CHAPTER 15

OPTIMAL DESIGN FOR WAVE SPECTRUM ESTIMATES

M. A. Tayfun[1], C. Y. Yang[2], and G. C. Hsiao[3]

ABSTRACT

Non-stationarity in an actual wave field restricts the application of the existing methods of estimating spectra. Despite the enormous amount of research work in the past, an analyst today is still faced with the lack of a unique procedure capable of providing a spectrum estimate which can be considered as the most accurate for the wave data collected under conditions where the stationarity assumption is in doubt. In this paper a generalized method is presented for estimating one dimensional frequency spectrum considering the non-stationarity. The generalized method and the associated design relations provide an effective measure for assessing the statistical quality of spectrum estimates, and a natural criterion as to how to select an optimal sample size. Concepts are illustrated by actual wave data analysis, and the validity of the procedure is demonstrated by simulation. In a simple manner, all concepts and methods developed for the non-stationary frequency spectrum apply to the wave number spectrum with spatial inhomogeneity. For simplicity, the presentation here will be primarily directed to the frequency spectrum.

INTRODUCTION

The computation of one dimensional frequency or wave number spectra is of fundamental importance in a statistical description of the ocean surface. However, the applicability of the existing methods [Blackman and Tukey, 1959; Hinich and Clay, 1968; Jenkins and Watts, 1969; Otnes and Enochson, 1972] is restricted by the basic assumption of stationarity or homogeneity. This constraint is violated in many cases of considerable interest such as the storm-generated waves, and the shallow water waves undergoing spatial modifications due to bottom friction, refraction and shoaling. The results given by Ploeg [1972] clearly indicate that, during the history of a storm generated wave field, major spectral components change in magnitude as much as 200% within twelve minutes. In the absence of a physically meaningful spectral theory for non-stationary processes, the selection of a sample size consistent with stationarity or spatial homogeneity becomes a major concern. In the time domain this selection is limited to 15-20 minutes [Harris, 1972; Borgman, 1972], based on the general experience in wave analysis but without a formal criterion. Tayfun et al. [1972] have shown that, even in a seemingly stationary wave field, significant differences in magnitude and shape exist between the stationary and non-stationary spectral estimates computed from the same set of data at various times. Realistic wave fields have a general time-dependent character, and a sweeping assumption of stationarity cannot be justified for a wave field on a visual or, an intuitive basis. The selection of a sample size in space is even more subjective and ambiguous due to the lack of experience [Schule et al., 1971; Collins, 1972].

[1]Assistant Professor of Civil Engineering
[2]Professor of Civil Engineering and Marine Studies
[3]Associate Professor of Mathematics, University of Delaware, Newark, Delaware 19711

Having limitations on the available sample size for spectral analysis presents serious difficulties in the description of the statistical quality of spectrum estimates, and, therefore, in establishing a uniquely determined design rationale for estimating a spectrum which can be considered as the most accurate available from the data. The basic criteria for the statistical quality of a spectral estimate are its bias (or resolution) and variability (or stability). Bias is a measure of how well an estimate approximates the true spectrum Variability is a consistency description for spectral estimates. The former arises as a direct consequence of the imperfections of various lag windows or spectral filters, and the latter essentially as a result of employing a single sample record A good quality estimate is therefore characterized by negligible bias (or high resolution) and low variability (or high stability). In the present state of the art variability of estimates is described in terms of probability confidence intervals in analogy with the properties of a chi-square variate. This analogy has proven satisfactory under fairly general conditions [Borgman, 1972]. However, since chi-square confidence intervals are constructed with reference to the spectral estimates themselves, the requirement that an estimate have negligible bias is clearly of paramount importance in this approach. With no limitation on the sample size and in the absence of periodic components, it is in principle possible to construct spectrum estimates with negligible bias by taking larger sample sizes. In this case a variability criterion based on the chi-square confidence intervals alone constitutes an adequately simple design criterion for spectrum estimations. On the other hand, with limitations placed on the sample size on account of either computational practicality or non-stationary conditions, spectrum estimates should realistically be expected to have bias as well as variability errors. In such cases a spectral design in terms of the chi-square confidence intervals alone cannot be justified, and a more effective design criterion based on the optimal balance between bias and variability errors is required.

The purpose of this study is, therefore, to present a generalized method and an optimal design rationale for wave spectrum estimations under realistic conditions in an objective manner. The approach is based on the non-stationary spectral theory developed by Priestley [1965, 1966, 1967] from a smooth extension of the classical stationary concepts. Previous work in this area [Brown, 1967, Tayfun et al., 1972] indicated the applicability of this concept to ocean waves. Further investigations of the non-stationary spectral theory reveals that a generalized approach and a uniquely determined design rationale for estimating spectra are possible based on an optimization of the statistical errors concisely contained in a relative mean-square error criterion. This criterion consists of bias of estimates in both time and frequency domains (or, space and wave number domains) as well as variability, and, therefore, provides an effective measure to describe the overall statistical quality of spectrum estimates. A minimization of the relative mean-square error expressed as a function of a general filter or, lag window characteristics and various wave field parameters yields a unique set of design relations in terms of shapes and parameters of filters and the sample size. The general concept and the associated design relations are presented primarily in physical terms and emphasis is placed on the application to ocean waves.

GENERALIZED SPECTRAL REPRESENTATION OF NON-STATIONARY PROCESSES

In a random wave field the surface oscillations from the mean water level observed at a fixed position is a one-dimensional zero-mean random process. If the wave field is stationary, this process admits a stochastic Fourier representation of the form

$$\eta(t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dZ(\omega) \tag{1}$$

where $i = \sqrt{-1}$, $Z(\omega)$ is a zero-mean random process continuously indexed with respect to a frequency parameter ω and with orthogonal increments such that, for a non-negative even function $S(\omega)$,

$$<dZ(\omega)d\overline{Z}(\omega')> = \begin{cases} 0, & \omega \neq \omega' \\ S(\omega)d\omega\ , & \omega = \omega' \end{cases} \tag{2}$$

where the overbar denotes the complex conjugate.

The mean energy per unit horizontal area of wave motion is proportional to the mean-square of the surface oscillations given, using (1) and (2), by

$$<|\eta(t)|^2> = \int_{-\infty}^{\infty} S(\omega)d\omega$$

The function $S(\omega)$ is recognized as the two-sided energy spectral density of the wave process. Replacing the time t with a spatial variable x and the frequency ω with the wave number k in the preceding equations yields the representation of a homogeneous wave field with the wave number spectral density $S(k)$.

