WAVE SHOALING CALCULATED FROM COKELET'S THEORY.

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ABSTRACT

Cokelet's numerical non-linear theory for progressive, periodic gravity waves is applied to the two-dimensional shoaling of finite amplitude waves on a beach up to breaking. The shoaling curves so obtained are compared with existing shoaling curves calculated from different finite amplitude wave theories, and with existing experimental data. It was found that the shoaling curves calculated from Cokelet's theory predict higher wave height ratios than other curves. The agreement between the present curves and the experimental results is good except near the breakpoint, where the wave height of the present curves is larger than the experimental wave height.

INTRODUCTION

In recent years, significant developments have taken place in the field of non-linear theories for progressive, periodic gravity waves of constant form.

Schwartz (1974) derived recurrence relations between successive coefficients in a Stokes-type power series, which made it possible to find computer-generated solutions of very high order.

Cokeket (1977) modified the procedure developed by Schwartz. He calculated the wave parameters with great precision, for arbitrary depth-length ratio and arbitrary wave steepness up to the highest wave of permanent form.

For all practical purposes, Cokelet's work can be seen as giving an exact solution to the classical nonlinear irrotational gravity wave problem addressed first by Stokes. It is worthwhile to apply it in situations involving highly nonlinear waves. One such application is given in the present paper, in which Cokeket's results are being used to calculate the two-dimensional shoaling of finite-amplitude waves on a beach, up to breaking.

Shoaling calculations have been made previously by several authors, using one or another approximate non-linear theory of the Stokes type or the cnoidal type, or a semi-numerical theory. The majority of these theories is for periodic waves of permanent form in water of uniform depth.

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They are applied locally in water of gradually varying depth on the assumption that the rate of change of the wave parameters is sufficiently small so that its effect on the local wave behaviour is negligible.

While it is difficult to make an a priori assessment of the errors resulting from this assumption, it is reasonable to expect that the approximation improves with decreasing value of the relative depth change within an interval of one wavelength. Thus, provided that the bed slope is sufficiently small, depending on the depth-to-wavelength ratio, it seems reasonable to expect also that the calculated wave height variation in the shoaling process is more nearly exact when it is based on a more nearly exact theory for waves of permanent form in water of constant depth.

The paper is made up as follows. Some relevant points of Cokelet's theory are summarized first. This is followed by an outline of the shoaling calculations, and by a presentation of the results. Finally, a comparison is made with previously published shoaling curves and with experimental data.

The presentation in this proceedings paper is kept rather brief. The reader is referred to Sakai and Battjes (1980) for more details and quantitative formulations.

COKELET'S THEORY

Cokelet (1977) deals with the classical problem of two-dimensional irrotational periodic gravity waves of constant form in a fluid of constant density (ρ) and uniform mean depth (h) and with a stress-free upper surface. The equations describing these conditions are well known and will not be repeated here. Neither shall we describe in detail the method of solution employed by Cokelet.

For given values of ρ and g, the motion is specified entirely by the wave length λ (or, equivalently, the wave number $k = 2\pi/\lambda$), the wave height H (or, equivalently, the semi-wave height $a = H/2$), and the mean depth h. Cokelet makes all quantities dimensionless in terms of k, ρ and g.

An orthogonal system of rectilinear axes is used, with Ox horizontal, positive in the direction of wave advance, and with Oy vertical, positive upward. The reference plane $y = 0$ is chosen near but not in the mean water level. The elevation of the free surface is given by $y = \eta(x,t)$, and that of the sea bed by $y = -d$, so that $d + \bar{\eta} = h$, the mean depth (an overbar denotes a time average).

The Oxy frame of reference is chosen such that the Eulerian mean velocity is zero in all points below the level of the wave troughs. A consequence of this choice is that the mean mass transport in this reference frame is not zero.

A Fourier series development was assumed for the complex velocity potential as a function of the complex coordinate. The Laplace equation and the kinematic boundary conditions at the bottom and the free surface can be made to be satisfied exactly. Satisfaction of the condition of constancy of pressure at the free surface leads to a set of nonlinear algebraic equations which determine the Fourier coefficients completely.

The values of these could be computed on a digital computer up to very
high order (>100) and with great accuracy (4 to 6 significant figures). In
his calculations, Cokelet used a perturbation parameter ε, whose value
ranges from zero for waves of vanishing steepness to 1 for waves of
limiting height.

Cokelet has tabulated numerical values of several dimensionless
dependent wave parameters as function of ε^2 and the relative depth kd.
These parameters include the semi-wave height (a), the square of the phase
speed (c^2), and a number of integral properties, such as the mean densities
and fluxes of mass, momentum and energy.

