On the Synthesis of Realistic Sea States

by

E.R. Funke* and E.P.D. Mansard**

1.0 INTRODUCTION

Recent investigations by some researchers (Johnson et al, 1978; Burchart, 1979; Gravesen and Sorensen, 1977) have indicated that it is no longer sufficient to match the variance spectral density of a simulated sea state to that of the prototype. When testing models of various fixed and floating structures, it appears to be most important to simulate the wave grouping phenomenon as well. Some researchers also believe that the wave steepness, the particular sequencing of high and low waves (Burchart, 1979) and the ratio of the maximum to the significant wave height within a wave train are also of significance.

Methods for the generation of 'random' waves throughout the world vary greatly. One may, however, categorize these in terms of two substantially different approaches. These may be referred to as "probabilistic" on the one hand and "deterministic" on the other.

In the former, a random or pseudo-random noise source is used which will never repeat or which has a very long repetition period. The assumption is then made that, in the course of the long testing period, all possible outcomes of wave heights, wave periods and wave groups will occur. The only constraint, which is usually placed on the synthesis, is the shape of the variance spectral density and its zeroth moment. The "deterministic" approach, on the other hand, attempts to create very specific and typically extreme conditions. Subsequent analysis of structural response to these conditions must, of course, be related to the likelihood of these conditions occurring in the prevailing climate. The old standby method of testing with monochromatic waves is a typical example of this category. However, other techniques such as Funke and Mansard (1979a) and the reproduction of prototype wave trains as favoured by several laboratories (Gravesen and Sorensen, 1977) may also be described as deterministic.

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This paper describes a method which can synthesize a wave train with a well defined grouping characteristic. Although, in principle, the wave train could be made arbitrarily long, for practical reasons it will normally be limited to a recycling period of 5 or 10 minutes in the laboratory. The method, which may also be identified as "deterministic", can synthesize a wave train with well defined grouping characteristics and, at the same time, with a good approximation to a specified continuous variance spectral density. Although these two features may not be sufficient for a completely realistic simulation of a natural sea state, the method is believed to be a step in the right direction.

The method produces a time series which describes the wave train as it is to be monitored at a prespecified location in a wave flume or basin and in the absence of reflections. It is therefore assumed that there exists a real time signal generator, such as an on-line digital computer (Funke, Crookshank and Wingham, 1980) which is capable of converting a number sequence (after suitable amplitude and phase compensation) into a smooth driving signal to a servo-controlled wavemaking machine. The method described here addresses itself only to the synthetic creation of the time series. Its conversion into a train of water waves at a specified location in the flume is the subject of other publications (Funke and Mansard, 1979a; Funke and Mansard, 1980).

As an input to the procedure, it is necessary to specify:

a) the variance spectral density $S(f)$ of the desired water surface displacement (this includes, by implication, the characteristic wave height),
b) the groupiness factor, GF,
c) the SIWEH spectral density, $\varepsilon(f)$, and
d) the desired repetition (or recycling) period of the wave train, T_n.

Both the SIWEH and the groupiness factor are concepts which had to be developed as descriptors for the sea state in order to realize a systematic procedure for synthesis. Whereas this paper addresses itself to the subject of how wave group activity may be defined and how this definition can be used to synthesize a wave train, it remains to be determined how important these parameters are in describing a wave climate and in causing severe and damaging structural response.

2.0 THE SIWEH

SIWEH is an abbreviation for Smoothed Instantaneous Wave Energy History (Funke and Mansard, 1979b). It is a function of time which describes the distribution of energy along the time axis. This somewhat awkward name is proposed (with apologies) in order to distinguish this energy function of time from the more commonly used energy function of frequency and it is believed that the word history does suggest a time function.

The SIWEH is proposed as an alternative to 'envelope functions' for the purpose of describing group activity within a wave train. Fig. 1 illustrates a typical wave train and two envelope functions. The 'half-wave rectified' envelope has been constructed by joining all peaks of the wave record whereas the 'full-wave rectified' envelope requires first a folding of all negative values about the mean before the peaks are joined. The former does not properly account for wave troughs and the latter is always affected by the presence of non-linear waves which have sharp peaks and flat troughs. Therefore, the concept of computing the average wave energy over the period of the dominant wave appeals intuitively. If $\eta(t)$ is the water surface displacement with zero mean value, then the smoothed instantaneous wave energy history may be defined initially as:

$$E^*(t) = \frac{1}{T_p} \int_{\tau=-T_p/2}^{T_p/2} \eta^2(t+\tau) \cdot d\tau \tag{1.1}$$

This function provides a uniform, running average of the squared water surface displacement over the interval $T_p = 1/f_p$, where f_p is the frequency at which the variance spectral density of $\eta(t)$ is a maximum. The operation defined by equation 1.1 may be considered convolution of a rectangular data window, Q_0, with the function $\eta^2(t)$, i.e.

$$E^*(t) = \frac{1}{T_p} \int_{-\infty}^{\infty} \eta^2(t+\tau) \cdot Q_0(\tau) \cdot d\tau \tag{1.2}$$

Another window function is the Bartlett window, Q_1, which has a triangular shape with a base of $2 \cdot T_p$. The result of this smoothing operation is also included in Fig. 1.