The general representation (1) is in an abstract form in which neither an explicit probability structure nor any specific physical considerations are taken into account. It simply states that the process $\eta(t)$ may be regarded as a superposition of many harmonic components with different frequencies and time-independent random amplitudes $dZ(\omega)$. Realizing that $\eta(t)$ is real and negative frequencies have no physical meaning, the representation (1) can be rewritten as

$$\eta(t) = \int_{0}^{\infty} \{\cos\omega t\ dV_1(\omega) + \sin\omega t\ dV_2(\omega)\} \tag{3}$$

where

$$\begin{aligned} dV_1(\omega) &= dZ(\omega) - dZ(-\omega) \\ dV_2(\omega) &= i\{dZ(\omega) + dZ(-\omega)\} \end{aligned} \tag{4}$$

are mutually orthogonal processes both real and such that

$$<|dV_1(\omega)|^2> = <|dV_2(\omega)|^2> = 2<|dZ(\omega)|^2> = 2S(\omega)d\omega$$

Furthermore, if the processes V_1 and V_2 are chosen so that

$$\begin{aligned} dV_1(\omega) &= 2\{S(\omega)d\omega\}^{1/2} \cos\phi_\omega \\ dV_2(\omega) &= -2\{S(\omega)d\omega\}^{1/2} \sin\phi_\omega \end{aligned} \tag{5}$$

in which ϕ_ω are independent random variables identically and uniformly distributed in the interval $[0,2\pi]$, it is seen that (1) reduces to

$$\eta(t) = \sqrt{2} \int_0^\infty \cos(\omega t + \phi_\omega) \sqrt{2S(\omega)d\omega} \tag{6}$$

The above form of (1) corresponds to Pierson's [Pierson and Marks, 1952] well-known stationary Gaussian model where the quantity $2S(\omega)$ is the one-sided energy spectral density.

When the wave field is non-stationary the process $\eta(t)$ can be represented in the generalized form [Priestley 1965, 1966, 1967, 1973],

$$\eta(t) = \int_{-\infty}^{\infty} A(t,\omega)\, e^{i\omega t}\, dZ(\omega) \tag{7}$$

where the new quantity $A(t,\omega)$ is a deterministic modulating function of time and frequency. Equation (7) states that $\eta(t)$ is the superposition of many harmonic components, with different frequencies and time-dependent random amplitudes $\{A(t,\omega)dZ(\omega)\}$. In the limiting case when $A(t,\omega)\to 1$, equation (7) reduces to (1) for the stationary wave process.

The mean-square of the process $\eta(t)$ is readily obtained, using (2) and (7), as

$$<|\eta(t)|^2> = \int_{-\infty}^{\infty} |A(t,\omega)|^2\, S(\omega)d\omega \tag{8}$$

Hence, the non-stationary spectral density of the wave process is given by

$$S(t,\omega) = |A(t,\omega)|^2\, S(\omega) \tag{9}$$

As in the stationary case, equation (7) for the non-stationary wave process $\eta(t)$ may be rewritten in the form [Brown, 1967],

$$\eta(t) = \sqrt{2} \int_0^\infty \cos(\omega t + \phi_\omega) \sqrt{2S(t,\omega)d\omega} \tag{10}$$

ESTIMATION OF NON-STATIONARY SPECTRA

Having developed the above theoretical basis, the attention may now be focused on the main problem which is to estimate, for a given wave record $\eta(t)$, the non-stationary spectral density $S(t,\omega)$. This estimation is based on a filtering technique with two fundamental concepts [Priestley 1965, 1966, 1967]. One is the concept of resonance. It is well-known in system response theory that when a disturbance is applied to a linear system whose natural frequency is ω, the output response will be primarily in the neighborhood of that frequency ω. In this manner, when the sample record $\eta(t)$ (disturbance) is passed through a linear filter (system) with a central frequency ω, the output is a record with Fourier components primarily around ω. The second concept concerns the time lag between the input and output records. The response of a system to an impulsive disturbance usually lasts for a short time. Quantitatively, this system behavior is described by the width of its system impulse response function. For a system or filter with a narrow impulse response function, the output record at any instant t for a general input sample record depends only on that part of the input sample in the immediate neighborhood of t. In applying the above two concepts in the filtering technique, one may conclude that when $\eta(t)$ is passed through a filter with a narrow frequency response centered around ω, the output will be a record whose frequency composition is primarily around ω, and whose magnitude at each instant relates only to those of the input sample $\eta(t)$ in the neighborhood of the same time. To put these ideas in a mathematical form, let $g(t)$ be the narrow impulse response function of a filter with a natural frequency centered around $\omega = 0$. Assume further that $g(t)$ is a continuous function identically zero for values $|t| \geq h$, where h is a positive constant (width parameter), and that it is normalized so that

$$2\pi \int_{-h}^{h} |g(u)|^2 \, du = 1$$

Then, the function $g(t)\exp(-i\omega t)$ corresponds to the impulse response function of a filter which has the same form as that of $g(t)$ but whose natural frequency is shifted to the central frequency ω. The output record

$$U(t,\omega) = \int_{-h}^{h} g(u)\eta(t-u) \, e^{-iu\omega} \, du \tag{11}$$

represents at each time t that part of the sample $\eta(t)$ in the neighborhood of frequency ω. Now, if one further processes the output record $U(t,\omega)$ by a squaring and an averaging (or weighting) operations over the neighboring values in time, the end result is the mean-square in the vicinity of frequency ω and time t. This is identical with the definition of power spectrum density with the addition of a time trend. To formalize the squaring and averaging operations in a mathematical form, let $w(t)$ be a non-negative weighting function identically zero for values $|t| \geq T'$, where T' is a positive width parameter, and properly normalized so that

$$\int_{-T'}^{T'} w(u) \, du = 1$$

The estimated non-stationary spectral density at frequency ω and time t is

$$\hat{S}(t,\omega) = \int_{-T'}^{T'} w(u) \, |U(t-u,\omega)|^2 \, du \tag{12}$$

It is evident from (11) and (12) that the minimum sample length that is required in this procedure is

$$T_{min} = 2(h + T') \tag{13}$$

In the filtering process (11), it is required that the impulse response function $g(t)\exp(-i\omega t)$ be narrow (small h) so that the output retains the instantaneous behavior of the input sample $\eta(t)$. In the mean time it is required that the output record $U(t,\omega)$ consist of primarily components with frequencies in the neighborhood of ω. Since the frequency composition of the output record through the filtering process is directly related to the frequency response function of the filter in the sense that a narrow frequency response function provides an output with high frequency concentration around ω, it is evident that a filter which has both a narrow frequency response function and a narrow impulse response function is required. In other words, the function g(t) is required to have high "resolving" powers over both frequency and time domains. Unfortunately, the two requirements are conflicting because the impulse and frequency response functions are a Fourier transform pair, and, therefore, if one is narrow, the other must be wide. This leads to the problem of an optimal choice for g and its width h so that both criteria are satisfied to a certain degree. These characteristics are quite unlike those of a filter in the stationary analysis where there is no inherent requirement on the local time, and the filter is required to be narrow in the frequency domain only.

In the selection of the weighting function w(t) it is required that its effective width T' be much wider than the width h of g(t) so that whereas g(t) operates on the input sample record locally in time, w(t) will do so over a substantially larger time interval to provide a sufficient averaging or smoothing effect. On the other hand smoothing over a very large interval in time introduces a smudging error and, therefore, decreases the resolution of the estimates $\hat{S}$ over time. Consequently, the selection of the weighting function w(t) and its width T' should be made on the basis of a trade off between the two conflicting requirements, a satisfactory resolution in time and an adequate stability for the estimates $\hat{S}(t\Delta t,\omega)$.

