CHAPTER 26

VOCOIDAL THEORY FOR ALL NON-BREAKING WAVES

by D.H. Swart and C.C. Loubser[1)]

ABSTRACT

The ideal theory for the prediction of wave-induced phenomena should be sufficiently accurate in all relative water depths, easy to apply and extensively tabulated. No available theory meets all these requirements. Vocoidal theory has been developed for application in all water depths. An extensive evaluation against experimental data proves that this theory meets the above requirements.

1. INTRODUCTION

During the past two decades great progress has been made in the development of new techniques for the prediction of wave-induced phenomena, such as the flow field in the vicinity of a coastal structure, longshore currents, forces on marine structures, ship motions, coastal sediment transport and combined wave diffraction and refraction. With the development of these more sophisticated techniques came the need for more accurate wave theories. Coastal sediment transport is an example of such a need.

Until about 10 years ago, the only way of predicting longshore sediment transport was to use an overall predictor [26], i.e. a formula yielding the total longshore sediment transport across the breaker zone. An example of such a theory is the well-known CERC-formula [22]. In 1967, Bijker published the first detail predictor [2], based on a uniform flow sediment transport formula. This method predicts the local longshore sediment transport rate. Subsequently, various researchers refined the Bijker technique, although the basic principle has remained the same. As the predictors become more sophisticated, the input required gets more extensive and it becomes more important to have accurate values for these input parameters. For example, because the transport is proportional to the longshore current raised to some power (between 3 and 6 [32]), a wrong prediction of the current by 10% will result in an error of 33% to 77% in the estimate of the transport. The longshore current, in turn, is determined from the balance between wave thrust (radiation stress) and dissipative forces (bed friction and lateral mixing). The Airy wave theory, which is the theory most frequently applied to water wave problems, can result in a radiation stress estimate at the breaker line up to 300% higher than the actual radiation stress. However, this does not imply that the longshore current will be overpredicted by the same amount, because the current velocity is linked to the radiation stress via an empirical roughness coefficient. It does, however, underline the importance of finding a more accurate description of the wave characteristics in shallow water.

Both for the prediction of erosional/depositional patterns in the vicinity of a coastal structure, by making use of a composite mathematical model, and for

1) National Research Institute for Oceanology, Stellenbosch, South Africa.

the computation of longshore sediment transport on a straight coast, accurate wave characteristics are required over the whole area of active transport. This zone can extend to depths of more than 50m [26], depending on the wave and sediment characteristics. For the prediction of erosional/depositional patterns wave characteristics and sediment transport computations are needed at numerous grid points. This requires large computer storage [8]. Both for this application and for the hand-computation of sediment transport rates, it must be possible to calculate the required wave characteristics easily. This can be done only if the wave theory under consideration either has simple algebraic expressions or is extensively tabulated.

A review was made of the available water wave theories [27] and the conclusion drawn that no existing theory meets all the above requirements. Vocoidal theory has therefore been developed in answer to these needs. The analytical and experimental validity of this theory is compared in Section 4 with that of other theories.

2. DERIVATION OF EXISTING WATER WAVE THEORIES

An insight into the techniques used for the derivation of the various existing theories is crucial for the understanding of the key differences between the results obtained. It is also the determining factor for the choice of a technique for the derivation of the new wave theory in Section 3.

2.1 Assumptions

The following well-known assumptions are common to all existing water wave theories and are therefore given below without any discussion, namely : (1) only non-breaking waves are considered; (2) the water movement is two-dimensional; (3) the water depth is constant, i.e. the bed is horizontal; (4) the flow is frictionless; (5) the fluid density is invariant; (6) surface tension effects are neglected; and (7) the wave motion is periodic and the waves propagate with constant velocity in water of constant depth.

2.2 Basic governing equations

The basic governing equations for the wave boundary value problem can be derived by making use of the above-mentioned assumptions. These governing equations are : (1) equation for the conservation of fluid mass, (2) equations of motion and (3) expression for the rotation of a fluid particle. These equations are given below without further explanation, as they can be found in numerous text books (e.g. [11] and [20]).

Conservation of mass (CM)

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad \ldots (1)$$

where x and z are the horizontal and vertical coordinates and u and w are the horizontal and vertical orbital velocity components.

Equations of motion (EM)

In the x-direction :

$$\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = 0 \qquad \ldots (2)$$

In the z-direction :

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = 0 \qquad \ldots (3)$$

where p is the pressure, g the gravitational acceleration, ρ the fluid density and t time.

Rotation of a fluid particle (RF)

The mean angular velocity R of an element of water is :

$$R = \tfrac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) \qquad \ldots (4)$$

If $R = 0$ the flow is irrotational, otherwise it is rotational. For cases where $R \neq 0$ Van Hijum [29] shows, by using the equations of motion, that

$$\frac{DR}{Dt} = \frac{\partial R}{\partial t} + u\frac{\partial R}{\partial x} + w\frac{\partial R}{\partial z} = 0 \qquad \ldots (5)$$

Equations (1) to (4) are the basic governing equations common to all solutions of the water wave problem.

2.3 Boundary conditions

All solutions to the system of equations (1) to (4) above should satisfy a number of boundary conditions, namely : (1) a *kinematic boundary condition* at both (a) the bed and (b) free surface, stating that no particle can pass through these boundaries (i.e. both bed and free surface are stream lines) and (2) a *dynamic boundary condition* which specifies that the pressure just inside the water mass should equal the atmospheric pressure.

re (1a) The *kinematic bed boundary condition (KBBC)* implies that

$$w = 0 \quad \text{at} \quad z = 0 \qquad \ldots (6)$$

re (1b) The *kinematic free surface boundary condition (KFSBC)* can be transformed to :

$$w = (u - c)\frac{\partial \eta}{\partial x} \quad \text{at } z = d + \eta \qquad \ldots (7)$$

where η denotes the free surface wave profile, d the mean water depth and c the wave celerity

re (2) The *dynamic free surface boundary condition(DFSBC)* states simply that

$$p = 0 \quad \text{at } z = d + \eta \qquad \ldots (8)$$

where p is the amount by which the pressure exceeds atmospheric pressure.

2.4 Solutions to the problem

The solutions found to date for the wave boundary value problem can be classified as follows : (1) irrotational theories, (2) rotational theories and (3) modified theories, which are discussed under these headings below. Discussions of these classes of theories can be found in numerous text books and original papers so that only those characteristics of the theories which are relevant to the present study are given in the following discussion. Table 5, in turn, is a summary of the discussion below.

Irrotational wave theories

When $R = 0$ in equation (5) it is possible to define a velocity potential Φ (or stream function Ψ) which, in a reference system moving along with a celerity the same as that of the wave, is given by

$$u - c = -\frac{\partial\Phi}{\partial x} = -\frac{\partial\Psi}{\partial z} \qquad \ldots (9)$$

$$w = -\frac{\partial\Phi}{\partial z} = +\frac{\partial\Psi}{\partial x} \qquad \ldots (10)$$

