CHAPTER 34

DIFFRACTION OF GRAVITY WAVES BY LARGE ISLANDS

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Abstract. The combined refraction and diffraction of long gravity water waves for certain depth variations around large islands is investigated analytically in a circular-symmetric geometry. Creeping waves are shown to exist for bottom profiles less convex than required by the trapping criterion due to Longuet-Higgins and Shen et al. From an asymptotic representation of the solution to the scattering problem the decay exponent and the diffraction coefficient is extracted. These "canonical" quantities may then be used for the construction of diffracted fields around smooth islands of more complex shape in accordance with J.B. Keller's Geometrical Theory of Diffraction.

INTRODUCTION

Analytical investigations of refraction and diffraction effects for gravity water waves are not too frequent in the literature. Sager [1-4] has studied the pure refraction phenomenon for various bottom profiles while the pure diffraction problem for constant depth is identical with well-known scattering problems in acoustics and electromagnetic theory. Combinations of the two phenomena have been treated by Homma [5], Vastano and Reid [6], and Lautenbacher [7]. In the present paper another example is added to this collection of solutions.

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GRAVITY WAVES ON FREE WATER SURFACE

Small waves on shallow water. Waves on free water surfaces can propagate under the influence of gravity. Such gravity waves satisfy the equations

$$(1) \quad \nabla\eta + \frac{1}{g}\bar{v}_t = 0$$

and

$$(2) \quad \nabla\cdot(h\bar{v}) + \eta_t = 0$$

where $\eta = \eta(\bar{r};t)$ is the elevation of the water surface above the undisturbed level, $\bar{v} = \bar{v}(\bar{r};t)$ is the horizontal velocity field, $h = h(\bar{r})$ is the water depth, and g is the constant acceleration due to gravity. The position vector of the field point is $\bar{r}$ and t is the time. The equations are valid for small amplitudes ($\eta \ll h$) and shallow water ($h \ll L$) where L is the wave length. Combination of (1) and (2) yields the modified two-dimensional wave equation for η

$$(3) \quad \nabla\cdot(h\nabla\eta) = \frac{1}{g}\eta_{tt} .$$

We shall consider time-harmonic solutions with angular frequency ω

$$(4) \quad \eta(\bar{r};t) = \eta(\bar{r})e^{-i\omega t} .$$

Here $\eta(\bar{r})$ must satisfy the modified two-dimensional Helmholtz's equation

$$(5) \quad \nabla\cdot(h\nabla\eta) + \frac{\omega^2}{g} \quad = 0 .$$

A special depth profile. In the present paper we consider a special circular-symmetric depth profile (see Fig. 1)

$$(6) \quad h(\bar{r}) = h_o\left(\frac{r}{r_o}\right)^{2\alpha} \qquad 0 \leqslant \alpha < 1$$

where r is one of the polar coordinates (r,θ) for $\bar{r}$. The power of r has been denoted 2α for convenience. On the circle $r = r_o$ $h = h_o$, the reference depth. Insertion of (6) into (5) yields

$$(7) \qquad \nabla\cdot\left(\left(\frac{r}{r_o}\right)^{2\alpha} \nabla\eta\right) + k_o^2\eta = 0$$

where we have introduced the reference propagation constant

$$(8) \qquad k_o = \frac{\omega}{\sqrt{h_o g}} .$$

The advantage of our choice (6) for $h(\bar{r})$ is that the solutions to (7) can be expressed in terms of well-known functions of the polar coordinates.

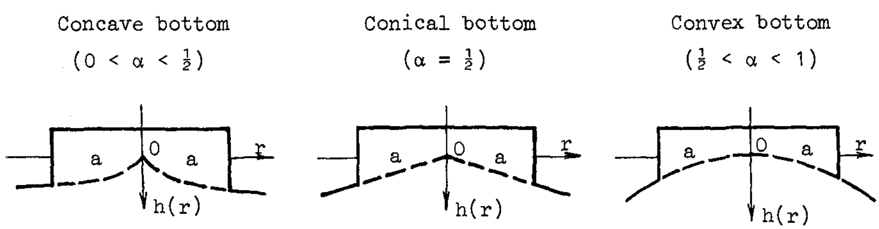

Fig. 1. Circular island (radius a and water depth $h(\bar{r}) = (\frac{r}{r_o})^{2\alpha}$).