OPTIMAL DESIGN RELATIONS

In general, the estimate $\hat{S}$ will have errors on account of the imperfections of the filter and weighting functions g(t) and w(t), the non-stationarity in the wave process, and the analysis of only one sample or one realization. The imperfections of g and w, and the non-stationarity of a wave process introduce bias or resolution errors to the estimate $\hat{S}$ over both time and frequency. The consequence of using only one sample is reflected in the variability or stability errors. The overall statistical quality of the spectral estimates of the form (12) is characterized by the relative mean-square error function defined, at a prescribed time t and frequency ω, by

$$M = \frac{\langle\{\hat{S}-S\}^2\rangle}{S^2} = \frac{\text{bias}^2\{\hat{S}\}}{S^2} + \frac{\text{var}\{\hat{S}\}}{S^2}, \tag{14}$$

where bias $\{\hat{S}\} = \langle\hat{S}\rangle - S$, and var$\{S\} = \langle\hat{S}^2\rangle - \langle S\rangle^2$. The function M depends on the functional forms of g(t) and w(t), the parameters h and T', and spectral bandwidth characteristics in time and frequency associated with the wave process [see, e.g., Priestley, 1966]. In an implicit manner, the functional form of M can be written concisely in the form

$$M = M\{\tilde{C},h,T',B_o(t,\omega),\ B_f(t,\omega)\}, \tag{15}$$

where $\tilde{C} = (C_{g1},C_{g2},C_{w1},C_{w2})$ denotes a set of coefficients which determine the characteristic shapes of g(t) and w(t), and $B_o(t,\omega)$ and $B_f(t,\omega)$ are defined as bandwidth parameters of the theoretical spectral density $S(t,\omega)$ regarded as a distribution over time and frequency, respectively. These parameters are given by

$$B_o(t,\omega) = \left|\frac{S}{\partial^2 S/\partial t^2}\right|^{1/2} \quad \text{and} \quad B_f(t,\omega) = \left|\frac{S}{\partial^2 S/\partial \omega^2}\right|^{1/2} \tag{16}$$

The spectral bandwidth over frequency $B_f(t,\omega)$ with the dimension $(\text{time})^{-1}$ is a well-known concept (with an exclusion of time dependency) in the stationary analysis. It is a measure of the shape of $S(t,\omega)$ as a function of frequency. A small value $B_f(t,\omega)$ indicates a highly peaked spectral density over frequency. The spectral bandwidth $B_o(t,\omega)$ with the dimension (time) is similarly defined, and it provides the measure for the temporal variation of the spectral density $S(t,\omega)$. Therefore, this parameter reflects the non-stationarity in a wave process. The smaller $B_o(t,\omega)$ is, the stronger is the non-stationarity. In the limiting case, when $S(t,\omega)\to S(\omega)$, corresponding to a stationary process, $B_o(t,\omega)\to\infty$.

Insomuch as the mean-square error M reflects the overall errors associated with the imperfections of $g(t)$ and $w(t)$, and the parameters h and T', the optimal procedure must be based on those parameters that minimize the error M for a given wave process characterized by the bandwidth parameters $B_o(t,\omega)$ and $B_f(t,\omega)$. This approach yields a set of unique optimal design relations in terms of the minimum estimation error for a given wave situation, the optimal shapes $g(t)$ and $w(t)$, the parameters h and T', and the optimal sample size [see Appendix 1 for derivations]. These design relations in general have a time and frequency dependent nature. However, of the various possible choices of the optimal design criterion, that which minimizes the maximum possible error over the ranges of both the frequency and the time of interest provides the simplest one amenable to practical computations. Specifically denoting the optimal values by the subscript zero, these relations are summarized as follows.

$$M_o = 3\left\{\frac{C_{g1}C_{g2}C_{w1}C_{w2}}{2\,B_oB_f}\right\}^{2/3} \tag{17}$$

is the minimal estimation error, where B_o and B_f denote respectively the values $B_o(t,\omega)$ and $B_f(t,\omega)$ which minimize the product $\{B_o(t,\omega)B_f(t,\omega)\}$, i.e.,

$$B_oB_f = \min_{\substack{T_1\le t\le T_2\\ \omega}} \{B_o(t,\omega)B_f(t,\omega)\} \tag{18}$$

where the selected wave record covers the interval (T_1,T_2). The optimal values of the parameters h and T' are given respectively by

$$h_o = \left\{\frac{3}{M_o}\right\}^{1/4}\frac{C_{g1}}{B_f} \quad\text{and}\quad T_o' = \left\{\frac{M_o}{3}\right\}^{1/4}\frac{B_o}{C_{w1}} \tag{19}$$

with the ratio

$$\left[\frac{h_o}{T_o'}\right] = \left[\frac{C_{w1}^2\,C_{g1}^2}{B_o^2\,B_f^2\,C_{w2}C_{g2}}\right]^{1/3} \tag{20}$$

The optimal sample size $(T_{min})_o$ is obtained from (13) using the above values of h and T'. In the preceding results, the constants $C_{w1} = \sqrt{0.2}$, $C_{w2} = 6\pi/5$, $C_{g1} = \pi/\sqrt{3}$, and $C_{g2} \simeq .1528$ relate to the optimal weighting and filter functions given by

$$w(t) = (3/4T_o')\,\{1-(t/T_o')^2\}\ ; \quad |t|\le T_o' \tag{21}$$

and

$$g(t) = (6\pi h_o)^{-1/2}\,\{1 + \cos(\pi t/h_o)\}\ ; \quad |t|\le h_o \tag{22}$$

corresponding, respectively, to the Parzen and Hanning windows in the stationary spectral theory [Jenkins and Watts, 1969].

The salient features of the optimal design relations can be pointed out as follows:

(1) The minimal error M_o consists of bias and variability errors in a one to two ratio. This is a consequence of the fact that bias errors are more sensitive to changes in the design parameters h and T' as compared to variability errors. Hence, the proper balance between bias and variability should be maintained in this estimation procedure in contrast with the general tendency in the routinely used procedures such as the autocorrelation and the Fast Fourier Transform techniques to disregard bias errors completely and to base the spectral design purely on a variability criterion.

(2) In the limiting case of a homogeneous process; i.e., as $B_o \to \infty$, it is readily seen that $M_o \to 0$, h_o and $T_o' \to \infty$, as the ratio $(h_o/T_o') \to 0$. The spectral estimates with the asymptotic property $(M_o \to 0)$ are referred to as consistent estimates in the classical spectral theory. The results in this particular limiting case are very much in accord with the general character of the conventional analysis. Therefore, there is, in principle, no bounded optimal spectral design relations in the homogeneous case. However, from a practical point of view, the limiting form of the design relations implies that in such cases one should take longer sample sizes, while decreasing the ratio (h_o/T_o'), until the spectral estimates attain a consistent, or simply convergent behavior.

(3) One particularly interesting feature about the time and frequency independent design relations is that, given a record length T, they provide the functions g(t) and w(t) with constant width parameters h_o and T_o', and therefore require a fixed minimum record length $(T_{min})_o$ to construct spectral estimates which are, in statistical quality, at least equal to or better than the estimate characterized by the maximum M_o.