With the aid of the above definition for Φ, the equation for conservation of mass (equation (1)) is transformed into the well-known Laplace equation ($\nabla^2\Phi=0$) and in addition the rotation is identically zero when equations (9) and (10) are substituted into equation (5). Similarly, when the stream function Ψ is used, the rotation R in equation (5) is transformed into the Laplace equation ($\nabla^2\Psi=0$) and the conservation of mass equation (equation (1)) is exactly satisfied when equations (9) and (10) are substituted into it. Therefore

$$\nabla^2\Phi = 0 \quad \text{or} \quad \nabla^2\Psi = 0 \qquad \ldots (11)$$

In all irrotational theories, the Laplace equation is used instead of equation (1) when solving the governing equations. The velocity potential Φ (or stream function Ψ) is determined by the domain in which it exists, i.e. it is a function of η, which in turn specifies the free surface. It is, consequently, not possible to find a unique solution to equations (6), (7), (8) and (11) above. These equations must be simplified by means of approximations to linearize the problem in such a manner that the non-linear terms in the defining equations become small when compared with the linear terms. These approximations determine the specific form of the irrotational theory. Three main methods of solution are used, namely:(i) analytical perturbation of Φ, η and c, as is performed in the Stokesian wave theories, which are valid for deep water, (ii) analytical perturbation of Φ, leading to the cnoidal wave theories, which are valid for shallow water and (iii) numerical perturbation of Φ or Ψ, which leads to the so-called numerical theories, valid for $H/d \leq 0.78$.

re (i) Stokesian wave theories

Taylor series expansions are assumed for Φ, η and c. By assuming that $H/d \ll 1$ (where H is the wave height) and thus that the free surface boundary conditions (7) and (8) can be applied at the mean water level, i.e. at $z = d$, a first-order solution, called Stokes I, is found. A second-order solution is obtained by making use of the first-order solution, etc. The higher the order of the expansion, the greater the mathematical effort required to obtain a solution. Although higher-order Stokesian theories do exist, only Stokes I (Airy), II and V are used frequently. The first-order Stokes theory is valid for deep water only because of the assumption that $H/d \ll 1$. In addition, higher-order Stokesian theories become unstable in shallow water because of the magnitude of the expansion parameter.

Earlier comparisons with experimental data show that predictions given by Stokes I are good for deep water but get progressively poorer as the water depth decreases.

re (ii) Cnoidal wave theories

The assumption is made that both d/λ and H/d are <<1 where λ is the wave length, i.e. small-amplitude waves in shallow water are considered. The choice of an appropriate expansion parameter allows the expansion of Φ into a Taylor series. The Laplace equation and kinematic bed boundary condition (equation (6)) are used to simplify this Taylor expansion to an equation in terms of the first-order term in Φ. Only the terms necessary to obtain the first-order cnoidal theory are retained. Substitution of this expression for Φ into the free-surface boundary conditions (equations (7) and (8)) yields the well-known long-wave equations, which are simplified to the Korteweg-de Vries equation and solved for η and c. Cnoidal theories can only be applied in shallow water, because of the assumptions made regarding the expansion parameter which yields imaginary wave celerities for $\lambda_o/d \lesssim 10$. However, comparison with measurements show that, provided the wave length is known, the cnoidal theory offers a good representation of the wave profile for deep to shallow water. This is not surprising since the cnoidal wave profile has as limiting cases a sinusoidal profile in deep water and a solitary wave profile in shallow water.

re (iii) Numerical wave theories

The rapid development of digital computers over the last three decades has opened up a new way of solving the defining equations (6) to (8) and (11) above by finding a solution, using direct numerical calculation. The best known numerical irrotational theories published to date are those of Chappelear [3], Dean [6], Von Schwind and Reid [31] and Cokelet [4]. The principle used is straightforward, namely : (1) find a Fourier series solution to the Laplace equation, and (2) optimize iteratively the coefficients in the Fourier series by means of the free-surface boundary conditions. The order of the Fourier series determines the accuracy to which the boundary conditions can be approximated. Each additional term of the Fourier series expansion added reduces the error in the free-surface boundary conditions. Dean used a variable number of terms in the Fourier expansion, and terminated the iterative procedure as soon as consecutive iterations (additions of further terms) resulted in small improvements in the error in the free-surface boundary conditions. The number of Fourier terms in Dean's solution varies between 2 (for deep water) and 19 (for shallow water). Dean showed that his stream function solution corresponds more closely to the free-surface boundary conditions than the theories of Chappelear and Von Schwind/ Reid do. Cokelet raised the level of accuracy by using 110 Fourier terms. As a result,his solution to Laplace is more accurate than any other to date. However, the advantage of Dean's method over that of Cokelet is that Dean's results are tabulated more conveniently for engineering use. Furthermore, the resulting celerities for 24 arbitrarily chosen sets of initial conditions (T, d, H) as computed by using Dean's and Cokelet's theories differed by a maximum of 2.8%; the mean difference being 1.4%. In all cases Cokelet's theory yielded larger celerities. The conclusion is that the improvement obtained in the celerity by extending the Fourier series to the 110th order is not significant for engineering purposes. Furthermore, Cokelet's theory has not yet been tabulated to permit the prediction of time-dependent wave properties, such as the wave profile and orbital velocities.

Earlier comparisons with measurements have shown Dean's stream function theory to be in good agreement with data for all water depths within its tubulated range.

Rotational wave theories

The assumption of frictionless flow can normally be regarded as a good approximation over the greater portion of the water column. Frictionless flow implies irrotationality. It is possible to obtain an easy solution for the horizontal orbital velocity by making use of the conservation of mass equation (1). Because frictionless flow is also assumed for this approach, the resulting theory should be an irrotational theory. However, it is possible that due to approximations made in finding a solution, a finite value of R, as given by equation (4), remains. Such theories will be called rotational theories. The equation for conservation of mass (equation (1)) is integrated over the depth. After substitution of the kinematic boundary conditions (equations (6) and (7)) and assuming no net mass transport (which is valid as long as no set-up occurs) an equation is found for the mean horizontal orbital velocity $\bar{u}$, namely :

$$\bar{u} = \frac{c\eta}{d+\eta} \qquad \ldots (12)$$

As no assumptions are made regarding rotationality, equation (12) is equally valid for irrotational wave theories and can, in fact, be used to check whether any given theory adheres to the original continuity equation (1) (see Table 5). Any expression for the instantaneous horizontal orbital velocity u which satisfies equation (12) therefore also satisfies equation (1). Expressions for the wave celerity c and wave profile η are found from the equations of motion, by making use of an assumed form of the orbital velocity u. Two recent examples of this approach are the rotational wave theories of Van Hijum [30] and Mejlhede [19].

To date no comparison has been made between actual measurements and predictions given by rotational theories.

Modified wave theories

As stated above, the simple-to-apply Airy wave theory yields predictions which become poorer as the water depth decreases. Various modifications have therefore been suggested to improve Airy theory's correspondence to wave characteristics in shallow water. The two most successful modifications are (1) Goda's empirical modification [9] of the Airy horizontal orbital velocity and (2) Hedges' theoretical modification [10][18] of the Airy wave celerity. Rather than modify Airy, another approach is to simplify the cnoidal solution, which is in any case applicable to shallow water conditions. In this manner, a good theory is obtained for shallow water which at the same time is easy to apply. The best example of such a simplified theory is Van Hijum's simplification [30] of the cnoidal wave profile, in which the cnoidal function is approximated by a cosine function raised to a variable power (depending on T, d, H).

Earlier comparisons with measured horizontal orbital velocities showed that Goda's approximation for the orbital velocity under the wave crest is superior to that predicted by Airy theory and is surpassed only by predictions for horizontal orbital velocity made with Dean's stream function theory.

2.5 Summary

The discussion above indicates that :

(1) Numerical theories show the greatest overall adherence to boundary conditions.

(2) None of the available theories agrees well with data for all water depths. The best theory which is still applicable in practice is Dean's numerical stream function theory.
(3) No one analytical theory is clearly superior to any of the other analytical theories as far as experimental validity is concerned.

The derivation of a new theory is described in the next section. This theory incorporates the properties of the best contemporary theories and is easy to apply.