From Fig. 1 it may be observed that the SIWEH provides superior identification of groups and that the Bartlett window achieves better smoothing without an apparent loss of contrast.

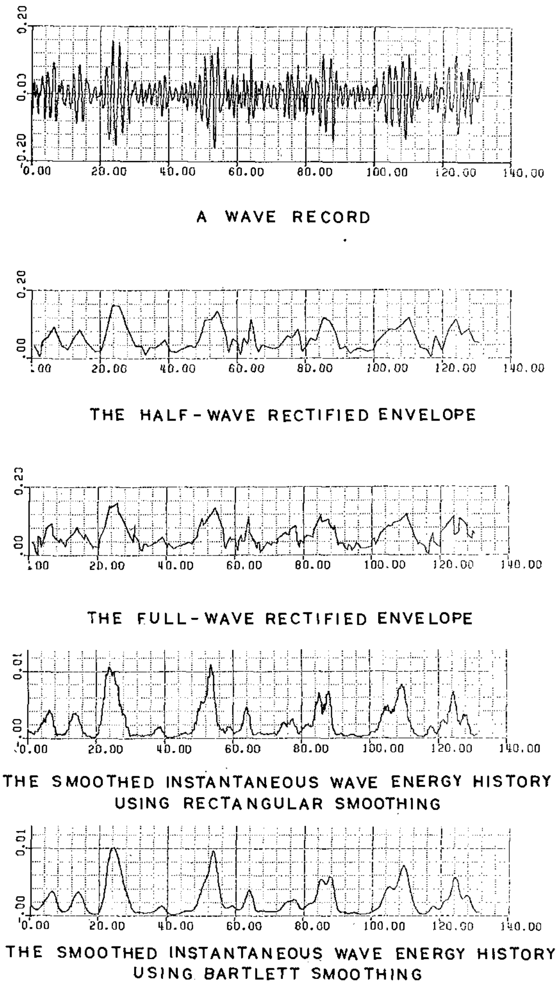

FIG. 1 COMPARISON BETWEEN FUNCTIONS FOR WAVE GROUP IDENTIFICATION

3.0 SPECTRAL CONSIDERATION TO SMOOTHING OF THE SIWEH

Convolution between two functions in the time domain is equivalent to multiplication between their respective Fourier transforms in the frequency domain. For this reason it is instructive to consider the Fourier transform of the smoothing windows, which were used for Fig. 1. Fig. 2 shows these frequency functions as a function of normalized frequency f/f_p where f_p is the frequency at which the variance spectral density is a maximum. For comparison purposes the square root of a JONSWAP spectrum is also included. It should be noted that the rectangular window has a frequency response which is quite oscillatory. The consequence of using this function as a digital filter is leakage of energy from outside its nominal pass band. The Bartlett window, on the other hand, has smaller side-band leakage but suffers from reduced contrast in the pass band.

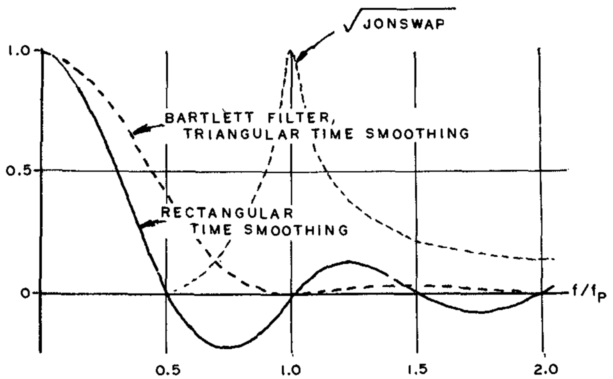

FIG. 2 FREQUENCY CHARACTERSTICS OF SMOOTHING FUNCTIONS

As the filters are applied to the square of the water surface displacement, it is essential to assess them in terms of the spectrum of $\eta^2(t)$. Fig. 3 illustrates first the function $\eta(t)$ and then its square after removal of the mean value $\overline{\eta}^2$. The SIWEH function, which was smoothed using the Bartlett window, is also shown with its mean value removed.