(4) It is recalled from the definitions (16) that $B_o(t,\omega)$ and $B_f(t,\omega)$ relate inversely to the second partial derivatives of a spectral density over time and frequency, respectively. Hence, being derived from the minimal product $B_o(t,\omega)B_f(t,\omega)$, the parameters B_o and B_f are associated with the narrowest peaks and valleys in a spectrum in time and frequency. Moreover, it is the simultaneous occurence of these peaks and/or valleys over both time and frequency, as evidenced by the product B_oB_f in (18), which characterizes the maximum error M_o. Therefore, the dimensionless product B_oB_f serves as an overall measure of significant spectral characteristics of a wave process in the sense that the larger this product is, the more accurate the estimation procedure becomes. On the other hand, it is realized that a background knowledge on the parameters B_o and B_f is required before one can proceed to compute the spectral density in a given realistic situation. These parameters must be estimated approximately either on a valid theoretical basis, or from "pilot" estimates of a spectral density

DIGITAL COMPUTATIONS

In digital computations of the non-stationary spectrum of a wave process $\eta(t)$, the optimal design relations remain invariant provided that a few simple modifications are made in the estimation procedure as follows. Consider a wave record $\eta(t)$ digitally sampled at intervals of Δt so that one has a sequence $\eta_1, \eta_2,$, where $\eta_n = \eta(n\Delta t)$. To make sure that no errors will be introduced in the digital computation due to aliasing errors, assume that the interval Δt is at most equal to the Nyquist interval. Under this assumption it is convenient to regard the sequence $\{\eta_n\}$ as if it consisted of points at unit time intervals. This is equivalent to transforming the original frequency scale into a standarized dimensionless frequency $\omega^* = \omega\Delta t$ defined in the interval $(-\pi,\pi)$.

Consequently, the estimated spectral density, say $S_d(n,\omega^*)$, of the discrete sequence $\{\eta_n\}$ and that of the actual wave process $\eta(n\Delta t)$ are related to one another in the form

$$\hat{S}(n\Delta t,\ \omega^*/\Delta t) = \Delta t\ \hat{S}_d(n,\omega^*),\quad |\omega^*|<\pi \tag{23}$$

where

$$\hat{S}_d(n,\omega^*) = \sum_{j=-M}^{M} w_j\ |U_{n-j}(\omega^*)|^2 \tag{24}$$

and

$$U_n(\omega^*) = \sum_{j=-N}^{N} g_j\ \eta_{n-j}\ e^{-ij\omega^*} \tag{25}$$

are the discrete time analogues of (11) and (12), with $g_j = g(j\Delta t)$ and $w_j = w(j\Delta t)$ derived from the continuous time versions $g(t)$ and $w(t)$. The integer width parameters N and M are now interpreted as the largest integers smaller than $(h_o/\Delta t)$ and $(T_o'/\Delta t)$, respectively, where h_o and T_o' are as previously defined as in (19). Hence, it follows from (21) and (22) that

$$g_j = (6\pi N)^{-1/2}\ \{1 + \cos(j\pi/N)\}\ ;\ j = -N,\ldots,-1,0,1,\ldots,N \tag{26}$$

and for M >>1

$$w_j = (3/4M)\ \{1 - (j/M)^2\}\ ;\ j = -M,\ldots,-1,0,1,\ldots,M \tag{27}$$

It is evident that the optimal length of the sequence $\{\eta_n\}$ is now given by $2(N + M)$.

ILLUSTRATIVE APPLICATIONS

EXAMPLE 1: A Storm Generated Non-stationary Wave Field. A significant application of the proposed spectral estimation procedure and the associated optimal design relations is in the analysis of storm generated extreme waves. A particularly interesting analysis of such a non-stationary wave field has been illustrated by J. Ploeg [1972] in terms of the time history of one dimensional frequency spectra collected on Lake Ontario during a storm on October 24-25, 1971. The reference covers about 13 hours of the storm with approximately 200 sequential spectral analyses performed through the Fast Fourier Transform technique using 16 minute overlapping segments, with the starting points approximately 4 minutes apart. With a proper caution to the non-stationary effect, Ploeg presents the results of Lake Ontario study in three figures [Figures 7 through 9 - J. Ploeg, 1972], and draws various conclusions as follows. The spectral history of the storm-generated wave field [Figure 7 - Ploeg] shows the familiar build-up with the peak of the spectrum density continuously shifting to the lower frequencies, while [Figure 8 - Ploeg] the higher frequency spectral components reach saturation and remain essentially stationary. The non-stationarity of the wave process at lower frequency spectral components, in particular at the peak spectral component is clearly observed. The time history of the spectral peak frequency [Figure 9 - Ploeg] shows a tendency to jump between discrete frequencies, while shifting towards lower values in a manner consistent with the shifting of the entire spectral density function to lower frequencies as the wave field builds up, with the higher frequency components remaining saturated. There is no physical explanation offered for the jumps of the spectral peak frequency. It is quite plausible that this effect is due to the "overshoot" effect observed in the typical growth of a spectral component [Barnett and Sutherland, 1968; Plate, 1971].

It is of interest to apply the optimal design relations to such non-stationary wave records. However, before one can proceed to do so, it is necessary to know the spectral bandwidth parameters B_o and B_f of the wave field. Hence, the first step is to estimate these parameters on a theoretical basis and/or from pilot estimates of the spectral density. In the particular case of a storm generated wave field, a theoretical basis is provided by the concept of equilibrium range [Phillips, 1965] which asserts that the spectral growth of a wave component is limited by breaking. Neglecting the overshoot effect, and for the general case of finite fetch, and variable wind conditions, the concept requires that the spectral density function $S(t,\omega)$ have the form

$$S(t,\omega) = \beta\, g^2\omega^{-5} \ ; \quad \omega \geq \omega^*(t) \tag{28}$$

where β is a dimensionless constant dependent in general on wind speed and fetch [Strekalov, et al., 1972], g is the gravitational acceleration, and $\omega^*(t)$ denote the time-dependent spectral peak frequency. In other words, $\omega^*(t)$ provides the required time-dependency, and, therefore, $(d\omega^*/dt)$ relates in a fairly simple manner to the rate at which the front face of the spectrum builds up as a result of the net rate of energy input between the generative wind and the dissipative breaking and friction effects. Hence, from (16), and (28), it follows that

$$B_f(t,\omega) \simeq \omega/\sqrt{30} \ ; \quad \omega \geq \omega^*(t) \tag{29}$$

and

$$B_o(t,\omega) \simeq \begin{cases} \dfrac{\omega}{\sqrt{51}} \left| 6\left[\dfrac{d\omega}{dt}\right]^2 - \omega\,\dfrac{d^2\omega}{dt^2} \right|^{-1/2} & , \ \omega \simeq \omega^*(t) \\ \infty & ; \ \text{otherwise} \end{cases} \tag{30}$$

The minimum of the product of the above two parameters as defined by equation (18) is then

$$B_oB_f = \begin{cases} \min\limits_{T_1 \leq t \leq T_2} \{B_o(t,\omega)B_f(t,\omega)\} & ; \ \omega \simeq \omega^*(t) \\ \infty & ; \ \text{otherwise} \end{cases} \tag{31}$$

The spectral bandwidths B_o and B_f must be estimated numerically on the basis of the preceding guideline (31) and from pilot estimates of the time history of the spectral peak frequency ω^*. Such pilot estimates can be obtained in a heuristic fashion either through conventional techniques by using overlapping segments of a sample or, through the generalized filter method described in this paper in an iterative manner [see, e.g., Tayfun et al., 1972]. The time history of the spectral peak frequency $\omega^*(t)$ obtained by Ploeg has been reproduced in part in the lower part of Figure 1 here. This data is associated with the extreme wind-wave conditions and, for the purpose of illustrating the optimal design considerations, is used here as the pilot estimates. The spectral bandwidth characteristics computed numerically on the basis of these pilot estimates and the relation (31) are likewise presented in Figure 1. Table 1 summarizes the optimal design considerations and the expected quality of spectral estimates in various intervals (T_1,T_2) = (1800, 1900), (1900, 2000), and (2000, 2100). The design relations in each of the preceding intervals are based on the peak spectral component in the neighborhood of an instant where non-stationarity is the most stringent, indicated by the vertical dashed lines A, B, and C, respectively (Figure 1).