For constant relative depth (kd), the wave steepness ka was found to
be a monotonic function of ε^2, but the phase speed and the integral
properties appeared to have a maximum for some value of ε^2 a little less
than 1, that is, for waves slightly less high than the highest wave possible
for the given value of kd. This had been discovered earlier for the
solitary wave by Longuet-Higgins and Fenton (1974), and for deep-water
waves by Longuet-Higgins (1975).

CALCULATION OF WAVE SHOALING

Assumptions

We consider the classical, idealised wave shoaling problem: a purely
periodic, two-dimensional wave motion, an impermeable, rigid sea bed of
gentle slope and ending in a dry beach, zero reflection, and zero
dissipation. As stated in the introduction, the bed slope is supposed to
be sufficiently gentle for the local applicability of a theory for
progressive waves of constant form in uniform depth.
    The preceding assumptions imply a zero mean mass transport, and
constancy of wave frequency and of onshore energy flux.

Adaptations of Cokelet's results

The conditions stated above required an adaptation of Cokelet's
results in two ways.
    First, we have transformed the results to a new reference frame by
adding a vertically uniform mean counter-current, such that the mean mass
flux would be zero. The transformed variables are indicated by a prime
(e.g. σ', the wave frequency, and F', the mean onshore energy flux per unit
span).
    Secondly, we have non-dimensionalised the results with respect to ρ,
g and σ', instead of ρ, g and k, which had been used by Cokelet. The latter
set is less suitable since k varies in the shoaling process, while σ' does
not. Varialles which have been made non-dimensional in terms of ρ, g and
σ' are indicated by an asterisk.

Procedure of shoaling calculation

The onshore energy flux F', suitably normalized for our purposes, is
expressed entirely in terms of dimensionless quantities tabulated by Cokelet
as functions of ε^2 and kd. We can write therefore

$$\overset{*}{F}{}' = \overset{*}{F}{}'(\varepsilon^2, \; kd) \; . \tag{1}$$

Since the (dimensional) frequency σ' and energy flux F' are constant in
the shoaling process, so is $\overset{*}{F}{}'$. It is everywhere equal to its value in deep
water, designated with a subscript "o":

$$\overset{*}{F}{}'(\varepsilon^2, \; kd) = \text{constant} = \overset{*}{F}{}'_o \; . \tag{2}$$

The procedure of the shoaling calculation is broadly as follows. A
deep-water steepness $H_o/\lambda_o = \pi^{-1} k_o a_o$ is chosen; this determines the deep-
water value ε_o^2, and subsequently all other dimensionless parameters of the
motion in deep water, including $\overset{*}{F}{}'$. For a chosen value of kd, Eq. 2 can be
solved for ε^2, as illustrated in Fig. 1. The local values of ε^2 and kd in
turn determine the local values of all other dimensionless parameters,
including ka, $k\bar{\eta}$, and k/k_o, so that the wave height ratio $H/H_o = a/a_o$ is
known as a function of relative depth $h/\lambda_o = (d+\bar{\eta})/\lambda_o$, with the chosen
value of H_o/λ_o as a parameter.

The variation of $\overset{*}{F}{}'$ with ε^2 and kd was found to be qualitatively
similar to that of F of Cokelet's theory itself, in the sense that for
each relative depth (kd) it has a maximum $(\overset{*}{F}{}'_m$, say) for some value of ε^2
less than 1 (see Fig.1). Eq. 2 has (one) real root(s) for sufficiently
large kd-values only, such that the associated maximum of $\overset{*}{F}{}'$ is at least
as large as the given deep-water value $\overset{*}{F}{}'_o$ (see Fig. 1). In smaller relative
depths, where $\overset{*}{F}{}'_m(kd) < \overset{*}{F}{}'_o$, the waves cannot deliver the required energy
flux. The point where $\overset{*}{F}{}'_m(kd) = \overset{*}{F}{}'_o$ (whose position is determined entirely
by $\overset{*}{F}{}'_o$, or by the deep-water wave steepness H_o/λ_o) is provisionally called
the breakpoint. The shoaling calculations have been performed only up to
that point. In the range of kd where Eq. 2 has two distinct roots in ε^2,
the smallest one was taken for reasons of continuity (see Fig. 1).