Short waves. In the short-wavelength limit ($k_o r \gg 1$) the elevation can be represented by

$$(9) \qquad \eta(\bar{r}) \sim A(\bar{r})e^{ik_o S(\bar{r})}$$

where the phase S and the amplitude A are reel functions of $\bar{r}$ that remains bounded for $k_o r \to \infty$. The assumption (9) which is used in geometrical optics (see [8], e.g.) is sometimes called Debye's assumption. Insertion of (9) in (7) yields the eiconal equation

$$(10) \qquad (\nabla S)^2 = \left(\frac{r_o}{r}\right)^{2\alpha}$$

and the transport equation

$$(11) \quad \frac{2\alpha}{r}\frac{\partial S}{\partial r} + 2\nabla A \cdot \nabla S + A\Delta S = 0 .$$

The ray tracing and the phase variation. The wave fronts are the curves $S(\bar{r})$ = const. where $S = S(\bar{r})$ are solutions to (10). Is is, however, easier to determine the wave orthogonals or rays as extremal curves to the Fermat's principle

$$(12) \quad \delta\int_{P_1}^{P_2} (\frac{r_o}{r})^{\alpha} ds = 0 .$$

According to (10) the water surface can be viewed as an inhomogeneous two-dimensional medium with refractive index $(r_o/r)^{\alpha}$. The distance element measured along the curves $r = r(\theta)$ is denoted ds. The variational principle (12) therefore states that the rays from a point P_1 to a point P_2 proceed in such a manner that the passage time becomes stationary. The obvious wave orthogonals, θ = const., through the top of the profile at the origin of the coordinate system, 0, are not included in (12).

By solving of the Euler equation corresponding to (12) we find that rays are sine spirals. Thus the family of rays through P_s (r_s, θ_s) away from 0 becomes

$$(13) \quad (\frac{r_s}{r})^{1-\alpha} = \frac{\sin(\phi_s - (1-\alpha)(\theta-\theta_s))}{\sin\phi_s} \qquad \text{for} \quad \theta > \theta_s$$

where the parameter ϕ_s is the angle between the unit vectors $\hat{r}_s$ and $\hat{t}_s$ at P_s (see Fig. 2). The angle ϕ between the unit vectors $\hat{r}$ and $\hat{t}$ at P can be shown to be

$$(14) \quad \phi = \phi_s - (1-\alpha)(\theta-\theta_s) .$$

Integration of (10) along the rays through P_s given by (13) yields

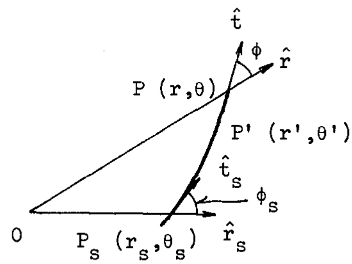

Fig. 2. Sine spirals through P_s (r_s,θ_s).

$$(15)\quad S(P) = S(P') + \frac{r_o^{\alpha}\, r_s^{1-\alpha}}{1-\alpha} \sin\phi_s(\mathrm{ctn}(\phi_s - (1-\alpha)(\theta-\theta_s)) - \mathrm{ctn}(\phi_s - (1-\alpha)(\theta'-\theta_s))) .$$

Here $S(P)$ $(S(P'))$ is the phase at a point P (r,θ) $(P'$ $(r',\theta'))$. The points P and P' lie on the same ray through P_s. (See Fig. 2.)

Energy conservation. By means of (10) the transport equation can be converted into

$$(16)\quad \frac{dA}{A} = -\tfrac{1}{2}\left(\frac{\alpha}{r}\,\hat{r}\cdot\hat{t} + \nabla\cdot\hat{t}\right)ds ,$$

where $\hat{r}$ and $\hat{t}$ are the unit vectors shown in Fig. 2. Integration of (16) along the rays (13) yields after some calculation

$$(17)\quad A(P) = A(P') \left(\frac{\sin(\phi_s - (1-\alpha)(\theta-\theta_s))}{\sin(\phi_s - (1-\alpha)(\theta'-\theta_s))}\right)^{\frac{\alpha}{1-\alpha}+\frac{1}{2}} \times \left(\frac{\sin((1-\alpha)(\theta'-\theta_s))}{\sin((1-\alpha)(\theta-\theta_s))}\right)^{\frac{1}{2}} .$$

Here P and P' must lie on the same side of P_s. Like (11) equation (17) expresses the fact that the energy is conserved along a pencil of rays.