EXAMPLE 2: A Spatially Inhomogeneous Wave Field. As an example of a spatially inhomogeneous wave process, consider a unidirectional wave field where waves in a fully developed state at deep water propogate towards the shore into intermediate and shallow water regions over a variable depth topography with a mean slope s. Assuming no dissipative effects and no wind-generation, the wave field is stationary in time, but

Time Interval (hrs)	B_o (sec)	B_f (sec^{-1})	B_oB_f	Quality, M_o (%)	h_o (sec)	T_o' (sec)	Sample Size, $(T_{min})_o$ (sec)
1800 - 1900	500	.20	100	$\leq$ 3.12	29	358	774
1900 - 2000	170	.22	37	$\leq$ 6.4	22	145	334
2000 - 2100	280	.20	56	$\leq$ 4.9	25	223	496

TABLE 1. Optimal spectral design considerations in a storm-generated wave field.

inhomogeneous in space due to wave shoaling. Using a reference frame with a positive x-axis extending from the deepwater surface toward the shore, the rate of change of energy spectral density along a way ray (the x-axis) is [Collins, 1972]

$$d\{cS(x,k)\}/dx = 0, \tag{32}$$

where $S(x,k)$ and $c = \omega/k$ are respectively the inhomogeneous spectral density and the phase speed. It is noted that the frequency of a wave component is conserved whereas the corresponding wave number is depth-dependent. Based on this result and from an integration of (32) between two points, one at deep water denoted by the subscript zero ($x_o = 0$), and the other towards the shore at a location x where the water depth is $D = D(x)$, it follows that

$$S(x,k) = k\,k_o^{-1}\,S_o(k_o). \tag{33}$$

Using the relation between the shallow water wave number k and deep water k_o,

$$k_o = k \tanh kD \tag{34}$$

equation (33) can be written in the form

$$S(x,k) = \coth kD\; S_o(k_o). \tag{35}$$

A comparison of (35) with (9) indicates that the function coth kD corresponds to the squared modulating function $|A|^2$ in the general definition of an inhomogeneous spectrum density.

The spatial and wave number bandwidth parameters $B_o(x,k)$ and $B_f(x,k)$, respectively, are derived from (16) and (33) (see Appendix 2). The dimensionless forms of these parameters and the spectral product $\{B_o(x,k)B_f(x,k)\}$ are presented in Figure 2 as functions of the dimensionless depth kD. For a given mean slope s, for all wave numbers $k \geq k^*$(the spectral peak wave number), and in the region of interest $\infty > D(x) \geq D$, it is seen from Figure 2 that

$$B_o = \lim_{kD(x)\to k^*D} [B_o(x,k)] \quad \text{and} \quad B_f = \lim_{kD(x)\to k^*D} [B_f(x,k)] \tag{36}$$

In deep water, $D \to \infty$, and $B_o \to \infty$, as the wave field becomes spatially homogeneous. For shallow water with small k^*D values

$$B_o \simeq \frac{2D}{\sqrt{3}^{\,1}\,s} \tag{37}$$

Hence, in this region the spatial inhomogeneity of the wave process is proportional directly to the local slope s, and inversely to the depth D as intuitively expected. A similar argument for the wave number spectral bandwidth B_f indicates that, as $D \to \infty$,

$B_f \to (k_o^*/2\sqrt{3})$, the spectral bandwidth at deep water. Moreover this limiting value is an upper bound, and B_f monotonically decreases as the water depth becomes shallower In other words, the spectral density $S(x,k)$ becomes increasingly peaked. In general then, since the product B_oB_f decreases as the slope s becomes larger and/or the depth D becomes shallower, the estimation procedure is expected to be increasingly less accurate. To illustrate the effect of variations in the bottom slope s, consider the density $S(x,k)$ at a location with depth D = 10 ft, deep water spectral peak wave number $k_o^* = 0.076$ (ft^{-1}). Hence, it follows from (34) that $k^*D = 1.0$. The dimensionless values B_o, B_f, and therefore the product B_oB_f are obtained from Figure 2 by using $kD(x) = k^*D = 1.0$. For mean bottom slope s (1/120, 1/90, 1/60, 1/30), the optimal design parameters are summarized in Table 2. It is seen that as the slope increases, the wave process becomes more inhomogeneous, the available sample size is increasingly limited, and the expected quality of estimates rapidly diminishes. It should be emphasized that this is true for the spectral estimates in the vicinity of the peak wave number $k_o^* = 0.1$ (ft^{-1}) at depth D = 10 ft where the spatial inhomogeneity is the most stringent. The accuracy of estimates corresponding to other wave numbers at the same depth, and of all estimates at depths larger than D = 10 ft will be equal to or better than M_o.

Slope, s	B_o (ft)	B_f (ft^{-1})	B_oB_f	Quality, M_o (%)	h_o (ft)	T_o' (ft)	Sample Size, $(T_{min})_o$ (ft)
1/120	1400	0.0154	21.6	$\le$ 14	250	1450	3400
1/90	1050	0.0154	16.2	$\le$ 18	240	1280	3020
1/60	700	0.0154	10.8	$\le$ 24	220	830	2100
1/30	350	0.0154	5.4	$\le$ 37	200	460	1320

TABLE 2 Optimal spectral design considerations in a shoaling wave field.

The preceding discussion provides a simple theoretical basis to investigate the validity of the proposed estimation procedure and the associated design relations. With this purpose in mind, consider the case with variable depth profile illustrated in Figure 3, and assume that the fully developed deep water conditions can be characterized in terms of a Pierson-Moskowitz spectrum [Pierson and Moskowitz, 1964] given in the wave number domain by

$$S_o(k_o) = 0.5\ \alpha\ k_o^{-3} \exp\{-\gamma g^2 (v\ k_o)^{-4}\}, \tag{38}$$

where $\alpha(= 8.1 \times 10^{-3})$ and $\gamma(= 0.74)$ are dimensionless constants, v is the wind speed, with the spectral peak wave number k_o^* given by

$$k_o^* = (0.52\gamma)^{1/2}\ (g/v^2) \tag{39}$$

The inhomogeneous wave number spectral density $S(x,k)$ at a given depth $D(x) = D$ is readily obtained from (33) and (38). For the profile shown in Figure 3 with $k_o^* = 0.076$ (ft^{-1}) (i.e., $v \approx 18$ ft. sec^{-1}), the theoretical forms of the inhomogeneous density $S(x,k)$ at locations (1,2,3) corresponding to the values D = 50, 20, and 10 (ft) are plotted in Figure 3. The spectral density $S(x,k)$ at D = 50(ft) is not influenced by shoaling effects, and, therefore, is the same as the deep water density $S_o(k_o)$. As the depth gets shallower at locations 2 and 3, the deep water spectral components are modified from lower wave numbers to the higher in a non-uniform manner as the form of the spectral density $S(x,k)$ becomes increasingly more peaked.

With the foregoing theoretical results on inhomogeneous spectra, it is then possible to generate realization of inhomogeneous wave series by a Gaussian wave surface model and a simulation technique (see Appendix 3). Finally the optimal estimation procedure is applied to the simulated wave series to obtain wave spectra. The estimated spectra obtained through (23), (24), and (25), and based on the required design relations in Table 2 corresponding to s = 1/90 and D = 10 (ft), are presented in Figure 3 for comparison with the theoretical forms. It is seen that, although the accuracy in the vicinity of a spectral peak, in particular at D = 10 (ft), is relatively poor as expected, the estimates in general agree very favorably with the theoretical forms. Thus the validity of the spectral design considerations is demonstrated.