3. VOCOIDAL WATER WAVE THEORY

3.1 General

As indicated in Section 2 above, the theoretical derivation of a water wave theory involves the approximation of the free-surface boundary conditions. Depending upon these approximations, various classes of solutions are found. A different approach is used below, namely, (1) qualitative assumptions are made regarding the expressions for η, c and u in terms of unknown parameters (Section 3.2), (2) by substituting these qualitative expressions into the free-surface boundary conditions, three equations are found from which the above-mentioned parameters can be solved numerically (Sections 3.3 and 3.4) and (3) algebraic formulae for the parameters are determined by means of curve-fitting techniques (Section 3.5).

3.2 Qualitative assumptions

Wave profile η

A wave profile of the type given by the cnoidal theory provides the best fit to data, mainly because it has as theoretical limiting forms a sinusoidal wave in deep water and a solitary wave in extreme shallow water. Van Hijum [30] showed that the following expression has the same characteristics :

$$\frac{\eta}{H} = (\cos^2\pi X)^P - \eta_{*t} \qquad \text{... (13)}$$

where $X = x/\lambda$, η_{*t} is the dimensionless ($\div$ H) trough elevation and P is a parameter which depends on the wave conditions (H/d, λ/d). By expressing $(\cos^2\pi X)^P$ as an infinite series, η_{*t} is found, after integration, to be :

$$\eta_{*t} = (\pi P)^{-\frac{1}{2}}\{1 - (8P)^{-1} + \tfrac{1}{2}(8P)^{-2}\} \qquad \text{... (14)}$$

When P = 1 equation (13) represents a sinusoidal profile and when P>>1 the profile given by equation (13) tends towards a solitary profile. Equation (13) is used here to define the wave profile η.

A new function, which simplifies the representation of formulae in the new wave theory, namely the variable order cosine function, or *voc*-oidal function, is defined :

$$\text{voc}\ (P,X) = (\cos^2\pi X)^P \qquad \text{... (15)}$$

where P is the order of the function.

Wave celerity c

Hedges [10] has shown that when applying the free-surface boundary conditions at some fixed "effective" level d_e the wave celerity is given by :

$$\frac{c^2}{gd} = \frac{1}{kd} \tanh Nkd \qquad \ldots (16)$$

where $N = d_e/d$. He shows that N tends to unity in deep water (same result as given by Stokes I) and to $1 + \eta_c/d$ in very shallow water (η_c = crest elevation of wave - same result as given by solitary theory). Equation (16) is used here to define the wave celerity c. N remains to be determined in terms of the wave conditions (H/d, λ/d).

Orbital velocity u

The approach leading to equation (12) is used to define the horizontal orbital velocity u. Mejlhede [19] shows that the following expression for u satisfies equation (12) :

$$u = \frac{c\eta\, M(X)k \cosh [M(X)kz]}{\sinh [M(X)k\,(d+\eta)]} \qquad \ldots (17)$$

where M(X) is a function of the wave conditions (H/d, λ/d) and also of X. Equation (17) is used to define the orbital velocity u in the new wave theory. Thus the new theory is not derived from the Laplace equation. It does not, however, necessarily imply that the theory is rotational, as will be seen in Section 3.3.1. Equation (17) represents an exact solution to the continuity equation which satisfies both kinematic boundary conditions.

3.3 Formulation of the governing equations

The wave boundary-value problem has now been reduced to a set of equations in which the unknowns are not η, c and u but the wave profile parameter P, the wave celerity parameter N and the orbital velocity parameter M(X). An expression for M(X) in terms of P is found below by using equation (4), after which optimum values of P and N are found by evaluating the pressure p from the equations of motion and equating it to the pressure in Bernoulli's equation.

3.3.1 A value for M(X) in terms of P

Equation (5) specifies that any particle retains its rotation. For the free surface, which is a stream line, it is possible to write $R[z=d+\eta]$= constant. An expression for R at the free surface is found by combining equations (4), (7) and (17) :

$$[R]_{z=d+\eta} = \tfrac{1}{2}\left[\frac{\partial}{\partial x}\left\{(u-c)\frac{\partial\eta}{\partial x}\right\} - c\eta(M(X)\;k)^2\right]_{z=d+\eta} \qquad \ldots (18)$$

which can be rewritten as

$$M(X)^2 = \left[\frac{\dfrac{\partial u}{\partial x}\dfrac{\partial \eta}{\partial x} + (u-c)\dfrac{\partial^2\eta}{\partial x^2} - 2R}{c\eta k^2}\right]_{z=d+\eta} \qquad \ldots (19)$$

At the wave trough, that is at X = 0.5, equation (19) reduces to

$$M(0.5)^2 = \left[\frac{-2R}{c\eta_t k^2}\right]_{z=d+\eta} \qquad \ldots (20)$$

where subscript t denotes the wave trough.

R $[z=d+\eta] = 0$ (and consequently $M(0.5) = 0$) is a solution to equation (20). Therefore $R[z=d+\eta] = 0$ is used . At the wave trough, where $M(0.5) = 0$, equation (17) reduces to equation (12). Substitution of this equation (12) into equation (4) indicates that R $[X=0.5] = 0$ for all $0 \leq z \leq d + \eta$. *As all stream lines in the flow field pass through $X = 0.5$, this implies that the flow will be irrotational, provided that $M(X)$ is determined exactly from equation (18).* However, since small approximations have to be made to obtain a usable solution (see below) the flow will not be completely irrotational but will retain a small rotation, which can be neglected when secondary wave properties are derived.
Substitution of zero rotation into equation (19) and rearrangement yields :

$$M(X)^2 = \left[\frac{\frac{\partial(u/c)}{\partial X}\,\frac{\partial(\eta/H)}{\partial X} + (u/c - 1)\,\frac{\partial^2(\eta/H)}{\partial X^2}}{4\pi^2(\eta/H)} \right]_{z=d+\eta} \qquad \ldots (21)$$

Equation (21) provides an implicit equation for $M(X)$ in terms of P and X.

3.3.2 Optimum values of P and N

Integration of equation (3) to z and substitution into the resulting equation of the dynamic free-surface boundary condition results in an expression for the pressure p, which at the bed (z=0) becomes :

$$\left[\frac{p}{\rho} \right]_{z=0} = g(d+\eta) + \left[\frac{u^2 + w^2}{2} - uc \right]_0^{d+\eta} \qquad \ldots (22)$$

Since the bed is a stream line,the pressure at the bed should also satisfy Bernoulli's equation, that is

$$\left[\frac{p}{\rho} \right]_{z=0} = \left[gd - \frac{(u-c)^2}{2} \right]_{z=0} + k_1 \qquad \ldots (23)$$

where k_1 is a constant.

The pressure can be eliminated from equations (22) and (23) to yield, after substituting the kinematic bed boundary condition into the resulting equation, an expression for c^2/gd :

$$\frac{c^2}{gd} = \left[\frac{\eta/d}{(u/c) - \frac{1}{2}\{(u/c)^2 + (w/c)^2 - (w_o/c)^2\}} \right]_{z=d+\eta} \qquad \ldots (24)$$

where $w_o = - c \left.\frac{\partial\eta}{\partial x}\right|_{\eta=0,\ z=d+\eta}$, that is, w_o is the value of w at the free surface when $\eta=0$.

Because of assumption (7) in Section 2.1 above, the wave celerity defined by equation (24) must be a constant, that is

$$\frac{\partial}{\partial X} \left[\frac{c^2}{gd} \right] = 0 \qquad \text{for } 0 \leq X \leq 0.5 \qquad \ldots (25)$$

Equations (21), (24) and (25) are the governing equations used to find solutions for P, N and M(X).

3.4 Solving the governing equations

The solution is determined in two phases, namely (1) a first-order solution is found by assuming $u/c \ll 1$, and (2) a second-order correction is applied to compensate for the fact that u/c is not negligible with respect to unity. This two-phase approach is necessary because M(X) cannot be determined at wave breaking.

