

$$\begin{aligned} 2\pi r \phi_r &= -\frac{r}{c} \int_0^\infty f'(\xi) \cosh u \, du = -\frac{r}{c} \int_0^\infty (\sinh u + e^{-u}) f'(\xi) \, du \\ &= -\left(\int_0^\infty \frac{\partial}{\partial u} f(\xi) \, du + \frac{r}{c} \int_0^\infty e^{-u} f'(\xi) \, du \right) \\ &= f(t - r_0/c + r/c) - \frac{r}{c} \int_0^\infty e^{-u} f'(\xi) \, du, \end{aligned} \quad (16)$$

Therefore, the following limit gives the source strength

$$\lim_{r \rightarrow 0} 2\pi r \phi_r = f(t - r_0/c), \quad (17)$$

$r \rightarrow 0$

or in other words we have a two dimensional source of strength $f(t')$, where $t' = t - r_0/c$, whose initiation time is $t' = -r_0/c$.

The corresponding wave profile to the above two-dimensional potential is found by substituting equation (14) into equation (6). Then

$$n\left(\frac{r-r_0}{c}, t\right) = \frac{1}{2\pi g} \int_0^\infty f'(t - r_0/c + \frac{r}{c} \cosh u) \, du, \quad (18)$$

Now at $t = 0$ we have

$$n\left(\frac{r-r_0}{c}, 0\right) = \frac{1}{2\pi g} \int_0^\infty f'\left(\frac{r}{c} \cosh u - \frac{r_0}{c}\right) \, du. \quad (19)$$

Therefore, the problem is reduced to solving the above integral equation of the first kind when the initial wave profile,

$n((r-r_0)/c, 0) = n_1((r-r_0)/c)$, is known.

The solution of equation (19) is

$$f'\left(\frac{r}{c} \cosh u - \frac{r_0}{c}\right) = -4g \frac{r}{c} \cosh u \int_0^\infty n_1'\left(\frac{r}{c} \cosh v - \frac{r_0}{c}\right) \, dv, \quad (20)$$

provided that $n_1(\infty) = 0$

To prove equation (20), let's substitute equation (20) into equation (19)

$$n_i \left(\frac{r-r_0}{c} \right) = - \frac{2}{\pi} \int_0^\infty \frac{r}{c} \cosh u \int_0^\infty n'_i \left(\frac{r}{c} \cosh u \cosh v - \frac{r_0}{c} \right) dv du \quad (21)$$

Let $z = r/c \cosh u \cosh v$ and $y = r/c \cosh u \sinh v$ so that $J(u, v) = (r/c)^2 \sinh u \cosh u$ transforming the u and v independent variables in the double integral of equation (21) to z and y variables, then we have

$$\begin{aligned} n_i \left(\frac{r-r_0}{c} \right) &= - \frac{2}{\pi} \int_0^\infty \frac{r}{c} \int_0^\infty \frac{n'_i(z-r_0/c)}{(z^2 - y^2 + r^2/c^2)^{1/2}} dy dz \\ &= - \frac{2}{\pi} \int_0^\infty \frac{r}{c} n'_i(z-r_0/c) (\sin^{-1} \left(\frac{y}{(z^2 - r^2/c^2)^{1/2}} \right))^{(z^2 - r^2/c^2)^{1/2}} dz \\ &= - \frac{2}{\pi} \int_0^\infty \frac{r}{c} n'_i(z-r_0/c) dz \\ &= n_i \left(\frac{r-r_0}{c} \right), \end{aligned} \quad (22)$$

Since $n_i(\infty) = 0$, and equation (20) has being proved.