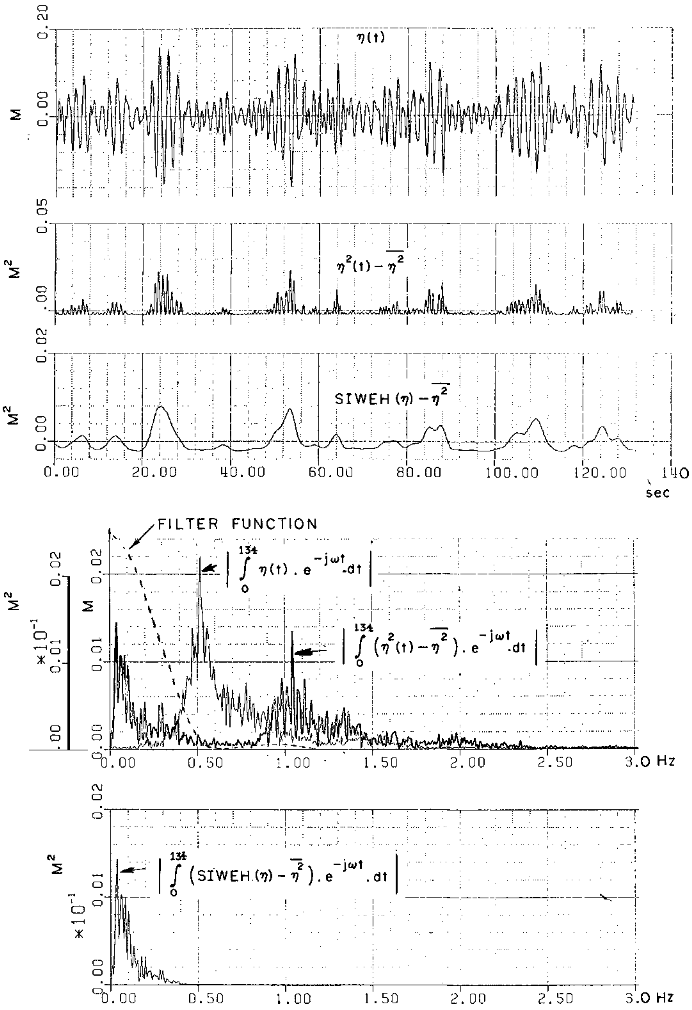

FIG. 3 SPECTRUM OF SQUARED WATER SURFACE ELEVATION

The fourth function in Fig. 3 gives the absolute values of the Fourier transforms of both $\eta(t)$ and $(\eta^2(t) - \overline{\eta}^2)$. The frequency response of the Bartlett filter function is also shown. The most surprising observation is the redistribution of energy in the frequency domain as a result of squaring in the time domain. It appears that energy is moved both above as well as below the band that was previously occupied by the variance spectral density of $\eta(t)$ with a notable absence of energy remaining in this latter band. The Bartlett smoothing filter which was selected intuitively does appear to be quite effective in separating the low-frequency from the high frequency part of the spectrum, although as can be seen, with some attentuation in accordance with its filter characteristics.

These frequency considerations suggest that convolution with a Bartlett window is superior to rectangular smoothing. Therefore, the following formulation for the computation of the SIWEH is proposed,

$$E'(t) = \frac{1}{T_p} \int_{\tau=-T_p}^{T_p} \eta^2(t+\tau)\cdot Q_1(\tau)\cdot d\tau \quad \text{for} \quad T_p \leq t \leq T_n - T_p \tag{3.1}$$

and for the beginning and the tail end of the record:

$$E'(t) = \frac{2}{T_p + t} \int_{\tau=-t}^{T_p} \eta^2(t+\tau)\cdot Q_1(\tau)\cdot d\tau \quad \text{for} \quad 0 \leq t \leq T_p \tag{3.2}$$

$$E'(t) = \frac{2}{T_p + T_n - t} \int_{\tau=-T_p}^{T_n - t} \eta^2(t+\tau)\cdot Q_1(\tau)\cdot d\tau \quad \text{for} \quad T_n - T_p \leq t \leq T_n \tag{3.3}$$

where

$$Q_1(\tau) = 1 - |\tau|/T_p \qquad \text{for} \quad -T_p \leq \tau \leq T_p$$

$$= 0 \text{ everywhere else.}$$

4.0 AN ALTERNATE WAY OF DEFINING AND COMPUTING THE SIWEH

If the wave record of length T_n is given as a Fourier series expansion, i.e.

$$\eta(t) = \sum_{i=1}^{N} c_i \cdot \cos(\omega_i \cdot t + \phi_i) \tag{4.1}$$

with $\omega = 2\pi\cdot i\cdot t/T$ and
c_i being the Fourier coefficients,
then $\eta^2(t)$ may be shown to contain four distinctly different groups of terms (Naes, 1978) namely:

a) a group involving c_i^2,

b) a group including the frequencies $2\omega_i$,
c) another group with frequencies $\omega_j+\omega_i$, and
d) finally, the fourth group with frequencies $\omega_j-\omega_i$.