SUMMARY AND CONCLUSIONS

A generalized procedure was described for estimating one dimensional non-stationary frequency spectra or inhomogeneous wave number spectra. In constrast to the auto-correlation approach, the procedure is based on a direct filter method which carries a simple physical interpretation and is therefore convenient to apply to actual wave fields.

The optimal design relations for the estimation procedure are constructed on the basis of a simple objective design criterion to minimize the overall statistical estimation errors which arise as a result of the imperfections of the filter and the weighting functions. Prior to the application of the estimation procedure, various spectral characteristics of the sampled wave field in time-frequency or space-wave number have to be estimated on the basis of a valid theoretical guideline and/or from pilot estimates of spectra. One of these characteristics, B_f, the spectral bandwidth in frequency is a familiar concept in the stationary analysis. The spectral bandwidth, B_o, in time or space is defined similarly and provides a measure for the inherent non-stationarity in a wave record. The spectral product B_oB_f is the most significant quantity for assessing the feasibility of the proposed estimation procedure in a given situation in the sense that the larger this product is, the more feasible it is to achieve accurate spectrum estimates. In the limiting case as $B_oB_f \to \infty$ (i.e., as $B_o \to \infty$) corresponding to a dominantly stationary wave process, the procedure reduces to the stationary analysis accordingly.

Two illustrative examples are given. The first example is for the non-stationary storm wave records from Lake Ontario, October 24-25, 1971, by which the use of the estimation procedure is demonstrated. The second example is for the analysis of inhomogeneous wave number spectra of digitally simulated wave records in a shoaling wave field. By the second example, the validity of the design criterion and the estimation procedure is demonstrated through a comparison between theoretical and estimated spectra.

APPENDIX 1: DERIVATION OF OPTIMAL DESIGN RELATIONS

The relative mean-square error (14) is approximately given by [Priestley, 1969]

$$M \simeq (1/4)\ \{B_w^2/B_o^2(t,\omega) + B_\Gamma^2/B_f^2(t,\omega)\}^2 + C(T')^{-1} \int_{-\infty}^{\infty} |\Gamma(\omega)|^4\ \omega \tag{A40}$$

where $\Gamma(\omega)$ is the Fourier transform of the filter g(t), and

$$B_\Gamma = \left\{ \int_{-\infty}^{\infty} \omega^2 |\Gamma(\omega)|^2\ d\omega \right\}^{1/2} \tag{A41}$$

$$B_w = \left\{ \int_{-T'}^{T'} t^2\ w(t)\ dt \right\}^{1/2} \tag{A42}$$

$$C = \lim_{T'\to\infty} \left\{ 2\pi T' \int_{-T'}^{T'} w^2(t)\, dt \right\} \qquad \text{(A43)}$$

The error M is the sum of the squared bias of an estimate $\hat{S}(t,\omega)$ over both the time and frequency, and its variance, corresponding respectively to the first and second group of terms in (A40). Utilizing the normality properties of the functions g(t) and w(t), two characteristic functions, G(u) and W(u), may be defined such that

$$G(u) = h^{1/2}\, g(uh) \quad ; \quad |u| < 1 \qquad \text{(A44)}$$

$$W(u) = T'\, w(uT') \quad ; \quad |u| < 1 \qquad \text{(A45)}$$

By virtue of the above definitions, it is easily shown that

$$B_\Gamma = h^{-1} \left\{ 2\pi \int_{-1}^{1} |dG/du|^2\, du \right\}^{1/2} \overset{\text{def}}{=} h^{-1}\, C_{g1} \qquad \text{(A46)}$$

$$\int_{-\infty}^{\infty} |\Gamma|^4 d\omega = h \left\{ 4\pi \int_0^2 \left| \int_{-1}^{1-y} G(u+y)\bar{G}(u)\, du \right|^2 dy \right\} \overset{\text{def}}{=} h\, C_{g2} \qquad \text{(A47)}$$

$$B_w = T' \left\{ \int_{-1}^{1} u^2 W(u)\, du \right\}^{1/2} \overset{\text{def}}{=} T'\, C_{w1} \qquad \text{(A48)}$$

$$C = 2\pi \int_{-1}^{1} W^2(u)\, du \overset{\text{def}}{=} C_{w2} \qquad \text{(A49)}$$

The error M given by (A40) can now be rewritten in terms of the parameters h, T', and the set of coefficients $\tilde{C} \equiv (C_{g1}, C_{g2}, C_{w1}, C_{w2})$, which depend only on the characteristic functions, in the form

$$M \simeq (1/4) \left\{ C_{w1}^2\, (T')^2\, B_o^{-2}(t,\omega) + C_{g1}^2\, h^{-2}\, B_f^{-2}(t,\omega) \right\}^2 + C_{g2} C_{w2}\, h (T')^{-1} \qquad \text{(A50)}$$

An optimization of the preceding expression with respect to h and T' yields the following

$$M_o = 3 \left[\{C_{w1} C_{w2} C_{g1} C_{g2}\} / \{2 B_o(t,\omega) B_f(t,\omega)\} \right]^{2/3} \qquad \text{(A51)}$$

denotes the minimal estimation error M consisting of the bias and variability errors in a one to two ratio, and corresponding to the optimal values of h and T' given by

$$h_o = (3/M_o)^{1/4} \{C_{g1}/B_f(t,\omega)\} \qquad \text{(A52)}$$

$$T_o' = (M_o/3)^{1/4} \{B_o(t,\omega)/C_{w1}\} \qquad \text{(A53)}$$

Moreover, an examination of M_o indicates that the conditions

$$\min_{W(u)} \{C_{w1} C_{w2}\} \quad \text{and} \quad \min_{G(u)} \{C_{g1} C_{g2}\} \qquad \text{(A54)}$$

constitute two formal optimality criteria for choosing and possibly constructing the characteristic functions, W(u) and G(u), respectively. One such possibility is to expand $W(u)$ and $G(u)$ in series (e.g. Fourier series or Legendre functions) with unknown coefficients, and to determine these coefficients through (A54). In particular, by letting $W(u) = \Sigma A_i P_i(u)$, in which A_i snd P_i denote, respectively, the unknown coefficients and Legendre functions of the first kind, it is easily verified that a unique solution to the first functional in (A54) is obtained, with $A_o = 1/2$, $A_2 = 1/2$, and $A_i = 0$ for $i \neq 0,2$, as

$$W(u) = (3/4)\ (1 - u^2) \quad ; \quad |u| < 1 \tag{A55}$$

Therefore, $C_{w1} = \sqrt{0.2}$, $C_{w2} = 6\pi/5$, and the minimal product $\{C_{w1}C_{w2}\} = 1.686$. Interestingly, (A55) corresponds to the well-known Parzen window in the ststionary analysis [Jenkins and Watts, 1969].