3. THE REFLECTION OF A SOLITARY WAVE

Consider a single cylindrical wave moving through the origin in water of constant depth h_1 . Due to a discontinuous change in the depth by a Cylindrical shelf of depth h_2 and radius $r = a$, some of the incoming energy is transmitted beyond the step and the remaining part is reflected backwards. Let the incident wave be

$$n_i = \int_0^\infty g \left(t - \frac{r_0}{c_1} + \frac{r}{c_1} \cosh u \right) du \quad (23)$$

with $C_1 = \sqrt{g h_1}$ the wave celerity in the region $r > a$. In this region, $r > a$, besides the incident wave there must be a reflected wave propagating through infinity that we may choose it to be

$$n_r = \int_0^\infty R \left(t + \frac{r}{c_1} - \frac{r}{c_1} \cosh u \right) du. \quad (24)$$

In a similar way, in the region $r \leq a$ we have a transmitted wave propagating through the origin with wave celerity $C_2 = \sqrt{g h_2}$ that we may write it as

$$n_t = \int_0^{\infty} T(t - r_t/c_2 + \frac{r}{c_2} \cosh u) du. \quad (25)$$

The function g is known and R and T at $r = a$ are to be found from the matching of the pressure and volume flux at the edge of the cylinder $r = a$, as is usually done. Then, we have

$$\begin{aligned} n_I + n_r &= n_t \\ h_1 \frac{\partial}{\partial r} (\phi_I + \phi_r) &= h_2 \frac{\partial \phi}{\partial r} \quad , \quad a \leq r = a \end{aligned} \quad (26)$$

In this way, we get

$$R(t + r/c_1 - \frac{a}{c_1} \cosh u) = \frac{c_1 - c_2}{c_1 + c_2} g(t - r_0/c_1 + a/c_1 \cosh u), \quad (27)$$

and

$$T(t - r/c_2 + a/c_2 \cosh u) = \frac{2c_1}{c_1 + c_2} g(t - \frac{r_0}{c_1} + \frac{a}{c_1} \cosh u). \quad (28)$$

The prolongation of the relation (27) in the region $r > a$, can be given by

$$R(t + r/c_1 - \frac{r}{c_1} \cosh u) = \frac{c_1 - c_2}{c_1 + c_2} g(t - \frac{r_0}{c_1} - \frac{(r - 2a)}{c_1} \cosh u). \quad (29)$$

Substituting equation (19) into equation (24) for the reflected wave. We get

$$n_r = \frac{c_1 - c_2}{c_1 + c_2} \int_0^{\infty} g(t - \frac{r_0}{c_1} - \frac{(r - 2a)}{c_1} \cosh u) du, \quad (30)$$

The above relation is not a solution of the cylindrical wave equation, as can be seen by substituting the following potential

$$\phi = \int_0^{\infty} f(t - \frac{(r - r')}{c} \cosh u) du \quad (31)$$

into the cylindrical wave operator. Since

$$\begin{aligned} & (c^2 (\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}) - \frac{\partial^2}{\partial t^2}) \int_0^{\infty} f(t - \frac{(r - r')}{c} \cosh u) du = \\ & \int_0^{\infty} (\frac{c}{r} f' (t - \frac{(r - r')}{c} \cosh u) \cosh u - f'' (t - \frac{(r - r')}{c} \cosh u) \sinh^2 u) \\ & du = \frac{c^2}{(r - r')^2} \int_0^{\infty} (-\frac{\partial^2}{\partial u^2} (f(t - \frac{(r - r')}{c} \cosh u))) du - \frac{c}{(r - r')} \\ & \frac{r'}{r} \int_0^{\infty} f' (t - \frac{(r - r')}{c} \cosh u) \cosh u du \end{aligned}$$

$$\begin{aligned}
&= - \frac{c^2}{(r - r')} \frac{r'}{r} \int_0^\infty f' \left(t - \frac{r - r'}{c} \cosh u \right) \cosh u / c \, du \\
&= - \frac{c^2}{(r - r')} \frac{r'}{r} \frac{\partial \Phi}{\partial r} \neq 0 \quad \text{for } 0 < r < \infty, \quad (32)
\end{aligned}$$

where we used the condition that f' tends to zero as u tends to infinity. Therefore, these standard technique does not bring a solution of the reflection of a general cylindrical long wave. The way around this difficulty is a Fourier superposition of the reflection of a cylindrical periodic wave, where the above technique gives the following solution for the reflection and transmission of the incident periodic waves

$$n_I = A_I H_0^1(k_1 r) e^{i\omega t} \quad (33)$$

propagating through the origin, as:

$$n_r = A_I R(\omega) H_0^2(k_1 r) e^{i\omega t} \quad (34)$$

for the reflected wave and

$$n_t = A_I T(\omega) H_0^1(k_2 r) e^{i\omega t} \quad (35)$$

for the transmitted wave. The function $R(\omega)$ and $T(\omega)$, reflection and transmission coefficient respectively, are determined by the matching condition at $r = a$, and are found to be

$$\begin{aligned}
R(\omega) &= (H_1^1(k_1 a) H_0^1(k_2 a) - \sqrt{\frac{h_2}{h_1}} H_0^1(k_1 a) H_1^1(k_2 a)) / \\
&\quad (H_0^2(k_1 a) H_1^1(k_2 a) - \sqrt{\frac{h_2}{h_1}} H_1^2(k_1 a) H_0^1(k_2 a)) \quad (36)
\end{aligned}$$

and

$$T(\omega) = (H_0^1(k_1 a) + R(\omega) H_0^2(k_1 a)) / H_0^1(k_2 a), \quad (37)$$

where $k_1 = \omega / \sqrt{gh_1}$ and $k_2 = \omega / \sqrt{gh_2}$ and ω is the wave frequency. Therefore, the above equations give the dependency of the reflection and transmission coefficients with the wave frequency.

By a Fourier superposition of the above problem, we have that an incident wave

$$n_I = \int_{-\infty}^{\infty} A_I(\omega) H_0^1(k_1 r) e^{i\omega t} \, d\omega. \quad (38)$$

will be reflected and transmitted as

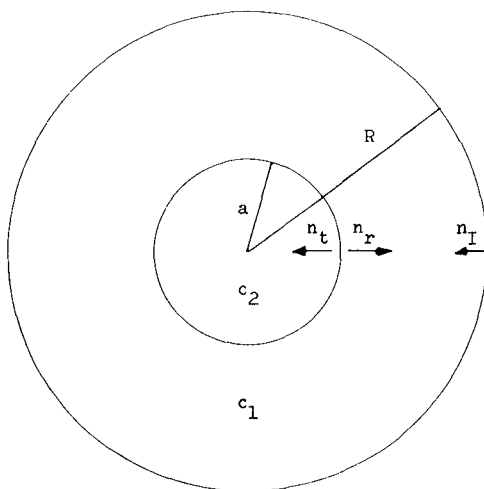
$$n_r = \int_{-\infty}^{\infty} A_I(w) R(w) H_0^2(k_1 r) e^{iwt} dw \quad (39)$$

and

$$n_t = \int_{-\infty}^{\infty} A_I(w) T(w) H_0^1(k_2 r) e^{iwt} dw. \quad (40)$$

Then, the incident, reflect and transmitted waves have different shapes. Because the Fourier coefficients of the above integrals are different functions of w , this is a new property characteristic of the three-dimensional effect of the fluid motion (planar waves are reflected and transmitted with the same shape as the incident waves).

The above technique leads to integrals that must be solved numerically. For this reason in the remaining of the section we will present the following approximate asymptotic analysis of this problem. To do it we will look first for the reflection of a spherical wave due to the presence of a sphere of radius $R = a$ of different density to that of the medium in which the incident wave is propagating.



For an incident wave potential

$$\phi_I = \frac{1}{4\pi R} f\left(t + \frac{R-r_0}{c_1}\right), \quad (41)$$

Whose maximum is at $R = r_0$ when $t = 0$, we will have a reflected wave potential

$$\phi_r = - \frac{1}{4\pi R} Y \left(t - \frac{R-r_r}{c_1} \right), \quad (42)$$

whose maximum is at $R = r_r$ when $t = 0$, and a transmitted potential

$$\phi_t = - \frac{1}{4\pi R} h \left(t + \frac{R-r_t}{c_2} \right), \quad (43)$$

whose maximum is at $R = r_t$ when $t = 0$, where r_r and r_t are to be found later.