The first and fourth group together make up the SIWEH of $\eta(t)$ which may thus be defined for the interval $0 \leq t \leq T$, $\mathrm{SIWEH}(\eta(t)) = E(t) =$

$$\frac{1}{2}\sum_{i=1}^{N} c_i + \sum_{i,j} c_i \cdot c_j \cdot \cos[(\omega_j-\omega_i) + (\omega_j-\omega_i)] \qquad 1 \leq i \leq j \leq N \qquad (4.2)$$

The first term of this expansion is evidently the mean square value of $\eta(t)$ and must therefore be equal to the area under the variance spectral density of $\eta(t)$. In other words:

$$\overline{E(t)} = m_0$$

Although the authors are not, at the time of writing, applying equation 4.2 for the computation of the SIWEH, it is believed that it will prove superior to equations 3.1 to 3.3 as it is not dependent on smoothing windows in the time domain or the treatment of start-up and end transitions in the associated convolution operation.

5.0 THE SIWEH SPECTRUM

The unsmoothed SIWEH variance spectral density may also be computed in two possible ways. By application of the Fourier transform to the SIWEH one obtains:

$$\varepsilon^*(f) = \frac{1}{2\Delta f}\left|\frac{2}{T_n}\int_0^{T_n}(E(t) - \bar{E})\cdot e^{-j\omega t}\cdot dt\right|^2 \qquad (5.1)$$

$$= \frac{2}{T_n}\left|\int_0^{T_n}(E(t) - \bar{E})\cdot e^{-j\omega t}\cdot dt\right|^2 \qquad (5.2)$$

where $\Delta f = 1/T_n$.

The second method may be derived from equation 4.2 by re-ordering the difference frequencies so that all terms with the same difference frequency are grouped together. Therefore, letting $j=i+k$ with $k=1,2,\ldots N-1$, equation 4.2 may be written thus,

$$E(t) = \frac{1}{2}\sum_{i=1}^{N} c_i^{\,2} + \sum_{k=1}^{N-1}\sum_{i=1}^{N-k} c_i \cdot c_{i+k} \cdot \cos(k \cdot \Delta\omega \cdot t + \theta_{i,k}) \qquad (5.3)$$

where $\Delta\omega = \omega_{i+1} - \omega_i = 2\pi/T_n$

and $\theta_{i,k} = \phi_{i+k} - \phi_i$

Equation 5.3 may be abbreviated as:

$$E(t) = \frac{1}{2}\sum_{i=1}^{N} c_i^{\,2} + \sum_{k=1}^{N-1} A_k \cdot \cos(k \cdot \Delta\omega \cdot t + \gamma_k) \qquad (5.4)$$

where

$$A_k = \sqrt{\left(\sum_{i=1}^{N-k} (c_{i+k} \cdot c_i \cdot \cos\,\theta_{i,k})\right)^2 + \left(\sum_{i=1}^{N-k} (c_{k+k} \cdot c_i \cdot \sin\,\theta_{i,k})\right)^2} \qquad (5.5)$$

and

$$\gamma_k = \mathrm{atan}\left(\frac{\sum_{i=1}^{N-k} c_{i+k} \cdot c_i \cdot \sin\,\theta_{i,k}}{\sum_{i=1}^{N-k} c_{i+k} \cdot c_k \cdot \cos\,\theta_{i,k}}\right) \qquad (5.6)$$

The unsmoothed SIWEH variance spectral density may then be obtained from equation 5.5 as follows:

$$\varepsilon(f) = [A(k \cdot \Delta f)]^2/(2 \cdot \Delta f) \qquad (5.7)$$

It is of particular interest to note that the SIWEH spectral density is completely defined in terms of the Fourier coefficients and Fourier phases of the water surface displacement $\eta(t)$. The term

$$\sum_{i=1}^{N-k} c_{i+k} \cdot c_i \qquad (5.8)$$

in equation 5.5 is reminiscent of the auto-covariance of the spectrum defined by c_i. It has been suggested by Nolte and Hsu (1972) that the spectrum of the envelope function is related to:

$$H(f) = \int_0^\infty S(x) \cdot S(x+f) \cdot dx \Big/ \int_0^\infty S^2(x) \cdot dx \qquad (5.9)$$

where $S(f)$ is the variance spectral density of $\eta(t)$. This proposition is, however, challenged on the grounds that according to equation 5.9

$$H(0) \geq H(f)$$

It is known from observations that this does not describe reality because highly periodic wave group phenomena lead to a peak in the SIWEH spectral density at non-zero frequencies and this cannot be reproduced with equation 5.9.

On the other hand, equation 5.5 indicates that the SIWEH spectral density is also affected by the phase differences between adjacent frequency components. This suggests for the first time how the phase spectrum of wave trains may be related to the grouping phenomenon.