The problem of finding a unique optimal $G(u)$ in a similar manner becomes cumbersome due to the more complicate nature of the second functional in (A54). Nonetheless, the criterion in (A54) serves as a figure of merit for choosing an optimal form $G(u)$ from the collection of spectral windows in the stationary analysis. Table 3 compares the relevant properties of most of the well-known window shapes consistent with the class of functions $\{G(u)\}$. Among the four different functions examined, the optimal one is the Hanning window given by

$$G(u) = (6\pi)^{-1/2}\ (1 + \cos u\pi) \quad ; \quad |u| < 1 \tag{A56}$$

Filter $G(u)$	C_{g1}	C_{g2}	$C_{g1}C_{g2}$
$(6\pi)^{-1/2}\ (1 + \cos\pi u)$	$\pi/\sqrt{3}$	.1528	.2778
$(2\pi)^{-½} \cos ½\ (\pi u)$	$\pi/2$	.1875	.2946
$(3\pi/4)^{½}\ (1 - \|u\|)$	$\sqrt{3}$	.1725	.2988
$\{15/(32\pi)\}^{½}\ (1 - u^2)$	$\sqrt{(5/2)}$	.1926	.3045

TABLE 3. Properties of spectral windows.

The optimal weighting function $w(t)$ and the filter $g(t)$ sre now readily obtained from (A55) and (A56), and by using the definitions (A45) and (A44), respectively.

It is noted that the optimal design relations developed above depend on time t and frequency ω through the bandwidth parameters $B_o(t,\omega)$ and $B_f(t,\omega)$. However, considering a slightly different design criterion to minimize the maximum possible mean-square error over both the time and frequency, i.e.,

$$\min_{h,T',\tilde{C}} \left[\max_{t,\omega} M\{\tilde{C},h,T',B_o(t,\omega)B_f(t,\omega)\} \right] \tag{A57}$$

provides the time- and frequency-independent design relations which are the simplest for practicsl estimations of ocean wave spectra. These are readily included in the above optimal design relations, and they correspond to replacing the product $\{B_o(t,\omega)B_f(t,\omega)\}$ in (A51) with

$$B_oB_f = \min_{\substack{T_1 \le t \le T_2 \\ \omega}} \{B_o(t,\omega)B_f(t,\omega)\} \tag{A58}$$

where the selected record covers the interval (T_1,T_2), and interchanging $B_f(t,\omega)$ and $B_o(t,\omega)$ in (A52) and (A53), respectively, with their corresponding vslues B_f and B_o which realize (A58).

APPENDIX 2: SPECTRAL BANDWIDTHS IN SHOALING WAVES

From (16) and (33), the spatial bandwidth parameter is

$$B_o(x,k) = (D/\sqrt{2}\,s)\{1 + (\sinh 2kD/2kD)\}^{3/2}\{1 + (3 + \cosh 2kD)(\sinh 2kD)(4kD)^{-1}\}^{-\frac{1}{2}} \quad \text{(A59)}$$

Similarly, from (16) and (33), the spectral bandwidth $B_f(x,k)$ is given by

$$B_f(x,k) = |2D^2 \operatorname{csch}^2 kD[1 - k\, S_o^{-1}(dS_o/dk_o)\tanh kD] + S_o^{-1}(d^2S_o/dk_o^2)$$
$$(\tanh kD + kD \operatorname{sech}^2 kD)^2|^{-\frac{1}{2}} \quad \text{(A60)}$$

Assuming that the deep water wave field is in a fully developed state, the wave number spectral density $S_o(k_o)$ can be expressed by using again the concept of equilibrium range, this time, in the form

$$S_o(k_o) = (\beta/2)\, k_o^{-3} \quad , \quad k_o \geq k_o^* \quad \text{(A61)}$$

where k_o^* is the spectral peak wave number. Hence, using (A61) in (A60), $B_f(x,k)$ is rewritten in an approximate but more tractable form as

$$B_f(x,k) \simeq (k/2\sqrt{3})\{(1/3)(kD/\sinh kD)^2 + [1 + (2kD/\sinh 2kD)]^2\}^{-\frac{1}{2}} \quad \text{(A62)}$$

The dimensionless forms of $B_o(x,k)$ and $B_f(x,k)$ given by (A59) and (A62), and the product $\{B_o(x,k)B_f(x,k)\}$ are illustrated in Figure 2 as functions of the dimensionless depth kD. By definition and from Figure 2, it follows that

$$B_oB_f = \min_{k,D(x)} \{B_o(x,k)B_f(x,k)\} = \lim_{kD(x)\to k^*D} \{B_o(x,k)B_f(x,k)\} \quad \text{(A63)}$$

APPENDIX 3: SIMULATION OF AN INHOMOGENEOUS WAVE SERIES

If the deep water wave field characterized by the spectral density $S_o(k_o)$ is assumed to be Gaussian, within the constraints of the linear wave theory and outside of the breaker zone, the surface oscillations $\eta(x,t)$ admit the pseudo-integral representation

$$\eta(x,t) = \sqrt{2} \int_0^\infty \cos(kx - \omega t + \phi_k)\, \sqrt{S(x,k)dk} \quad \text{(A64)}$$

as an inhomogeneous Gaussian process. In the above representation ϕ_k are independent random phases uniformly distributed in the interval $[0,2\pi]$ as in (6), $\omega^2 = gk \tanh kD(x)$, with $D(x)$ and $S(x,k)$ denoting, respectively, the variable depth profile and the associated one-sided inhomogeneous spectral density whose theoretical forms at various depths are illustrated in Figure 3. The representation (A64) characterizes the wave process completely as a bivariate random process stationary in time t, and inhomogeneous in space x. Moreover, it provides a convenient basis to digitally simulate samples of the surface as a bivariate (time-space) series $\eta(n\Delta x, m\Delta t)$ $(n = 1,2,\ldots;\ m = 1,2\ldots)$, where Δx and Δt denote respectively suitably chosen spatial and temporal sampling intervals, or, as a stationary univariate time series $\eta(x, m\Delta t)$ $(m = 1,2,\ldots)$ at a specified location x, or, as an inhomogeneous space series $\eta(n\Delta x, t)$ $(n = 1,2,\ldots)$ at a prescribed time t [see, e.g., Shinozuka and Jan; 1971]. The space series $\eta(n\Delta x, t)$

corresponds to a digitized stereophotographic or laser profile of the surface. In this instance, the reference time value t is immaterial as the process $\eta(x,t)$ is stationary. In particular, choosing $t = 0$ for convenience, and denoting the wave number interval outside of which $S(x,k)$ is of insignificant magnitude by $[k_L, k_R]$, such a space series is readily obtained from a digital analogue of (A64) by

$$\eta(n\Delta x) = \sqrt{2} \sum_{j=1}^{J} \cos(\tilde{k}_j n\Delta x + \phi_{k_j}) \{S(n\Delta x, k_j)\Delta k\}^{\frac{1}{2}} \qquad \text{(A65)}$$

in which $k_j = (k_L + j\Delta k)$ $(j = 1,2,\ldots,J)$ with $J = (k_R - k_L)/\Delta k$ corresponds to a proper discretization of the wave number k, ϕ_{k_j} $(j = 1,2,..,J)$ the associated random phases, and $\tilde{k}_j = (k_j + \delta k_j)$, where δk_j is a small random wave number introduced to avoid the periodicity of the simulated series, and is uniformly distributed in the interval $[-(\Delta k'/2), (\Delta k'/2)]$ with $\Delta k'$ chosen such that $\Delta k >> \Delta k'$. A sample space series $\eta(n\Delta x)$ of the wave process investigated here was simulated in the described manner implementing (A65) with $\Delta x = 4$ (ft), $k_L = 0.01$, $k_R = 0.41$, $\Delta k = 0.01$, $\Delta k' = 0.002$, and $J = 40$, with the sample extending from the deep water reference ($x_0 = 0$, corresponding to $n = 1$) towards the shore up to $x = 5400$ (ft) (corresponding to $n = 1350$), satisfactorily covering and well over the region of interest where $50 > D(x) \geq 10$ (ft).