The exact solution to equations (21), (24) and (25) above can only be obtained numerically for $u/c<1$. The following wave conditions (H/d, λ/d) are used :

$0.01 \leq H/d \leq 0.8$	$1 \leq \lambda/d \leq 75$
0.01, 0.02, 0.05, 0.1(0.1)0.8	1, 2, 5, 8(2) 20, 20(10) 50, 75

Equation (25) is satisfied by computing c^2/gd from equation (24) at 1000 discrete intervals in the area $0 \leq X \leq 0.5$ and choosing that combination of P and N for which the relative standard deviation of the computed c^2/gd-values is the least.

Although the maximum theoretical breaker index H/d is about 0.78, experimental data indicate that H/d-values as high as 1.3 can occur. To ensure that the theory developed herein is applicable in this area the numerical solution of equations (21), (24) and (25) is repeated with the assumption that $u/c \ll 1$ for an extended H/d range $0.01 \leq H/d \leq 1.3$. The following ratios are computed for those wave conditions at which an exact solution was found above.

$$R_P = P/P_1$$

$$R_N = N/N_1 \qquad \ldots (26)$$

and $$R_M = M(0)/M_1(0)$$

where subscript 1 denotes the solution with the small-amplitude assumption (that is $u/c \ll 1$) and P, N and M(0) denote the "exact" solution, that is the solution in which no assumption is made regarding the value of u/c.

The results obtained for the above-mentioned wave conditions indicate the following :

(1) It is possible to find solutions for P_1, N_1 and $M_1(0)$ for all wave conditions listed ($H/d \leq 1.3$).
(2) The values of R_M and R_N vary systematically with H/d for all λ/d-values.
(3) For any given λ/d-value N tends to $1 + \eta_c/d$ for large H/d.
(4) For any given λ/d-value M(0) tends asymptotically towards a constant value with varying H/d. For all practical purposes M(0) is constant for $H/d \geq 0.7$.
(5) R_P varies systematically with λ/d for all H/d-values. The variation of R_P with H/d is analogous to the variation of P_1 with H/d at large λ/d-values (say λ/d = 50).

The above observations imply that it is possible to use the small-amplitude solution to extend the range of the "exact" solution.

3.5 Algebraic expressions for P, N and M(X)

The values determined as described above for P_1, N_1, $\dot{M}_1(0)$, R_P, R_N and R_M are correlated to the wave conditions (H/d, λ/d) and algebraic expressions are determined by curve-fitting. The resulting algebraic expressions are listed below.

Wave profile parameter P

(1) $$P_1 = \begin{cases} 1.03\,F_s + 9 + 3.33\exp(-0.109(11 + F_s)) & \text{for } F_s > -11 \\ 1 & \text{for } F_s \le -11 \end{cases} \quad \ldots (27)$$

where $$F_s = \frac{U_r - U_{ro}}{\Theta} \quad \ldots (28)$$

$$U_r = (H/d)(\lambda/d)^2 = \text{Ursell parameter}$$

$$U_{ro} = 63 + 90\,(H/d)^{1.48} \quad \ldots (29)$$

$$\Theta = \begin{cases} 1.01\exp(3.31(H/d)) & \text{for } H/d \ge 0.505 \\ 5.38 & \text{for } H/d < 0.505 \end{cases} \quad \ldots (30)$$

(2) $$R_P = R_{Pi} - (R_{Pi} - 1)\exp(b\,\lambda/d) \quad \ldots (31)$$

where $$R_{Pi} = 1 + 0.0021\,(P_{1i} - 1) + 6.09 \times 10^{-7}(P_{1i} - 1)^{2.56} \quad \ldots (32)$$

$$b = -0.0916 + 2.718 \times 10^{-4} P_{1i} \quad \ldots (33)$$

$$P_{1i} = P_1\text{-value for } \lambda/d = 50$$

(3) $$P = R_P\,P_1 \quad \ldots (34)$$

Wave celerity parameter N

(4) $$N_1 = \begin{cases} 1 + 0.19\,F_N^{\,1.5} & \text{for } F_N < 0.72 \\ 0.6 + 0.72\,F_N & \text{for } F_N \ge 0.72 \end{cases} \quad \ldots (35)$$

where $F_N = (H/d)(\lambda/d)^{0.1}$ ... (36)

(5) $$R_N = \begin{cases} 1 & \text{for } F_{RN} < 0.093 \\ 0.67\, F_{RN} + 0.938 & \text{for } F_{RN} \geq 0.093 \end{cases}$$ ... (37)

where $F_{RN} = (H/d)(\lambda/d)^{-\frac{1}{4}}$ ... (38)

(6) $$N = R_N N_1$$ ... (39)

Orbital velocity parameter M(X)

(7) For any given wave condition (H/d, λ/d) equation (19) can be used to obtain numerical values of M(X) for $0 \leq X \leq 0.5$. These M(X)-values vary systematically from M_p at X = 0 to zero at X = 0.5. It is shown in [27] that M(X) can be closely approximated by

$$M(X) = \begin{cases} (M_p - M_t)\cos\pi X + M_t & \text{for } r_1 P_1 < 0.5 \\ M_p \ \mathrm{voc}(r_1 P_1, X) & \text{for } r_1 P_1 \geq 0.5 \end{cases}$$ ... (40)

M_p is defined in (8) to (10) below and M_t in (11) below and where r_1 as determined by curve-fitting equals :

$$r_1 = 0.4\ \{1 - P_1^{-0.9}\}$$ ... (41)

Maximum value M_p of orbital velocity parameter M(X)

(8) The maximum value M_p of M(X) as determined by equation (19) is always found at X = 0. An expression for $M_1(0)$ is determined analytically from equation (19) by setting X = 0 and rearrangement :

$$M_1(0) = \left(\frac{P_1}{2\eta_{*c1}}\right)^{\frac{1}{2}}$$ ... (42)

where $\eta_{*c1} = 1 - \eta_{*t1}$ and subscript $_1$ refers to the first-order approximation.

(9) $$R_M = \begin{cases} \dfrac{0.54}{1+6.7(H/d)^4} + 0.46 & \text{for } H/d \leq 0.7 \\ 0.667 & \text{for } H/d > 0.7 \end{cases}$$ ... (43)

(10) $$M_p = M(0) = \begin{cases} R_M\, M_{p1} & \text{for } H/d \leq 0.7 \\ (R_M\, M_{p1})_{H/d=0.7} & \text{for } H/d > 0.7 \end{cases}$$ ... (44)

Minimum value M_t of orbital velocity parameter M(X)

(11) The solution to equation (20) indicates that $M_t = M(0.5) = 0$. However, when $P_1=1$, that is, for very deep water, $r_1 = 0$, thereby indicating a constant M(X)-value for $0 \le X < 0.5$. The resulting discontinuity at X = 0.5 is unrealistic. To a lesser extent the same phenomenon occurs for $1 \le P_1 \le 2$. To prevent the discontinuity from occurring, the following approximations are made : (i) M(X)-values in the area $0.45 \le X \le 0.49$ are extrapolated to X = 0.5 and the value thus found is used as a minimum value M_t for M(X) where

$$M_t = R_M \exp \{ -4.2 (P_1 - 1)\} \qquad \ldots (45)$$

and (ii) the X-variation of M(X) is determined by equation (40a) rather than by equation (40b). It is shown in [27] that the effect of the approximations on the predicted horizontal orbital velocity amounts to less than 2% of the actual velocity for any location within the fluid.