Smoothing of SIWEH variance spectral densities may be implemented by the usual statistical techniques.

6.0 THE GROUPINESS FACTOR

The SIWEH or its spectral density may be used to describe the degree of group activity. The groupiness factor is defined as

$$GF = \left(\sqrt{\int_0^{T_n} (E(t) - \bar{E})^2 \cdot dt}\right) \Big/ \bar{E} \qquad (6.1)$$

which gives the standard deviation of the SIWEH about its mean value $\bar{E}$.

Since the variance of $E(t)$ about its mean must equal the area under the variance spectral density, the groupiness factor may also be given as

$$GF = \sqrt{m_{\varepsilon 0} / m_0} \qquad (6.2)$$

where m_{ε_0} is the zeroth moment of the SIWEH spectral density.

Groupiness factors for prototype wave data so far observed at one location over a period of six months (Sea of Japan, N38 44'33", E139 39'48") fall in the range $0.46 \leq GF \leq 0.94$.

7.0 A POSSIBLE MODEL FOR THE SIWEH SPECTRUM

It is expected that analysis of prototype wave data will reveal that SIWEH spectral densities will, occasionally exhibit a more or less pronounced peak indicating a periodicity of groups. However, the majority of SIWEH spectral densities, particularly for low wave heights, will decay almost exponentially with increasing frequencies. The broadness of the SIWEH spectral densities are expected to be inversely related to the width of the average wave groups.

A model which may provide enough control over the spectral shape has been borrowed from linear system analysis and is given by

$$\varepsilon(\ ) = \frac{1}{\sqrt{(1-\lambda^2)^2 + 4\cdot\zeta^2\cdot\lambda^2}} \cdot \frac{\lambda}{\sqrt{1+\lambda^2}}$$

This function is shown in Fig. 4 for $\lambda = 0.1$, $\zeta = 0.1, 0.3$ and 1 and the area under the function has been adjusted so that the groupiness factor according to the definition of equation 6.2 is 0.95.

8.0 SYNTHESIS OF A SIWEH FROM ITS SPECTRAL DENSITY

As it is expected that, in future, a sea state may be specified both in terms of its variance spectral density as well as its SIWEH spectral density, it must be determined if one can synthesize a SIWEH from the SIWEH spectral density. It is evident from inspection of a SIWEH that this function is highly non-Gaussian. This fact imposes a severe problem on the ability to synthesize such a function through inverse Fourier transformation as may be seen from the following argument.

When applying the inverse Fourier transform to an amplitude spectrum, some arbitrary phase spectrum must first be assumed. It is common practice to create such a phase spectrum by selecting phases for each of the constituent frequencies from a random number generator which has a uniform distribution of random numbers over the interval $-\pi$ to π. As a consequence of such a phase spectrum, the function resulting from inverse Fourier transformation can be shown to have a Gaussian amplitude probability density and non-Gaussian functions cannot be generated in this way.

The technique which has been used here to overcome this difficulty is an iterative procedure (see Funke and Mansard, 1979b). After the first inverse Fourier transform following a random phase selection, the resulting time function is then clipped below $-\alpha\cdot m_0$ so that the resultant time signal now looks highly distorted with all troughs being flat at $-\alpha\cdot m_0$. A subsequent forward Fourier transform of this clipped function will produce a new amplitude and phase spectrum. Evidently the new amplitude spectrum is wrong but the new phase spectrum is a better approximation of the unknown phase spectrum than the first guess. One therefore pairs up this new phase spectrum with the original amplitude spectrum and repeats the inverse Fourier transform. The resultant time function will now be non-Gaussian, however, there will still be exceedances below the $-\alpha\cdot m_0$ level. It