EXCITATION OF GRAVITY WAVES

Green's function. Gravity waves are usually excited by an incident plane wave. However, for $r \to \infty$ our model is not valid because $h \to \infty$ according to (6). Thus the shallow water condition ($h \ll L$) is violated. We shall therefore consider a point source placed at a finite point P_s (r_s, θ_s). The source has the somewhat artificial property that it adds the volume of water pr. length and time unit, $q(t)$, in a vertical thin column from the bottom of the sea to the water surface. Furthermore, the source oscillates harmonically with time such that

$$(18) \quad q(t) = q_s e^{-i\omega t},$$

where q_s is a constant. Mathematically, we have played for safety since the wave field excited by the source simply is a Green's function. This is determined as the solution to the inhomogeneous Helmholtz's equation

$$(19) \quad \nabla \cdot \left(\left(\frac{r}{r_o} \right)^{2\alpha} \nabla \eta \right) + k_o^2 \eta = - \ell_s \frac{\delta(r - r_s)\delta(\theta - \theta_s)}{r}$$

which is obtained by addition of the source term, $h(r_s)q(t)\delta(\bar{r}-\bar{r}_s)$, on the right hand side of (2). The position vector for P_s is $\bar{r}_s$ and δ is Dirac's delta function. The source strength (a complex length) has been denoted

$$(20) \quad \ell_s = - i\omega \left(\frac{r_s}{r_o} \right)^{2\alpha} \frac{q_s}{g} .$$

Furthermore, the Green's function must satisfy the radiation condition

$$(21)\qquad \lim_{r\to\infty} r^{\frac{3\alpha+1}{2}} \left(\frac{\partial\eta}{\partial r} - ik_o\left(\frac{r_o}{r}\right)^{\alpha}\eta\right) = 0 .$$

Separation of variables yields the Green's function

$$(22)\qquad \eta(r,\theta;r_s,\theta_s) = \frac{i\ell_s}{4}\frac{1}{1-\alpha}\left(\frac{r_o^2}{r_s r}\right)^{\alpha}$$

$$\times \sum_{n=-\infty}^{\infty} H^{(1)}_{\frac{\sqrt{\alpha^2+n^2}}{1-\alpha}}\left(\frac{1}{1-\alpha}(k_o r_o)^{\alpha}(k_o r_s)^{1-\alpha}\right)$$

$$\times J_{\frac{\sqrt{\alpha^2+n^2}}{1-\alpha}}\left(\frac{1}{1-\alpha}(k_o r_o)^{\alpha}(k_o r)^{1-\alpha}\right)e^{in(\theta-\theta_s)}$$

$$\text{for } r < r_s ,$$

where $H^{(1)}_{\nu}(z)$ $(J_{\nu}(z))$ is the Hankel function of first kind (Bessel function) of argument z and order ν. For the derivation of (22) a standard result from [9] has been used. The excitation for $r > r_s$ is obtained by interchange of r and r_s in (22) due to reciprocity.

Asymptotic representation. By transforming (22) into a contour integral we obtain the representation

$$(23)\qquad \eta(r,\theta;r_s,\theta_s) \sim \ell_s D_s(\phi_s)e^{ik_o(S(P)-S(P_s))}\frac{A(P)}{A(P'_{s,1})}$$

in the short-wavelength limit $(k_o r_o)^{\alpha}(k_o r_s)^{1-\alpha}/(1-\alpha) \gg 1$ by means of the Debye representations of the cylinder functions, the method of steepest descent and (13) after a lengthy calculation. We have written (23) in a form which shows that the field, η, can be viewed as being produced by a ray through P_s and the observation point P (r,θ) (see Fig. 2). The angle ϕ_s is then determined from (13) by insertion of the value of (r,θ) at P. In the phase factor, $\exp\{ik_o(S(P)-S(P_s))\}$, the difference between the phases S(P) and

$S(P_s)$ at P and P_s, respectively, is given by (15) with $\theta' = \theta_s$. The point $P'_{s,1}$ (r',θ') on the ray P_sP is a reference point with the property

$$(24) \quad r_s^{\frac{1}{2}}[\sin(1-\alpha)(\theta'-\theta_s)]^{\frac{1}{2}}[\sin(\phi_s-(1-\alpha)(\theta'-\theta_s)]^{-\frac{\alpha}{1-\alpha}-\frac{1}{2}} = 1 .$$