Because M(X) as determined from equation (40) is an approximate solution to equation (19), the flow will not be irrotational. However, computations covering the same wave conditions used in Section 3.4 above indicate that $\bar{R}$ never exceeds a few per cent of $(2\pi/T)$ and can for practical purposes be assumed equal to zero. This implies that although the flow is, as a result of the approximation of M(X), rotational, the rotation is negligible.

3.6 Presentation of other wave properties

As stated in Section 1, the aim of the study is the derivation of a simple-to-apply wave theory in which wave characteristics can be computed from algebraic expressions. For this reason, algebraic expressions were also derived in [27] for the following primary wave characteristics, namely, vertical orbital velocity, horizontal and vertical orbital excursions, mass transport velocity, group velocity, wave energy/unit surface area and the pressure within the wave. Where necessary, further parameters were introduced and expressions determined for them by curve-fitting techniques, in a similar manner as in Section 3.5 above.

A quantitative knowledge of secondary wave-induced phenomena, such as shoaling of water waves, wave refraction, diffraction and breaking, principal radiation stresses, wave set-up, generation of longshore breaker-zone currents, shear stresses exerted on the bed due to combined current and wave action, wave-induced bed-form generation, sediment entrainment and sediment suspension, is essential for the prediction of the effect of waves on the coastal environment. These aspects will be studied in follow-up reports to [27].

3.7 Tabulation of parameters

To facilitate the easy use of the theory, it is essential to tabulate the main parameters in terms of the wave conditions (H/d, λ/d), in a way similar to that for linear wave theory. It is intended to publish such a book of tables in 1979. However, abridged tables are included here as Tables 1 to 4.

4. VERIFICATION OF WATER WAVE THEORIES

4.1 Evaluation

The various theories mentioned in Section 2 are evaluated in this section (see Table 6). The data sets used in the evaluation are summarized in Table 7. There are various methods for the verification of wave theories. Dean [7] mentions two methods, namely (1) an analytical verification, that is, the adherence of each

TABLE 1 : λ/d AS A FUNCTION OF H/d AND $T\sqrt{g/d}$

$T\sqrt{g/d}$ \ H/d	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
1	.16	.16	.16	.16	.16								
2	.64	.64	.64	.64	.64	.64							
4	2.51	2.51	2.51	2.51	2.52	2.53	2.53	2.54	2.54				
6	4.91	4.91	4.91	4.92	4.97	5.05	5.13	5.21	5.28	5.39			
8	7.17	7.18	7.18	7.20	7.29	7.45	7.61	7.78	7.97	8.26	8.53	8.77	
10	9.34	9.34	9.35	9.37	9.49	9.72	9.95	10.20	10.52	10.98	11.42	11.84	12.23
12	11.45	11.45	11.47	11.49	11.63	11.92	12.22	12.54	13.00	13.62	14.21	14.78	15.33
14	13.53	13.53	13.55	13.59	13.73	14.08	14.45	14.83	15.44	16.20	16.94	17.66	18.35
16	15.59	15.59	15.61	15.66	15.81	16.22	16.65	17.09	17.84	18.75	19.63	20.49	21.33
18	17.64	17.64	17.66	17.71	17.88	18.34	18.82	19.34	20.23	21.27	22.29	23.29	24.26
20	19.67	19.68	19.70	19.76	19.93	20.45	20.99	21.57	22.60	23.78	24.94	26.07	27.17
25	24.74	24.75	24.78	24.86	25.06	25.68	26.37	27.09	28.52	30.03	31.51	32.96	34.39
30	29.78	29.79	29.83	29.93	30.19	30.89	31.71	32.58	34.41	36.24	38.05	39.82	41.58
35	34.82	34.83	34.88	34.99	35.30	36.08	37.04	38.05	40.29	42.45	44.57	46.68	48.73
40	39.84	39.85	39.91	40.05	40.41	41.25	42.34	43.63	46,17	48.65	51.11	53.51	55.88
45	44.86	44.87	44.94	45.10	45.51	46.42	47.64	49.19	52.05	54.86	57.64	60.35	63.02
50	49.88	49.89	49.97	50.14	50.61	51.59	52.94	54.75	57.94	61.07	64.16	67.19	70.17
75	74.93	74.95	75.07	75.36	76.10	77.35	79.37	82.60	87.44	92.18	96.87	101.44	105.95
100	99.95	99.99	100.15	100.55	101.60	103.08	105.76	110.57	117.07	123.43	129.70	135.83	141.86

TABLE 2 : WAVE PROFILE PARAMETER (P) AS A FUNCTION OF H/d AND λ/d

λ/d \ H/d	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
1	1.00	1.00	1.00	1.01	1.02								
2	1.00	1.00	1.01	1.01	1.03	1.05							
4	1.00	1.00	1.01	1.03	1.06	1.09	1.12	1.13	1.12				
6	1.00	1.01	1.02	1.04	1.08	1.13	1.17	1.18	1.18	2.91			
8	1.00	1.01	1.02	1.05	1.20	1.20	1.22	1.24	2.89	4.79	6.19	7.18	
10	1.00	1.01	1.04	1.48	2.26	2.92	3.42	3.67	5.92	7.48	8.44	9.00	9.34
12	1.00	1.01	1.33	2.10	3.68	5.31	6.91	8.24	10.17	11.09	11.36	11.31	11.14
14	1.00	1.01	1.67	2.87	5.50	8.46	11.60	14.50	15.78	15.71	15.01	14.14	13.31
16	1.00	1.14	2.09	3.80	7.75	12.41	17.57	22.54	22.82	21.39	19.43	17.52	15.88
18	1.00	1.31	2.57	4.90	10.46	17.19	24.84	32.37	31.30	28.15	24.64	21.47	18.86
20	1.02	1.51	3.12	6.18	13.63	22.81	33.40	43.98	41.24	36.00	30.65	26.00	22.26
25	1.30	2.11	4.85	10.23	23.61	40.41	60.28	80.69	72.35	60.34	49.11	39.82	32.58
30	1.66	2.88	7.10	15.51	36.41	62.85	94.75	128.44	112.34	91.27	72.36	57.15	45.47
35	2.09	3.83	9.91	22.03	51.86	89.87	136.68	187.42	161.23	128.67	100.28	77.86	60.87
40	2.61	4.97	13.31	29.76	69.83	121.35	185.99	257.80	219.03	172.48	132.77	101.89	78.72
45	3.21	6.32	17.29	38.63	90.25	157.22	242.61	339.66	285.69	222.63	169.78	129.18	98.96
50	3.90	7.88	21.85	48.61	113.09	197.51	306.46	433.01	361.15	279.04	211.26	159.70	121.58
75	8.90	19.06	52.77	114.46	263.18	462.16	731.16	1069.56	867.24	652.74	484.39	360.24	270.11
100	16.64	35.74	96.45	206.64	472.91	831.74	1326.40	1979.69	1580.47	1175.31	865.54	640.13	477.55