The numerical values of the functions referred to above were read
from Cokelet's tables. In each of these, kd is constant, and ε^2 is varied,
from 0 to 0.8 with increments of 0.1, and from 0.8 to 1 with increments of
0.01. The tables are given for ten values of e^{-kd}, ranging from 0 (deep
water) to 0.9, with increments of 0.1. This grid proved to be too coarse
for our shoaling calculations. Using cubic splines (Ahlberg et al, 1967),
interpolated values were calculated first with respect to ε^2, with a
constant increment of 0.01, subsequently with respect to kd, with a
constant increment of 0.02. This was necessary (and sufficient) to obtain
a well-defined curve through the calculated points. However, near the
breakpoint still greater accuracy with respect to kd is required because
of the rapid variation of the wave parameters in that region. To locate
the breakpoint, the equation $\overset{*}{F}{}'_m(kd) = \overset{*}{F}{}'_o$ was solved for kd to an accuracy
of 10^{-4}, using cubic spline interpolations, after which a conventional
shoaling calculation yielded the wave height at the breakpoint.

RESULTS

The calculations were performed for different values of the deep-water
steepness H_o/λ_o, ranging from 0.001 to 0.1. The calculated values of H/H_o
were plotted versus h/λ_o and a smooth curve was drawn through the points.
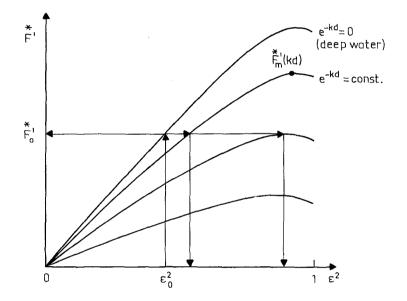
Fig. 1   Sketch illustrating procedure of shoaling calculation

The results are presented in Fig. 2 with a full line. The curve for H_o/λ_o = 0 is for the linear wave theory. The breakpoints of the non-linear shoaling curves have been connected by a smooth curve. Dashed lines indicate extrapolations.

It should be noted that the quantity λ_o is defined as the wave length in deep water, including non-linear effects. It must be distinguished from the deep-water wave length L_o in the linear approximation. Values of the ratio between the two as calculated by Cokelet (1977), are presented in Table 1, for those values of H_o/λ_o for which the shoaling curves were calculated.

Table 1   Non-linear effect on deep-water wavelength

H_o/λ_o	$0 \leq H_o/\lambda_o \leq 0.02$	0.04	0.06	0.08	0.10
λ_o/L_o	$1 \leq \lambda_o/L_o \leq 1.01$	1.02	1.04	1.07	1.10

DISCUSSIONS

The behaviour of the shoaling curves near the breakpoint deserves a special comment. The curves are seen to have a very steep gradient there. This is a consequence of the fact that (by definition) the energy flux has a maximum (in ϵ^2) at the breakpoint (see Fig. 1). This implies that an infinitesimal variation of ϵ^2 from its value at the breakpoint, at constant value of the energy flux F', gives zero variation of kd. Since for constant kd the wave steepness is a monotonically increasing function of ϵ^2, it follows that the gradient of the shoaling curve becomes infinite at the breakpoint.

The foregoing conclusion contrasts with the assumption of very gradual variation of the wave parameters, which was the justification for the use of a theory for waves of constant form in the calculation of wave shoaling. Notice also that it holds regardless of how small the bed slope is. The use of Cokelet's theory for wave shoaling calculation therefore leads to an internal inconsistency as the breakpoint is approached. Consequently, the behaviour of the waves near that point cannot be inferred from that theory, no matter how small the bed slope may be.

Comparisons with existing shoaling curves

There are several theoretical shoaling curves, calculated by using different finite amplitude wave theories.
Le Méhauté and Webb (1964) calculated shoaling curves of Stokes waves of the third order. Their curves are limited to rather large deep-water wave steepness due to the limitation of applicability of Stokes waves. Iwagaki and Sakai (1967) calculated shoaling curves for rather small deep-water wave steepness and small depth-length ratio. They used an approximate expression of Laitone's cnoidal wave theory of the second approximation which is called "hyperbolic waves". Fig. 2 shows the comparison of the shoaling curves calculated from Cokelet's theory with the above mentioned two kinds of shoaling curves. It is clear that the wave height ratio H/H_o of the present curves increases more rapidly with decreasing relative
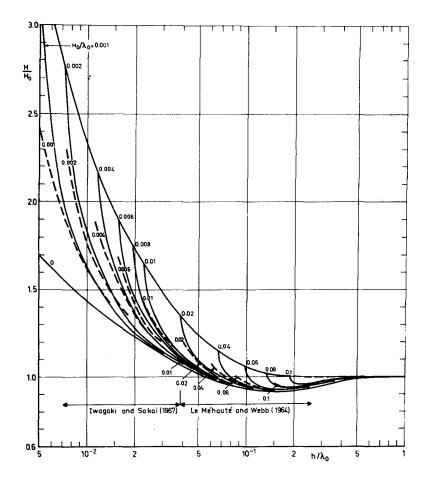
Fig. 2   Full lines: shoaling curves based on Cokelet's theory; values of
H_o/λ_o are shown.
Dashed lines, $h/\lambda_o > 0.04$: shoaling curves after Le Méhauté and Webb
(1964); values of H_o/λ_o are shown.
$h/\lambda_o < 0.04$: shoaling curves after Iwagaki and Sakai
(1967); values of H_o/λ_o are shown.