As a consequence of (17) the divergence factor for the pencil of rays emanating from P_s becomes

$$(25) \quad \frac{A(P)}{A(P'_{s,1})} = \frac{(\sin(\phi_s-(1-\alpha)(\theta-\theta_s))^{\frac{1}{2}+\frac{\alpha}{1-\alpha}}}{r_s^{\frac{1}{2}}(\sin((1-\alpha)(\theta-\theta_s)))^{\frac{1}{2}}} .$$

For $\alpha = 0$ this expression reduces to $r_{P_sP}^{-\frac{1}{2}}$ where r_{P_sP} is the distance from P_s to P. Finally, the source factor

$$(26) \quad D_s(\phi_s) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{8\pi k_o}} \left(\frac{r_o}{r_s}\right)^{\frac{3\alpha}{2}} (1-\alpha)^{\frac{1}{2}}(\sin\phi_s)^{-\frac{\alpha}{1-\alpha}}$$

describes the emission of rays from a unit source at P_s. For $\alpha \neq 0$ the radiation is anisotropic due to the sloping bottom at the source. The asymptotic representation of the Green's function (22) thus confirms the ray interpretation of the field and yields a determination of the source factor $D_s(\phi_s)$ (26).

SCATTERING OF GRAVITY WAVES

A circular island. As an example of a scattering problem we consider diffraction by a circular island with center at 0 and radius a (see Fig. 3). The mathematical formulation of the problem then consists of (19) for $r \geq a$ and $r_s \geq a$, (21), and the boundary condition

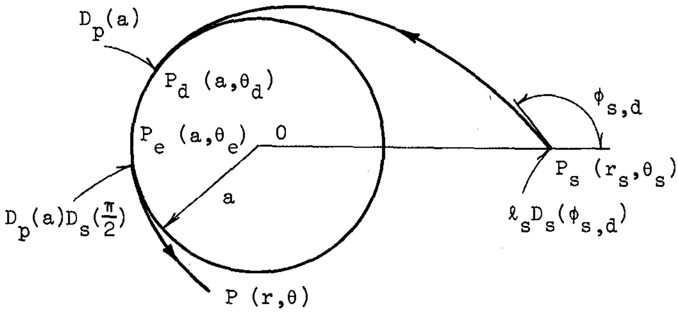

Fig. 3. Diffraction by a circular island. Propagation in negative direction and multiple circulations are not shown.

$$\frac{\partial \eta}{\partial r} = 0 \qquad \text{for } r = a \tag{27}$$

which expresses full reflection at the island.

Sommerfeld's method. The solution can be written in the following form

$$\eta(r,\theta;r_s,\theta_s) = \frac{\ell_s}{k_o a}\left(\frac{r_o}{a}\right)^{2\alpha} \sum_{p=1}^{\infty} \frac{\cos\nu_p(\theta-\theta_s\mp\pi)}{\sin\nu_p\pi} \tag{28}$$

$$\frac{R_{\nu_p}(k_o r_s)R_{\nu_p}(k_o r)}{R_{\nu_p}(k_o a)\left.\frac{\partial^2}{\partial k_o r\partial\nu} R_{\nu_p}(k_o r)\right|_{\substack{r=a\\ \nu=\nu_p}}} \qquad \text{for } \left.\begin{matrix}0\\-\pi\end{matrix}\right\} \leqslant \theta-\theta_s \leqslant \left\{\begin{matrix}\pi\\0\end{matrix}\right.$$

by means of Sommerfeld's method [10]. Here

$$(29)\quad R_\nu(k_or) = \frac{1}{(k_or)^\alpha} H^{(1)}_{\frac{\sqrt{\alpha^2+\nu^2}}{1-\alpha}} \left(\frac{1}{1-\alpha}(k_or_o)^\alpha(k_or)^{1-\alpha}\right)$$

and ν_p is the p'th root in the equation

$$(30)\quad \frac{\partial}{\partial k_or} R_\nu(k_or)\Big|_{r=a} = 0$$

in the first quadrant of the complex ν-plane. The advantage of an expression like (28) is that it gives a rapidly convergent representation of the diffracted field in the short-wave length limit $(k_or_o)^\alpha(k_or_s)^{1-\alpha}/(1-\alpha) >> 1$. The diffracted field is present everywhere outside the island. It is, however, only a dominant part of the total field in the geometric-optics shadow region, i.e. the region where no incident or reflected field is present (see [11], e.g.). Such a region can be shown to exist behind the island with respect to the source point when $0 \leq \alpha < \frac{1}{2}$. For larger values of α the rays become so curved that every point behind the island is reached by a direct ray from the source. The rest of this paper is devoted to an investigation of the diffracted field.