TABLE 3 : DIMENSIONLESS TROUGH DEPTHS (η_{At}) AS A FUNCTION OF H/d AND λ/d

λ/d \ H/d	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
1	.4980	.4978	.4974	.4966	.4951								
2	.4979	.4976	.4966	.4953	.4923	.4893							
4	.4977	.4973	.4957	.4930	.4874	.4819	.4775	.4759	.4763				
6	.4976	.4970	.4948	.4911	.4832	.4754	.4691	.4666	.4674	.3170			
8	.4975	.4967	.4941	.4891	.4649	.4646	.4618	.4583	.3180	.2511	.2223	.2069	
10	.4974	.4965	.4909	.4262	.3550	.3163	.2941	.2846	.2271	.2028	.1914	.1855	.1621
12	.4973	.4963	.4459	.3667	.2841	.2390	.2108	.1936	.1747	.1675	.1658	.1659	.1672
14	.4973	.4962	.4047	.3188	.2351	.1911	.1639	.1469	.1409	.1412	.1444	.1487	.1532
16	.4972	.4736	.3679	.2601	.1994	.1585	.1336	.1182	.1175	.1213	.1272	.1338	.1405
18	.4972	.4475	.3353	.2484	.1773	.1351	.1126	.0988	.1004	.1059	.1131	.1210	.1290
20	.4952	.4225	.3068	.2224	.1514	.1175	.0973	.0848	.0876	.0937	.1015	.1101	.1189
25	.4496	.3657	.2497	.1743	.1155	.0885	.0725	.0627	.0662	.0725	.0803	.0891	.0985
50	.4065	.3182	.2081	.1421	.0932	.0710	.0579	.0497	.0532	.0590	.0662	.0745	.0834
35	.3676	.2790	.1769	.1195	.0782	.0594	.0482	.0412	.0444	.0497	.0562	.0638	.0722
40	.3332	.2467	.1532	.1030	.0674	.0512	.0413	.0351	.0381	.0429	.0489	.0558	.0635
45	.3030	.2200	.1347	.0905	.0593	.0450	.0362	.0306	.0334	.0378	.0433	.0496	.0566
50	.2766	.1978	.1200	.0807	.0530	.0401	.0322	.0271	.0297	.0338	.0388	.0446	.0511
75	.1865	.1284	.0775	.0527	.0348	.0262	.0209	.0172	.0192	.0221	.0256	.0297	.0343
100	.1373	.0940	.0574	.0392	.0259	.0196	.0155	.0127	.0142	.0165	.0192	.0223	.0258

The dotted line indicates the approximate theoretical wave breaking limit

TABLE 4 : MAXIMUM ORBITAL VELOCITY PARAMETER (M_p) AS A FUNCTION OF H/d AND λ/d

λ/d \ H/d	.01	.02	.05	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00
1	1.000	1.000	1.000	1.000	1.000								
2	1.000	1.000	1.000	1.000	1.000	1.000							
4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000				
6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000			
8	1.000	1.000	1.000	1.000	1.018	1.000	1.000	1.000	1.005	1.104	1.104	1.104	
10	1.000	1.000	1.002	1.115	1.261	1.326	1.308	1.216	1.320	1.313	1.313	1.313	1.313
12	1.000	1.000	1.084	1.260	1.514	1.670	1.721	1.673	1.642	1.540	1.540	1.540	1.540
14	1.000	1.000	1.173	1.415	1.776	2.018	2.131	2.114	1.970	1.779	1.779	1.779	1.779
16	1.000	1.037	1.270	1.579	2.047	2.371	2.539	2.548	2.301	2.026	2.026	2.026	2.026
18	1.000	1.087	1.372	1.750	2.325	2.727	2.945	2.974	2.633	2.278	2.278	2.278	2.278
20	1.001	1.139	1.480	1.928	2.609	3.085	3.347	3.392	2.964	2.533	2.533	2.533	2.533
25	1.085	1.285	1.769	2.397	3.332	3.977	4.335	4.406	3.781	3.175	3.175	3.175	3.175
30	1.179	1.446	2.081	2.888	4.060	4.856	5.298	5.389	4.584	3.814	3.814	3.814	3.814
35	1.283	1.621	2.410	3.391	4.783	5.721	6.242	6.353	5.376	4.450	4.450	4.450	4.450
40	1.394	1.806	2.752	3.899	5.500	6.576	7.175	7.305	6.162	5.083	5.083	5.083	5.083
45	1.513	2.001	3.101	4.408	6.211	7.423	8.101	8.249	6.944	5.715	5.715	5.715	5.715
50	1.638	2.203	3.456	4.915	6.917	8.266	9.022	9.188	7.723	6.345	6.345	6.345	6.345
75	2.332	3.285	5.241	7.418	10.410	12.440	13.585	13.845	11.594	9.487	9.487	9.487	9.487
100	3.097	4.412	7.007	9.893	13.876	16.587	18.119	18.472	15.448	12.622	12.622	12.622	12.622

TABLE 5 : REVIEW OF AVAILABLE PERIODIC WATER WAVE THEORIES

NAMES OF THEORIES	GOVERNING EQUATIONS	IRROTATIONAL FLOW	ANALYTICAL VALIDITY ADHERENCE TO				ADVANTAGEOUS PROPERTIES	REFERENCE TO LITERATURE
			CONTINUITY	KBBC	KFSBC	DFSBC		
Stokes I, II, V, gravity cnoidal	La Place La Place	✓ ✓	x ✓	✓ ✓	x x	x x	c, η, u in *deep* water η in *all* water depths for known λ	[1], [5], [11], [23], [24], [15], [19], [25]
Dean's numerical Ψ	La Place	✓	✓	✓	✓	x	c, η in *tabulated range*	[6], [7]
Hedges modified c Goda's empirical u	La Place N/A	✓ -	x x	✓ ✓	x x	x x	c in *all* waterdepths u at wave crest in *all* depths	[10], [18] [9]
Van Hijum's simplified cnoidal (P-wave)	Conservation of mass and eq. of motion	x	✓	✓	✓	x	useful *simplification* of cnoidal η	[30 [
Mejlhede's cnoidal	Conservation of mass and eq. of motion	x	✓	✓	✓	✓	useful higher order expression of u	[19]
Van Hijum's numerical	rotational flow	x	✓	✓	✓	✓	all boundary conditions exactly satisfied for both [13] and [29]	[29]

TABLE 6 : THEORIES EVALUATED

NAME OF THEORY	THEORY NUMBER	REFERENCE TO LITERATURE
Stokes 1 (Airy)	1	Airy [1], Ippen [11]
Stokes II	3	Stokes [24], Ippen [11]
Stokes V	10	De [5], Skjelbreia [23]
Dean	11	Dean [6], [7[
Svendsen's cnoidal	4	Svendsen [25]
Keulegan et al's cnoidal	12	Keulegan et al [15]
Mejlhede's cnoidal	13	Mejlhede [19]
Mejlhede 1 ; β=0	6	Mejlhede [19]
Mejlhede 2 ; β≠0	7	Mejlhede [19]
Hedges	9	Hedges [10], Lewis [18]
Goda	2	Goda [9]
P-wave	8	Van Hijum [30]
Vocoidal	5	Swart [27]

TABLE 7 : NUMBER OF DATA SETS FOR EACH DATA ORIGIN

DATA ORIGIN	ORBITAL VELOCITY u	WAVE PROFILE η	WAVE LENGTH λ/d
Le Mehauté et al (1978)[17]	8	1	
Iwagaki and Sakai (1970)[12]	12	21	
Morison and Crooke (1953)[21]	5		
Goda (1964)[9]	18		
Van Hijum (1972)[30]	25		
Tsuchiya & Yamaguchi (1972)[28]		2	75
Knoesen (1976)[16]		36	
Wiegel (1960) [33]		2	
Dean (1965) [6]		1	
Iwagaki & Yamaguchi (1972)[13]			67
Karpul (1968)[14]	29		
NRIO (1977)	27	217	217
TOTAL	124	280	359

theory to the free-surface boundary conditions, and (2) an experimental verification, based on the adherence of each theory to observed data. Another possibility exists, namely, (3) a comparison of each theory with an exact solution to the governing equations, such as Cokelet's numerical theory. Possibilities (1) and (3) test the extent to which any given theory adheres to the governing equations and boundary conditions, but do not necessarily indicate the correlation with observed data. The reason for this is that the governing equations are in themselves approximations; they do not include the effects of, for example, surface tension and the bed slope, both of which are important near wave breaking. Provided that observations can be made with sufficient accuracy under controlled conditions, i.e. when errors due to both measuring equipment (techniques) and the generation of waves in too shallow water are minimized, experimental verification is the most reliable of the above methods. However, the available data are frequently contaminated, especially by secondary waves because of the method of generation used. Because of the large number of data sets used (see Table 7) errors due to both generation and measuring techniques are to some degree averaged out.