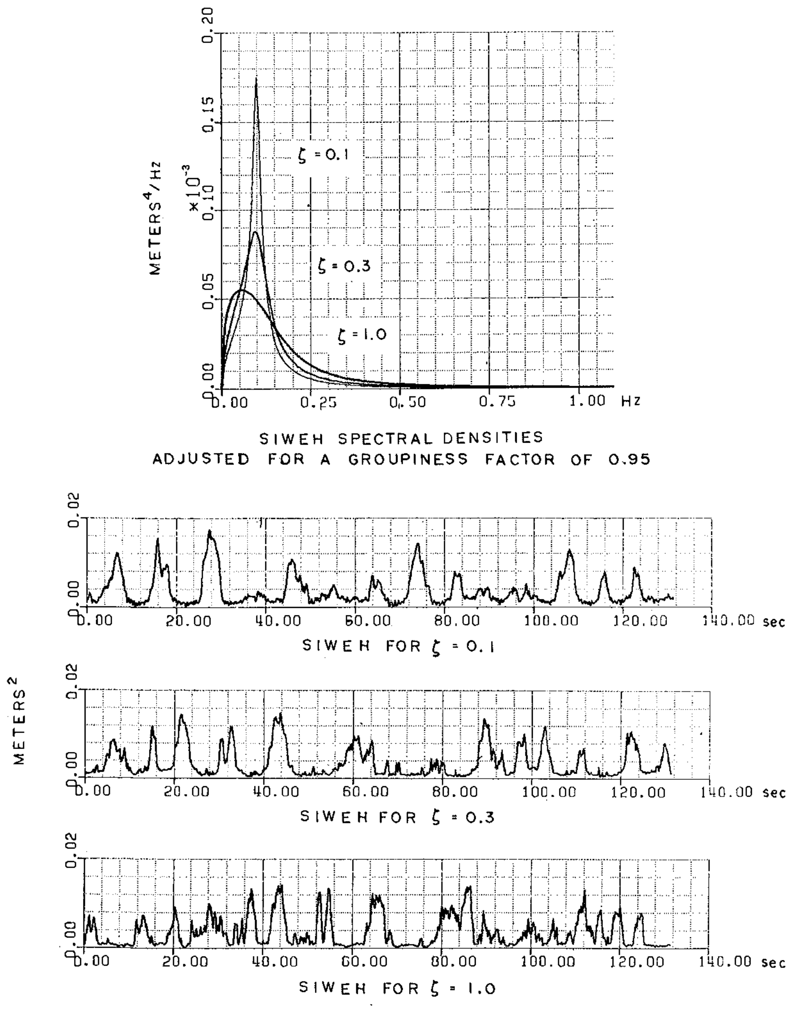

FIG. 4 THE SYNTHESIS OF A SIWEH

is therefore necessary to repeat this operation of clipping and transforming until these exceedances no longer occur.

Different values of α have been tried which appear to work equally well. For $\alpha = 0.6$ it was found that 20 iterations were required.

Fig. 4 gives three different SIWEHs which were synthesized from the three spectral densities also given in this figure. The SIWEH for $\zeta = 0.1$ is the one which has the most pronounced periodicity of groups.

9.0 THE SYNTHESIS OF A GROUPED WAVE

Fig. 5 illustrates the procedure which is now followed in order to create a wave train which not only has a given variance spectral density, but also has wave grouping as specified by a SIWEH, which may have been synthesized by the method described in section 8.0.

With reference to Fig. 5 one will recognize the desired variance spectral density and the desired SIWEH. From the latter a phase modulating function is created which is also shown in Fig. 5. It may be noticed that this phase modulator has large values when the SIWEH is small and vice versa. It is being used to phase modulate a constant amplitude sinusoid with a dominant frequency of 0.5 Hz, which is the peak frequency of the desired variance spectral density. Closer observation will reveal that the dominant frequency has been preserved in those intervals where the SIWEH has large values while everywhere else the frequency is increased. The reasoning behind this is that each wave group should have a dominance of energy in the peak frequency band if one wishes to match the desired variance spectral density by this initial approximation.

After this, the phase modulated carrier is also amplitude modulated by using the square root of the SIWEH as the modulating function. The result of these manipulations is a rough approximation of the synthesized wave train. Fourier transformation, as shown in Fig. 5, does in fact show that the amplitude spectrum approximates quite well the desired amplitude spectrum (which has been obtained by square-rooting the spectral density). The phase spectrum, which has been obtained from the Fourier transform of the rough approximator, is now paired with the desired amplitude spectrum. Subsequent inverse Fourier transformation will then produce the desired wave train or at least something which comes very close to the ideal.

By computing the SIWEH for this wave train, E2(t) and comparing it to the originally specified SIWEH, E1(t), a correcting function C(t) may be computed which is