Asymptotic representation. By insertion of the Debye representations and the transition region representations (see [12], e.g.) of the cylinder functions in (28) we have obtained the following result

$$(31)\quad \eta(r,\theta;r_s,\theta_s) \sim \ell_s D_s(\phi_{s,d}) e^{ik_o(S(P_d)-S(P_s))} \frac{A(P_d)}{A(P'_{s,1})}$$

$$\times \sum_{p=1}^{\infty} D_p(a)\Big[e^{(ik_o(\frac{r_o}{a})^\alpha - \beta_p(a))a(\theta_e-\theta_d)}$$

$$+ e^{(ik_o(\frac{r_o}{a})^\alpha - \beta_p(a))a(2\pi-\theta_e+\theta_d)}\Big] \times \sum_{q=0}^{\infty} e^{(ik_o(\frac{r_o}{a})^\alpha - \beta_p(a))a2\pi q}$$

$$\times D_p(a) D_s(\tfrac{\pi}{2}) e^{ik_o(S(Q)-S(P_e))} \frac{A(Q)}{A(P'_{e,1})}\;.$$

Below we shall discuss the symbols used in this formula. It turns out that the diffracted field at P (r,θ) can be viewed as being produced by the ray system partly illustrated in Fig. 3.

The creeping wave. The essential feature of the ray tracing is the existence of a creeping wave travelling around the island in positive and negative direction (from P_d (a,θ_d) to P_e (a,θ_e) in Fig. 3). The creeping wave which is excited by a tangentially incident ray, P_sP_d, continuously launches tangentially diffracted rays, P_eP e.g., into the surrounding sea. In this sense the creeping wave is a "semi-trapped" wave which only exists for $0 \leq \alpha < 1$. According to a criterion due to Longuet-Higgins [13] and Shen et al. [14] trapped waves around circular islands are found when $O(h(r)) > r^2$, i.e. for $\alpha > 1$. The creeping wave travels with the free space propagation constant at the boundary of the island ($= k_o(r_o/a)^\alpha$) and is strongly damped due to the energy loss to the diffracted rays. This phenomenon is described by the decay exponent, $\beta_p(a)$. As a consequence, the phase factor for the creeping wave, P_dP_e, becomes $\exp\{(ik_o(r_o/a)^\alpha - \beta_p(a))a(\theta_e-\theta_d)\}$ as seen in (31). From the asymptotic representation of (28) follows

$$(32) \qquad \beta_p(a) \sim 2^{-\frac{1}{3}} e^{-i\frac{\pi}{6}} x_p'(1-\alpha)^{\frac{2}{3}} (k_o(\frac{r_o}{a})^\alpha a)^{\frac{1}{3}} a^{-1}$$

where $-x_p'$ is the p'th zero of the derivative of the Airy function of first kind

$$(33) \qquad \mathrm{Ai}(x) = \frac{1}{\pi}\int_o^\infty \cos(\frac{1}{3}t^3 + xt)dt .$$

Finally, (31) shows that the creeping wave possesses a modal structure which is expressed in the summation with index p.

The incident ray and the diffracted ray. For the incident ray, P_sP_d, $\phi_{s,d}$ is given by ϕ_s in (13 and 14) with $(r,\theta) = (a,\theta_d)$ and $\phi = \frac{\pi}{2}$. The emission of rays from the source is described by ℓ_s (20) and $D_s(\phi_{s,d})$ given by (26) with $\phi_s = \phi_{s,d}$. The phase factor and the divergence factor becomes $\exp\{ik_o(S(P_d) - S(P_s))\}$ and $A(P_d)/A(P'_{s,1})$ given by (15) and (25), respectively, with $\theta' = \theta_s$, $\theta = \theta_d$ and $\phi_s = \phi_{s,d}$.