Dean's computations [7] regarding the adherence of various theories to the free-surface boundary conditions indicate that this measure (analytical verification) is not always a true reflection of the relative validity of the various theories. For example, the Stokes I (Airy) theory (derived for deep water) has a smaller dynamic free surface boundary condition error than the cnoidal theory (derived for shallow water) under shallow water conditions ($H/H_b = 1$ where b denotes breaking waves, $d/\lambda_o < 0.195$) (Dean [7]). Therefore, the analytical verification, in the way defined by Dean [7], is not discussed further in this paper. However, an extensive analysis of the analytical verification of the above 13 theories will be contained in a follow-up to [27], which is being prepared at present.

A practical comparison with Cokelet's theory is not yet possible, because the tabulation of parameters in Cokelet's theory permits only the calculation of wave characteristics which are independent of time. However, the comparison between the theories of Dean and Cokelet, mentioned in Section 2.4 above, indicates that Dean's theory can, for engineering purposes, be assumed to be an "exact" solution to the "approximate" governing equations.

The following procedure has been adopted to establish experimental validity :

(1) The relative error E_r for each wave characteristic W is given by :

$$E_r = \left| \frac{W_p - W_m}{W_m} \right| \qquad \ldots (46)$$

where subscripts p and m denote predicted and measured values, respectively.

(2) Relative errors for the orbital velocity u under the wave crest, wave length λ and wave profile η are computed by applying each theory to all data sets for which relevant data are available.

(3) To assist the evaluation of the application range of the various wave theories, an Ursell-like parameter which depends solely on the wave conditions T, d and H, is defined, namely,

$$F_c = (H/d)^{\frac{1}{2}}\, T_c^{\,2\frac{1}{2}} \qquad \ldots (47)$$

where $T_c = T\,(g/d)^{\frac{1}{2}} = (2\pi)^{\frac{1}{2}}\,(\lambda_o/d)^{\frac{1}{2}}$ ... (48)

A wave with a height H = 1m and period T = 8s has F_c-values of 0.3, 17, 934 and 3143 in water depths of 200m, 20m, 2m and 1m respectively, that is; F_c varies from zero in very deep water to a few thousand in very shallow water. The values of F_c for shallow water are limited by wave breaking.

(4) For each wave characteristic and wave theory the mean and standard deviation of the absolute values of all relative errors are computed for each of the following F_c-groups (0-50, 50-100, 100-200, 200-500, 500-1000 and > 1000). Those theories which do not differ at a 5% significance level from the theory with the smallest mean relative error in each of the above-mentioned F_c-groups are singled out as being the best theories for the prediction of the specific wave characteristic in that F_c-group. A theory is considered to fall within its validity area if it passes this statistical test.

(5) The resulting validity areas of the various theories for the prediction of u, λ and η are shown in Figures 1 to 3. Also shown in each F_c-group is the average of the mean relative errors of the equivalent good theories.

(6) Figure 4 indicates the overall validity areas of the evaluated theories and, as such represents a generalized experimental validity, which is not bound to a specific wave characteristic. The theories listed in this figure are those with a higher than average occurrence in any given F_c-group (that is those appearing two or three times in the specific F_c-group on Figures 1 to 3).

4.2 Results

(1) According to Figure 4 only two wave theories are valid for all F_c-groups, namely, Dean's stream function theory and Vocoidal theory, that is, only these two theories are suggested for application in all relative water depths (theory numbers are listed in Table 6).

(2) However, Dean's tabulated range restricts the application of his theory at both high and low H/d values.

(3) In the restricted range of Dean's tables there is no significant difference between Dean's results and those obtained by Vocoidal theory. It can therefore be concluded that Vocoidal theory is equivalent at a 5% significance level to Dean's theory, which in turn was shown, for engineering purposes, to be an "exact" solution to the governing equations.

(4) The wave profile predicted by Vocoidal theory for F_c<200 is not statistically equivalent to the best theories in this range. The relative errors in the 0-50, 50-100 and 100-200 F_c-groups for Vocoidal theory are 6%, 7% and 8% respectively, as compared to 5%, 5% and 7% for the best theories.

(5) Within its application range, i.e. for $(T\sqrt{g/d})^{-1}$ <0.126, Mejlhede's gravity cnoidal theory is statistically equivalent to Dean's stream function theory and Vocoidal theory. The same does not apply to the Keulegan/Patterson and Svendsen cnoidal theories [15], [25].

(6) Stokes I, II and V are statistically equivalent to the better theories for $F_c \le 200$, 100 and 200 respectively, which indicates that the use of Stokes II theory for anything but deep water is normally not advisable and that Stokes I is, generally speaking, equivalent to Stokes V.

(7) Goda's theory is listed in the $500 \le F_c \le 1000$ column only because of its good adherence to orbital velocity data.

(8) Mejlhede's rotational cnoidal theory is statistically equivalent to the better theories for $F_c \le 200$. Beyond this range, it yields unrealistic horizontal orbital velocities under the wave crest, which tend to zero at the bed and to some extremely high value (far greater than the wave celerity) at the free surface.