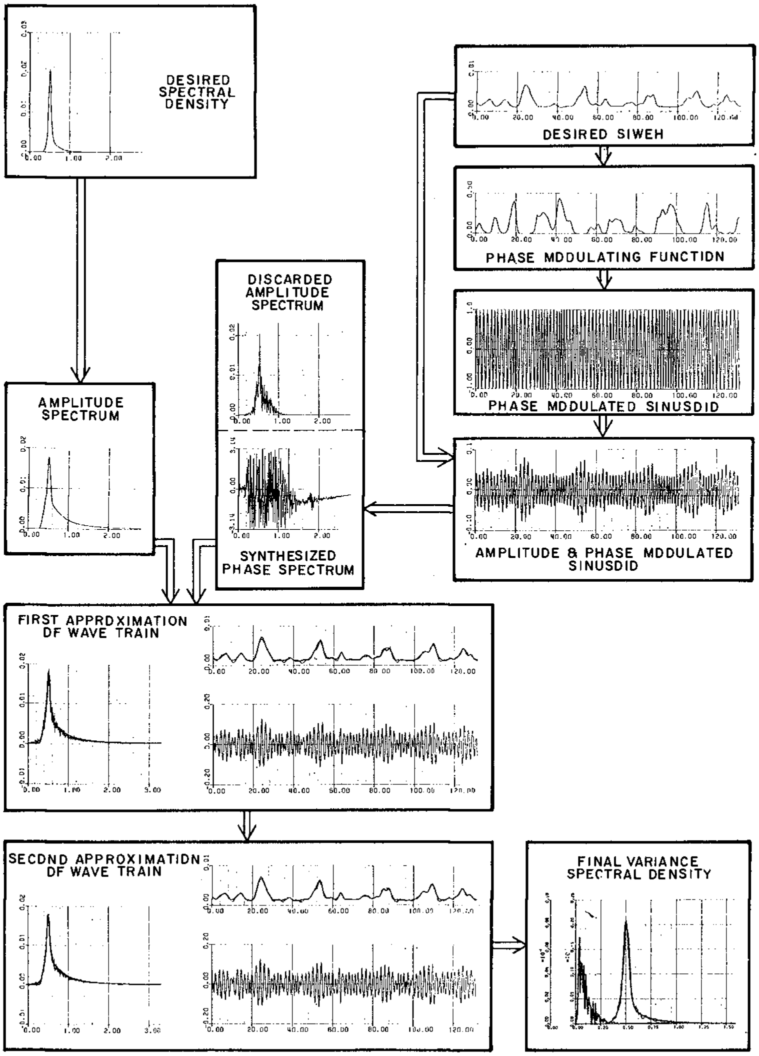

FIG. 5 THE SYNTHESIS OF A GROUPED WAVE

$$C(t) = \sqrt{E1(t)/E2(t)}$$

The first wave train is now multiplied by C(t) and the product is also Fourier transformed. Again one obtains an amplitude and a phase spectrum. The new phase spectrum is then paired with the original amplitude spectrum and an inverse Fourier transform will then provide the second approximation of the desired wave train. This may be repeated a few times wich occasionally leads to further improvements. Fig. 6 shows some other synthesized wave trains. Attention is drawn to the difference between the actual and the desired variance spectral density. It may be noticed that there are some differences, particularly in the higher frequency tail of the spectrum.

Beside this, Fig. 6 illustrates how one may generate any number of grouped wave trains from the same input specifications. In all three cases the same SIWEH spectral density and the same variance spectral densities were used. However, for the purpose of synthesizing the SIWEHs, three different random number sequences were generated.

10.0 SPECTRAL BROADNESS FACTOR AND GROUPINESS

With the availability of the synthesis tool described here, one may address oneself to the question of how the spectral broadness factor is related to wave groupiness. It has been generally accepted that waves with a narrow variance spectral density also have a more pronounced groupiness than broad-banded waves. It may, in fact be true that there is a strong correlation between these two which is a characteristic of natural gravity waves. This could be explained if strong grouping occurs after waves have travelled over long distances and, as is well known, waves due to swell do have narrow band spectra.

However, it should be noted that there is no necessary relationship between a spectral width parameter and groupiness. Fig. 7 illustrates three synthesized wave trains. Each of these were synthesized from the same SIWEH spectral density with the only difference that the area under the SIWEH spectral densities was rescaled according to equation 6.2 so that the groupiness factor could be varied from GF = 0.95 to 0.2. In all three cases the variance spectral densities are more or less the same. Whereas the wave train for GF = 0.95 shows very pronounced grouping, the wave train for GF = 0.2 is almost of constant amplitude.

It may, perhaps, appear unbelievable that this latter wave record can have the variance spectral density which is shown. It must, however, be remembered that the record shown is not a pure sinusoid and that there is an appreciable amount of phase modulation which causes the