The functions Y and h at $R = a$ are determined by the matching condition at the surface of the sphere of radius $R = a$. From the flux condition, we have

$$\frac{\partial}{\partial R} (\phi_I + \phi_r) = \frac{\partial}{\partial R} \phi_t \quad \text{at } R=a$$

or

$$\begin{aligned} \frac{1}{a} \left(f \left(t + \frac{a-r_0}{c_1} \right) - Y \left(t - \frac{a-r_r}{c_1} \right) \right) - \frac{1}{c_1} \left(f' \left(t + \frac{a-r_0}{c_1} \right) - Y' \left(t - \frac{a-r_r}{c_1} \right) \right) \\ = \frac{1}{a} h \left(t + \frac{a-r_t}{c_2} \right) - \frac{1}{c_2} h' \left(t + \frac{a-r_t}{c_2} \right), \end{aligned} \quad (44)$$

If a is large enough, the above equation simplifies to

$$\frac{1}{c_1} \left(f' \left(t + \frac{a-r_0}{c_1} \right) - Y' \left(t - \frac{a-r_r}{c_1} \right) \right) = \frac{1}{c_2} h' \left(t + \frac{a-r_t}{c_2} \right), \quad (45)$$

and from the pressure condition, we get

$$\frac{\partial}{\partial t} (\phi_I + \phi_r) = \frac{\partial}{\partial t} \phi_t \quad \text{at } R=a$$

or

$$f' \left(t + \frac{a-r_0}{c_1} \right) + Y' \left(t - \frac{a-r_r}{c_1} \right) = h' \left(t + \frac{a-r_t}{c_2} \right), \quad (46)$$

by combination of equation (45) and (46), we obtain

$$h' \left(t + \frac{a-r_t}{c_2} \right) = \frac{2 c_2}{c_1 + c_2} f' \left(t + \frac{a-r_0}{c_1} \right) \quad (47)$$

and

$$Y'(t-a-r_r) = - \frac{(c_1 - c_2)}{c_1 + c_2} f' \left(t + \frac{a-r_0}{c_1} \right). \quad (48)$$

The prolongation of equation (47) and (48) in the region $R < a$ and $R > a$ respectively can be given by

$$h' \left(\frac{t+R-r_t}{c_2} \right) = \frac{2 c_2}{c_1 + c_2} f' \left(t + R + \frac{a}{c_1} - \frac{a}{c_2} - \frac{r_0}{c_1} \right). \quad (49)$$

Therefore $r_t = a + c_2 (r_0 - a)/c_1$, and

$$Y' \left(\frac{t-R-r_r}{c_1} \right) = - \left(\frac{c_1 - c_2}{c_1 + c_2} \right) f' \left(t - \frac{R}{c_1} + \frac{2a}{c_1} - \frac{r_0}{c_1} \right), \quad (50)$$

Where $r_r = - (r_0 - 2a)$, and a is sufficiently large compared to the wave length.

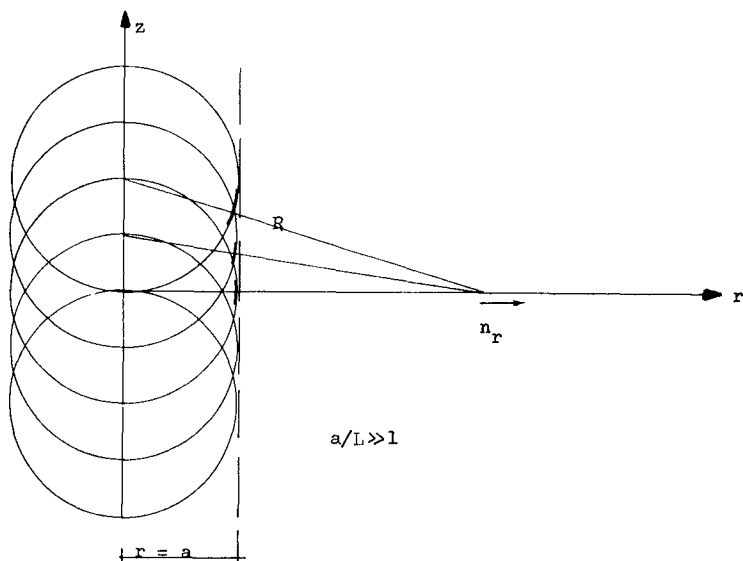
By a superposition of infinite sources on the z axis, line sources, as the ones given by equations (41) and (42), with the relation between y and f given by (50), we will have an outgoing wave

$$n_r = - \frac{1}{2\pi g} \left(\frac{c_1 - c_2}{c_1 + c_2} \right) \int_0^\infty f' \left(t - \frac{r_0 - 2a}{c_1} - \frac{r}{c_1} \cosh u \right) du \quad (51)$$