depth than that of the other two kinds of curves. In general, the difference between the values of the present and other curves becomes large with decreasing water depth. The breaking wave height of the present curves is much larger than that of the other two kinds of curves.

Svendsen and Brink-Kjaer (1972) presented shoaling curves of cnoidal waves. They used Laitone's cnoidal waves of the first approximation (1963). Fig. 3 shows the comparison with their shoaling curves. (The values were read from a table in Skovgaard, Svendsen, Jonsson and Brink-Kjaer, 1974.) In their shoaling curves, the small-amplitude deep-water wave length (L_o) was used, for which reason we have replotted our curves as H/H_o vs. h/L_o; however, we have not found it worthwhile to re-calculate a set of curves so as to obtain slightly different values of the deep-water steepness. The H/H_o-values given by Svendsen and Brink-Kjaer are considerably smaller than those of the present paper. It must be pointed out in this respect that the former were obtained by matching the calculated energy flux values of the cnoidal theory and the linear theory at $h/L_o = 0.1$, which gives a discontinuity in wave height. If the wave heights are matched instead, then the calculated H/H_o-values increase significantly (Svendsen and Buhr-Hansen, 1976).

Shuto (1974) derived several practically useful formulae for wave shoaling from an equation of non-linear long waves including effects of dispersion and variable depth. The trend of Shuto's curves is found to be nearly the same as that of Iwagaki and Sakai's curves. The values of H/H_o of Shuto's curves (not shown here) are smaller than those of the present shoaling curves.
Yamaguchi and Tsuchiya (1976) calculated shoaling curves from several kinds of finite amplitude wave theories. Their shoaling curves, calculated from Chappelear's (1962) cnoidal wave theory of the second approximation, which predict the highest wave height among their calculated curves, were compared with those of the present paper. The trend of these shoaling curves (not shown here) is also found to be nearly the same as that given by Iwagaki and Sakai.

All of the curves based on approximate shallow-water theories display a smaller rate of wave amplification as the breakpoint is approached, than the curves based on Cokelet's theory. But, as has been pointed out, the application of Cokelet's theory gives an inconsistency near the breakpoint. Therefore, although it may be an exact theory for waves of constant form, its use for the calculation of wave shoaling does not necessarily give more reliable results near the breakpoint than does the use of an approximate theory. Ironically, the inconsistency noted above has not appeared in the existing shoaling calculations, based on more approximate theories for waves of constant form, because in all of these theories the energy flux is an increasing function of wave height, up to the highest wave.

Comparison with experimental data

The theoretical shoaling curves calculated from Cokelet's theory are compared with experimental data. We have selected data published by Svendsen and Buhr-Hansen (1976), since these were obtained in waves which were relatively free from undesired disturbances. Svendsen and Buhr-Hansen's experiments were carried out on a plane beach of 1:35 slope. Six cases of
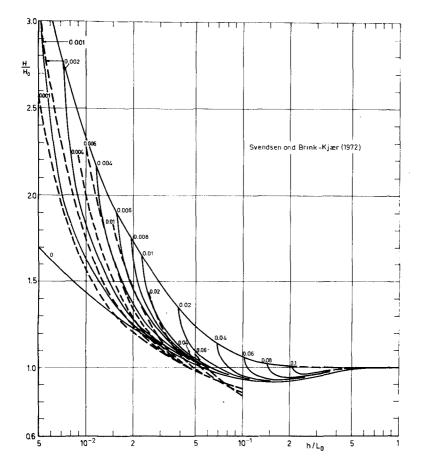
Fig. 3   Full lines: shoaling curves based on Cokelet's theory; values of
                H_o/λ_o are shown.
         Dashed lines: shoaling curves after Svendsen and Brink-Kjaer (1972);
                values of H_o/L_o are shown.

variation of wave height were discussed in their paper. These were used
for the present comparison. The data were read from data tables which were
kindly provided to the present authors.