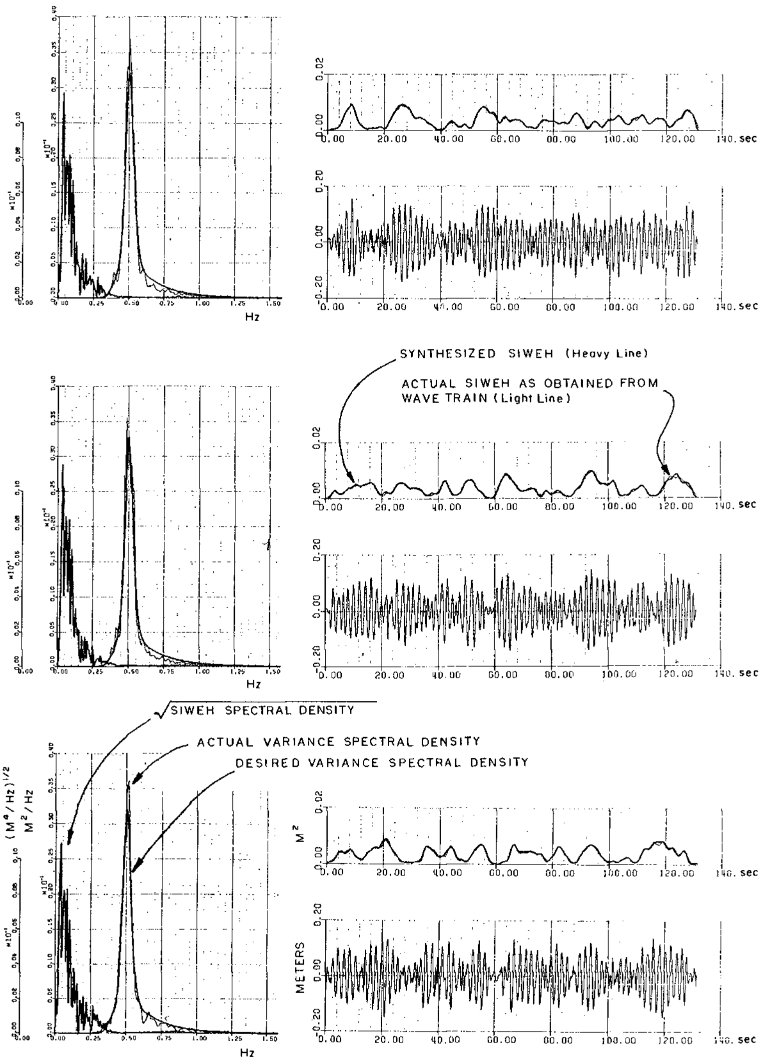

FIG. 6 THE SYNTHESIS OF GROUPED WAVES FROM COMMON SIWEH SPECTRAL DENSITIES

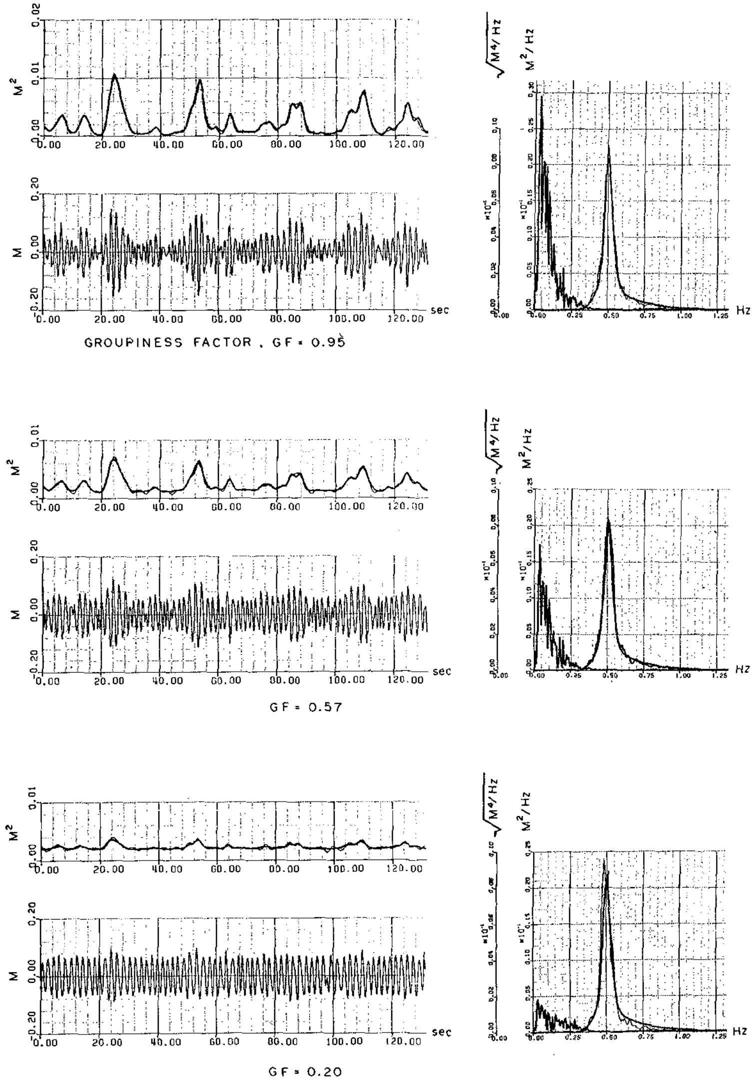

FIG. 7 SYNTHESIS OF GROUPED WAVE TRAINS FROM THE SAME SIWEH BUT WITH DIFFERENT GROUPINESS FACTORS

spectrum to be non-monochromatic. The authors have not attempted to reproduce this latter wave train in a wave flume, and it is therefore not known if this particular wave train is physically realizable.

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