That will be the approximate reflected wave, from a cylinder of radius $r = a$, of the incident wave

$$n_i = \frac{1}{2\pi g} \int_0^\infty f' \left(t - \frac{r_0}{c_1} + \frac{r}{c_1} \cosh u \right) du. \quad (52)$$

Since in the integration that we have to do in order to get the reflected wave given by (51), the integrand is proportional to $1/R$ and therefore the mayor contribution comes from the sources near the origin, where the waves generated by that sources are the reflected waves by a surface almost cylindrical of radius $r = a$ of the corresponding spherical incoming waves, the above approximation tends to the exact solution when the radius r goes to infinite (see the definition sketch given below).



Using the equation(20) of section 2, we get that f' in the above equation is a function of the initial incoming wave profile, given by

$$f' \left(\frac{r}{c_1} \cosh u - \frac{r_0}{c_1} \right) = -4g \frac{r}{c_1} \cosh u \int_0^\infty n_i' \left(\frac{r}{c_1} \cosh u \cosh v - \frac{r_0}{c_1} \right) dv \quad (53)$$

or

$$f' \left(-\frac{r}{c_1} \cosh u - \frac{(r_0 - 2a)}{c_1} \right) = 4g \frac{r}{c_1} \cosh u \int_0^\infty n_i' \left(-\frac{r}{c_1} \cosh u \cosh v - \frac{(r_0 - 2a)}{c_1} \right) dv \quad (54)$$

Therefore, when we have an initial incoming wave profile $n_i \left((r - r_0)/C_1, t = 0 \right)$, the reflected wave can be approximated, for a large r , by

$$n_r = \frac{2}{\pi} \frac{(c_1 - c_2)}{c_1 + c_2} \int_0^\infty \left(t - \frac{r}{c_1} \cosh u \right) \int_0^\infty n_i' \left(\left(t - \frac{r}{c_1} \cosh u \right) \cosh v - \frac{(r_0 - 2a)}{c_1} \right) dv du \quad (55)$$