From the experimental values of wave period, wave height and mean
water depth at the toe of the slope, a deep-water steepness (H_o/L_o) was
calculated on the basis of the shoaling theory of the present paper.
To estimate the damping due to boundary layers on the bed and the
sidewalls, we used Hunt's theory (1952). Although this theory is based on
the assumptions of small amplitude and laminar boundary layers, it is
thought to be sufficient for an estimate of the damping, which is relatively
small anyway. A damping factor C_d (ratio of local damped wave amplitude to
local undamped amplitude) was calculated at several points on the slope up
to the breakpoint. Its value was found to range from 2% to 7%. The
theoretical wave height H was then calculated as

$$H = C_d \cdot H_o \cdot (H/H_o)_t \; ,$$

where $(H/H_o)_t$ is the wave height ratio of the present theoretical curves,
and H_o is the deep-water wave height determined in the calculation of H_o/L_o.
The still water depth was corrected for wave set-down.

Fig. 4 shows the comparisons between the theoretical wave height
variation and experimental results obtained by Svendsen and Buhr-Hansen.
The theoretical curve ends at its breakpoint. It is clear that in all cases
the agreement with the experimental results is good except near the
breakpoint. The theoretical wave height increases more rapidly near the
breakpoint than the experimental wave height. The reason for this rapid
increase has already been explained.

For completeness' sake, it is noted that Svendsen and Buhr-Hansen
(1976) have compared Svendsen and Brink-Kjaer's shoaling curves (1972) to
their data. The agreement was found to be very good even near the
breakpoint, but only after the original curves were shifted upwards so as
to have continuity in wave height at the point of matching with the (linear)
theory for deeper water, at the expense of continuity in calculated energy
flux.

CONCLUSIONS

Wave shoaling curves were calculated by using Cokelet's (1977) non-
linear theory for progressive, periodic gravity waves of constant form in
arbitrary uniform depth.

It was found that the shoaling curves calculated from Cokelet's theory
show a rapid amplification as the theoretical breakpoint is approached,
and that they have an infinite slope at their breakpoints. The reason for
this is that the wave energy flux in Cokelet's theory has an intermediate
maximum before the highest wave is reached. The rapid variation of wave
height is inconsistent with the basic assumption of gradually varying wave
parameters. In this meaning, the accuracy of the present curves near the
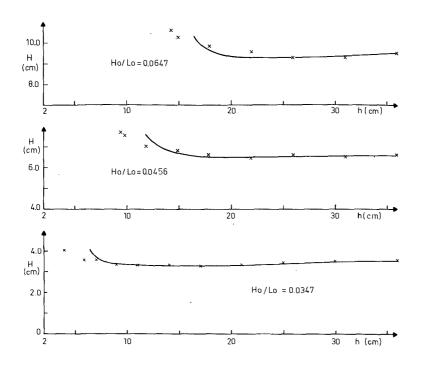breakpoint is poor.

Fig. 4,(1)   Comparison of shoaling curves based on Cokelet's theory
             (corrected for viscous damping) with experimental results of
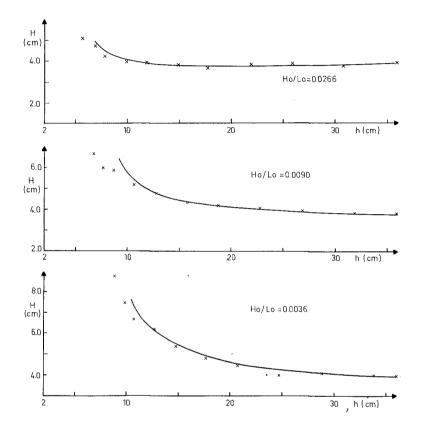   Svendsen and Buhr-Hansen (1976)

Fig. 4,(2)   Comparison of shoaling curves based on Cokelet's theory
(corrected for viscous damping) with experimental results of
Svendsen and Buhr–Hansen (1976)

The shoaling curves calculated from Cokelet's theory were compared with five kinds of existing shoaling curves calculated from different finite amplitude wave theories: Le Méhauté and Webb (1964), Iwagaki and Sakai (1967), Svendsen and Brink-Kjaer (1972), Shuto (1974), and Yamaguchi and Tsuchiya (1976). The curves of the present paper showed the largest amplification for all initial steepnesses and relative depths. However, except for the region near the breakpoint, the differences between most of the existing curves used in the comparison, and those from the present paper, are relatively minor.

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