We obtain the coordinate θ_e for P_e by letting $r_o = a$, $\phi_s = \frac{\pi}{2}$, and (r,θ) equal to the coordinates of P in (13). The emission of a pencil of diffracted rays from P_e is described by $D_s(\frac{\pi}{2})$ given by (26) with $\phi_s = \frac{\pi}{2}$. The phase factor and the divergence factor for the ray, P_eP, are $\exp\{ik_o(S(P) - S(P_e))\}$ and $A(P)/A(P'_{e,1})$, respectively, where $P'_{e,1}$ is a reference point on P_eP corresponding to $P'_{s,1}$ on P_sP_d. These factors are still given by (15) and (25), respectively, but now $\theta' = \theta_s = \theta_e$, $r_s = a$, and $\phi_s = \frac{\pi}{2}$.

The diffraction processes. The remaining factor in the asymptotic representation of (28) is viewed as describing the diffraction processes at P_d and P_e where the creeping wave starts and ends. Due to reciprocity the description of these two events must be the same function of the radius of curvature of the island (and of the depth at the coast line), $D_p(a)$. As a consequence $D_p^2(a)$ occurs in (31) and we find

$$(34) \qquad D_p(a) \sim \frac{2^{\frac{1}{3}} e^{-i\frac{\pi}{12}}}{\sqrt{x'_p}\, Ai(-x_p)} (1-\alpha)^{-\frac{1}{6}} \left(\frac{a}{r_o}\right)^{\frac{\alpha}{2}} \left(k_o\left(\frac{r_o}{a}\right)^{\alpha} a\right)^{\frac{1}{6}} .$$

The asymptotic representation of (28) thus confirms the ray interpretation of the field in terms of creeping waves and ordinary rays. Furthermore, a determination of the decay exponent $\beta_p(a)$ (32) and the diffraction coefficient $D_p(a)$ (34) is provided.

APPLICATION OF THE RESULTS

According to J.B. Keller's Geometrical Theory of Diffraction (see [15] e.g.) the decay exponent and the diffraction coefficient are "canonical" quantities which apply for the construction of the

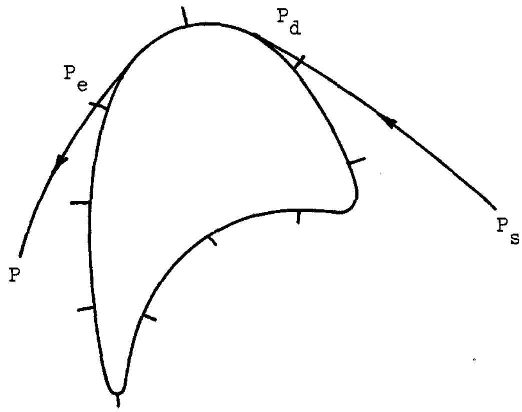

Fig. 4. Diffraction around an island of complex shape.

diffracted fields around scatterers of generalized shape. In the case of gravity water waves β_p (32) and D_p (34) are conjectured to be applicable at islands of more complex shape (see Fig. 4) at least when the gradient of the bottom profile is perpendicular to the coast line. The local variations in radius of curvature, depth, and profile shape must then be taken into account in the proper manner in (31).

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REFERENCES

[1] Sager, G., Gerlands Beitr. Geophysik, 82 (1973) 210.

[2] Sager, G., ibid., 82 (1973) 309.

[3] Sager, G., ibid., 82 (1973) 361.

[4] Sager, G., ibid., 82 (1973) 505.

[5] Homma, S., Geophys. Mag., 21 (1950) 199.

[6] Vastano, A.C. and R.O. Reid, J. Marine Res., 25 (1967) 129.

[7] Lautenbacher, C.C., J. Fluid Mech., 41 (1970) 655.

[8] Kline, M. and I.W. Kay, Electromagnetic Theory and Geometrical Optics, Interscience (1965) 11.

[9] Kamke, E., Differentialgleichungen I, Akademische Verlagsgesellschaft (1967) 440.

[10] Sommerfeld, A., Lectures on Theoretical Physics, Vol. 6, Academic Press (1964) 214.

[11] Lewis, R.M. and J.B. Keller, Asymptotic Methods for Partial Differential Equations, Courant Institute of Mathematical Sciences, New York University, EM-194 (1964) 70.

[12] Christiansen, P.L., Comparative Studies of Diffraction Processes, (to be published) 184.

[13] Longuet-Higgins, M.S., J. Fluid Mech., 29 (1967) 781.

[14] Shen, M.C., R.E. Meyer, and J.B. Keller, Phys. Fluids, 11 (1968) 2289.

[15] Keller, J.B., J. Opt. Soc. Amer., 52 (1962) 116.