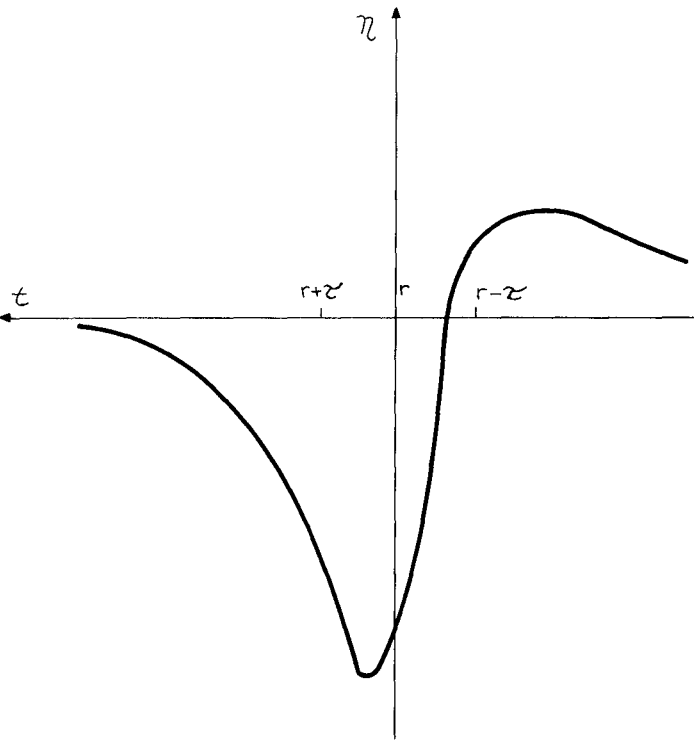


FIG. # 3

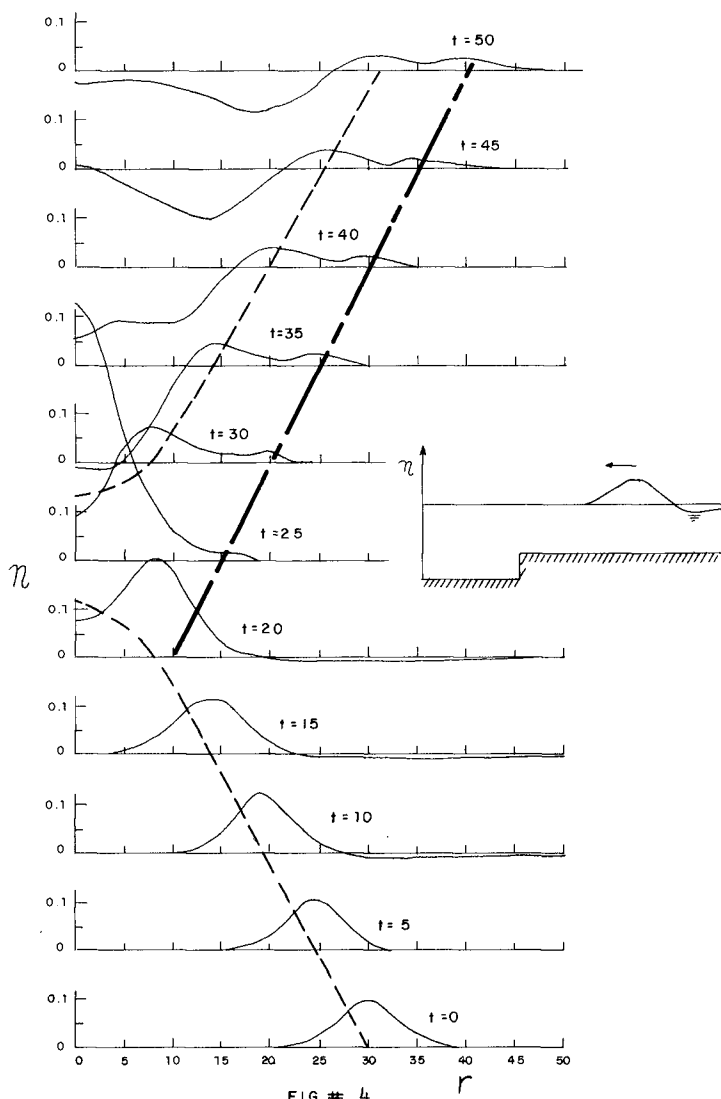


FIG. # 4

In particular a solitary wave, free of discontinuities, can be obtained if Lamb's (1902) source strength is assumed as

$$f(t') = \tau / (t'^2 + \tau^2) \text{ with } t' = t - \frac{(r_0 - 2a)}{c_1}, \quad (56)$$

where τ is a parameter, the function $f(t')$ has no definite beginning or ending, but the range of time within which it is sensible can be made as small as we please by diminishing τ . With the above source strength, the asymptotic form of the reflected wave is

$$\eta_r = - \frac{1}{4\sqrt{2}} \left(\frac{c_1 - c_2}{c_1 + c_2} \right) \sqrt{\frac{\tau}{r}} \frac{1}{g\tau} \sin \left(\frac{\pi}{4} - \frac{S}{2} \right) \cos^{3/2} S,$$

where

$$S = \tan^{-1} \left(\frac{1}{\tau} \left(t - \frac{r_0 - 2a}{c_1} - \frac{r}{c_1} \right) \right). \quad (57)$$

Equation (57) is plotted in figure 3 for the case $C_1 > C_2$, consisting of a negative main wave followed by a longer positive small tail, and it is in agreement with the numerical solutions for both cases $C_1 \geq C_2$, where for a positive shelf, $C_1 > C_2$, the reflected wave is negative and for a negative shelf, $C_2 > C_1$, the reflected wave is positive (see figures 1, 2 and 4).

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