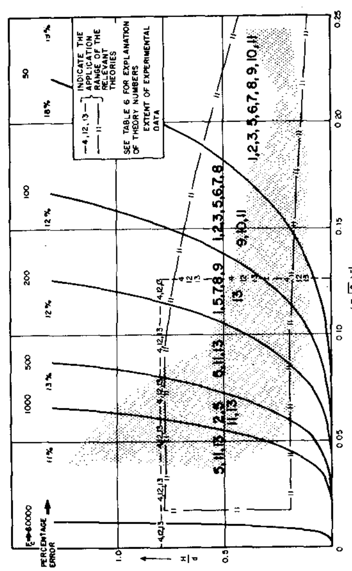

FIGURE 1 BEST THEORIES FOR THE PREDICTION OF THE ORBITAL VELOCITY UNDER THE WAVE CREST

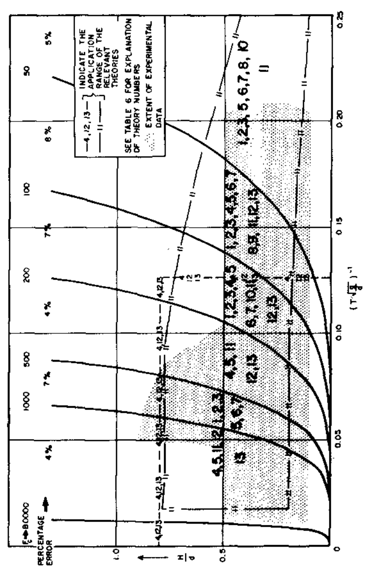

FIGURE 2 BEST THEORIES FOR THE PREDICTION OF THE WAVE LENGTH

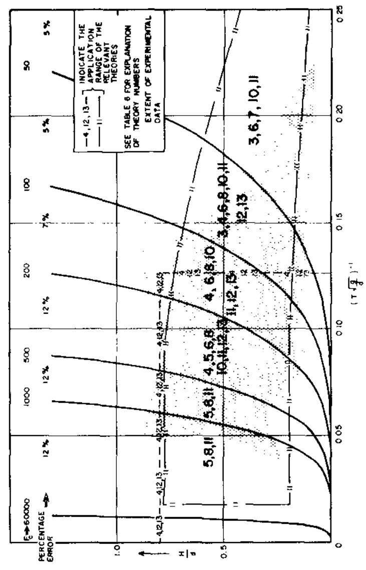

FIGURE 3 BEST THEORIES FOR THE PREDICTION OF THE WAVE PROFILE

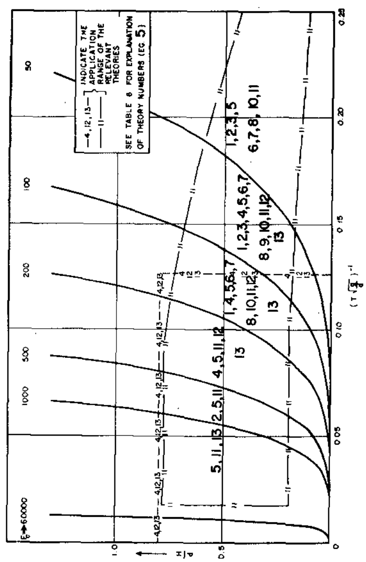

FIGURE 4 REGIONS OF OVERALL VALIDITY : VARIOUS WAVE THEORIES

(9) The simple-to-apply Van Hijum-approximation to cnoidal theory is statistically equivalent to the better theories for $F_c \leq 200$. It should preferably not be used beyond this range because the predicted horizontal orbital velocity tends to a constant value over the full depth from bed to free surface.

5. SUMMARY AND CONCLUSIONS

1. There is a need for a water wave theory which is (i) sufficiently accurate in all relative water depths, (ii) expressed in terms of algebraic expressions, (iii) relatively easy to apply/require little computer time and (iv) well-tabulated. None of the available theories meets all these requirements.

2. Vocoidal theory has therefore been developed from first principles in answer to these needs. The theory is simple to apply and is expressed algebraicly (see equations (13), (16) and (17)).

3. Frictionless flow was assumed in the derivation of Vocoidal theory, but because of an approximation made in the representation of the time-variation of the horizontal orbital velocity, the theory contains a small rotation, which can, for practical pruposes, be neglected.

4. An extensive analysis of the experimental validity of thirteen different theories indicates that only two of the evaluated theories consistently yield good results in all relative water depths, namely, Vocoidal theory and Dean's stream function theory.

5. Dean's theory is only applicable in a restricted H/d-range (see - 11 - line in Figure 4), which can never be extended beyond H/d = 0.78.

6. Vocoidal theory is therefore recommended for general application for all relative water depths.

LIST OF REFERENCES

1. AIRY C.B. Tides and waves; Encyc. Metrop. Art. No 192, 1845.
2. BIJKER E.W. Some considerations about scales for coastal models with movable bed; Delft Hydraulics Laboratory, Publ. No 50, 1967.
3. CHAPPELEAR J.E. Direct numerical calculation of wave properties; Journal of Geophysical Research, Vol. 66, No 2., 1961.
4. COKELET E.D. Steep gravity waves in water or arbitrary uniform depth; Phil. Trans. of the Royal Society of London, Series A, Vol 286, 1977.
5. DE S.C. Contributions to the theory of Stokes; Proc. Cambridge Phil. Soc., No 51, 1955.
6. DEAN R.G. Stream function representation of nonlinear ocean waves; Journal of Geophysical Research Vol 70 No 18, 1965.
7. DEAN R.G. Evaluation and development of water wave theories for engineering application; Vol 1 & 2, Special report No 1, prepared for U.S. Army, Corps of Engineers, CERC, 1974.
8. FLEMING C.A. The development and application of a mathematical sediment transport model; Ph. D. Thesis, University of Reading, 1977.
9. GODA Y. Wave forces on a vertical circular cylinder : Experiments and a proposed method of wave force computation; Report of Port and Harbour Technical Research Institute No 8, 1964.
10. HEDCES T.S. An empirical modification to linear wave theory; Proc. Instn. Civ. Engrs. Part 2 No 61, 1976.
11. IPPEN A.T. Estuary and coastline hydrodynamics; McGraw-Hill, New York 1966.
12. IWACAKI Y. and SAKAI T. Horizontal water particle velocity of finite amplitude waves; 12th Conf. Coast. Eng. Vol 1, 1970.
13. IWACAKI Y. and YAMACUCHI M. Studies on cnoidal waves (sixth report) - Limiting condition for application of cnoidal wave theory;Annuals,D.P.R.I., Kyoto University, No 11B, 1968.

14. KARPUL M. Investigation into aspects of orbital velocity of waves in shoaling water; M.Sc. Thesis, University of Cape Town, 1978.
15. KEULEGAN G.H. and PATTERSON G.W. Mathematical theory of irrotational translation waves; J. Res. Mat. Bur. Stand., No 24, 1940.
16. KNOESEN J.S. 'n Sistematiese ondersoek van golwe in 'n kanaal ten opsigte van die vorm daarvan, die orbitaalbeweging daaronder en die vergelyk van bogenoemde met die eerste orde teorie; Final year thesis, University of Stellenbosch, R.S.A., 1976.
17. LE MEHAUTE B., DIVOKY D. and LIN A. Shallow water waves : a comparison of theories and experiments; 11th Conf. Coast. Eng. Vol 1, 1968.
18. LEWIS A.W. Discussion on an empirical modification to linear wave theory by T.S. Hedges; Proc. Instn. Civ. Engrs. Part 2, 1977.
19. MEJLHEDE N. Cnoidal waves on arbitrary depth; Prog. Rep 36, Inst. Hydrodyn. and Hydraulic Engrg. Tech. Univ. Denmark, 1975.
20. MILNE-THOMSON L.M. Theoretical hydrodynamics; Chapter XIV, 2nd Ed. 1949.
21. MORISON J.R. and CROOKE R.C. The mechanics of deep water, shallow water, and breaking waves; U.S. Army, Corps of Engineers, Beach Erosion Board, TM. No 40, 1953.
22. SHORE PROTECTION MANUAL Vol 1, U.S. Army CERC, 1973.
23. SKJELBREIA L. and HENDRICKSON J.A. Fifth order gravity wave theory; Proc. 7th Conf. Coast. Eng., 1961.
24. STOKES G.G. On the theory of oscillatory waves; Trans. Cambridge Phil. Soc. Vol 8, and supplement, Sci. Papers Vol 1, 1847.
25. SVENDSEN I.A. Cnoidal waves over a gently sloping bottom; Series Paper 6, ISVA, Technical University of Denmark, 1974.
26. SWART D.H. Predictive equations regarding coastal transports; 15th Conf. Coast. Eng. Vol 2, 1976.
27. SWART D.H. Vocoidal water wave theory, Vol 1 : Derivation ; Coastal Engineering and Hydraulics Division, NRIO, CSIR, 1978.
28. TSUCHIYA Y. and YAMAGUCHI M. Some considerations on water particle velocities of finite amplitude wave theories; Coast. Eng. in Japan, Vol 15, 1972.
29. VAN HIJUM E. Niet-rotatie vrije benadering van de waterbeweging bij regelmatige korte golven in water met een konstante diepte; Speurwerkverslag S171 deel II, 1975.
30. VAN HIJUM E. P-golf; Unpublished note, 1972.
31. VON SCHWIND J.J. and REID R.O. Characteristics of gravity waves of permanent form; Journal of Geophysical Research, 1972.
32. WHITE W.R., MILLI H. and CRABBE A.D. Sediment transport : an appraisal of available methods; HRS, Wallingford, Report INT 119, Vols 1 and 2, 1973.
33. WIEGEL R.L. A presentation of cnoidal wave theory for practical application; Jour. Fluid. Mech. 7, Part 2, 1960.