ACKNOWLEDGEMENTS

The financial support of this work was provided by the Office of Naval Research, Geography Division, under constract No. N00014-69-40407 with the University of Delaware. The permission of Dr. J. Ploeg, Head, Hydraulics Division, National Research Council of Canada, for using Lake Ontario data is appreciated.

REFERENCES

Barnett, T. P., and A. J. Sutherland, A note on an overshoot effect in wind-generated waves, J. Geophys. Res., 73, p. 6879, 1968.

Blackman, R. B., and J. W. Tukey, The Measurement of Power Spectra, Dover Publications, 1959.

Borgman, L. E., Confidence intervals for ocean wave spectra, Proc. 13th Coastal Eng. Conf., 1 (10), 237-250, 1972.

Brown, L. J., Methods for the Analysis of Non-Stationary Time Series with Applications to Oceanography, Lab. Rep. HEL 16-3, 135 pp., Hydraulic Eng., University of California, Berkeley, California, 1967.

Collins, J. I., Prediction of shallow-water spectra, J. Geophys. Res., 77, (15), 2693 - 2707, 1972.

Harris, D. L., Characteristics of wave records in the coastal zone, Waves on Beaches, Academic Press, 1972.

Hinich, M. J., and C. S. Clay, The application of the discrete Fourier transform in the estimation of power spectra, coherence, and bispectra of geophysical data, Reviews of Geophysics, 6, 347 - 363, 1968.

Jenkins, G. M., and G. G. Watts, Spectral Analysis and Its Applications, Holden Day, 1969.

Otnes, R K., and L. Enochson, Digital Time Series Analysis, John Wiley and Sons, 1972.

Phillips, O. M., The Dynamics of the Upper Ocean, Cambridge University Press, London, 1966.

Pierson, W. J., and W. Marks, The power spectrum analysis of ocean wave records, Trans. Amer. Geophys. Union, 33 (6), 835 - 844, 1952.

Pierson, W. J., and L. Moskowitz, A proposed spectral form for fully developed wind seas based on the similarity theory of S. A. Kitaigorodskii, J. Geophys. Res., 69 (24), 5181 - 5190, 1964.

Plate, E. J., Limitations of spectral analysis in the study of wind-generated water surface waves, Proc. 1st Inter. Symp. on Stochastic Hydraulics, 522 - 539, 1971.

Ploeg, J., Some results of a directional wave recording station, Proc. 13th Coastal Eng. Conf., 1 (4), 131 - 144, 1972.

Priestley, M B., Evolutionary spectra and non-stationary processes, J. Roy. Statist. Soc., 27B (2), 204 - 237, 1965.

Priestley, M. B., Design relations for non-stationary processes, J. Roy. Statist. Soc., 28B (1), 228 - 240, 1966.

Priestley, M. B., Power spectral analysis of non-stationary processes, J. Sound Vibration, 6 (1), 86 - 97, 1967.

Priestley, M. B., and H. Tong, On the analysis of bivariate non-stationary processes, J. Roy. Statist Soc., 35B (2), 153 - 166, 1973.

Schule, J J., L. S. Simpson, and P. S. DeLeonibus, A study of fetch-limited wave spectra with an airborne laser, J. Geophys. Res., 76, 4160 - 4171, 1971.

Shinozuka, M., and C -M. Jan, Simulation of Multi-variate and Multi-dimensional Processes II, Civil Eng. -Eng. Mech Rep. 12, 41 pp., Columbia University, New York, N.Y., 1971.

Strekalov, S. S., V. Ph. Tsyploukhin, and S. T. Massel, Structure of sea wave spectrum, Proc. 13th Coastal Eng. Conf., 1 (14), 307 - 314, 1972.

Tayfun, M. A., C. Y. Yang, and G. C. Hsiao, Non-stationary spectrum analysis of ocean waves, Proc. 13th Coastal Eng. Conf., 1 (11), 251 - 269, 1972.

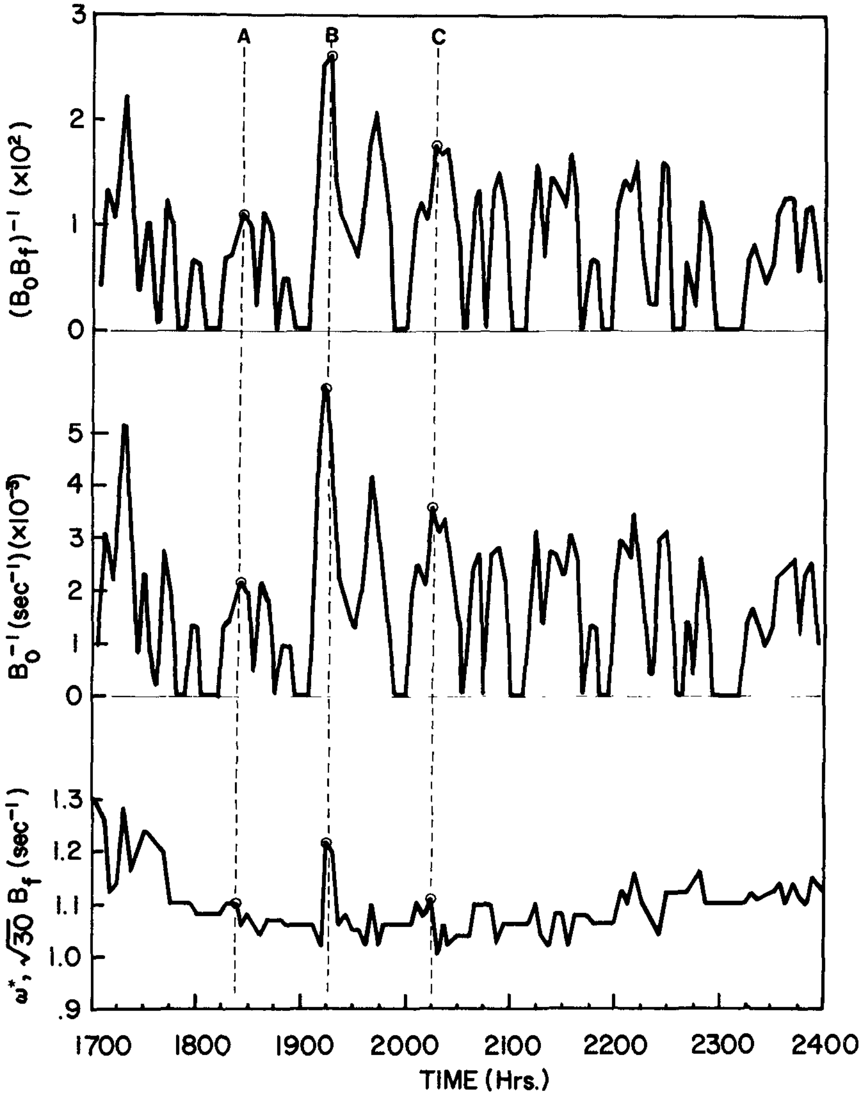

FIG. 1. Time Histories of Spectral Bandwidths and Spectral Peak Frequency

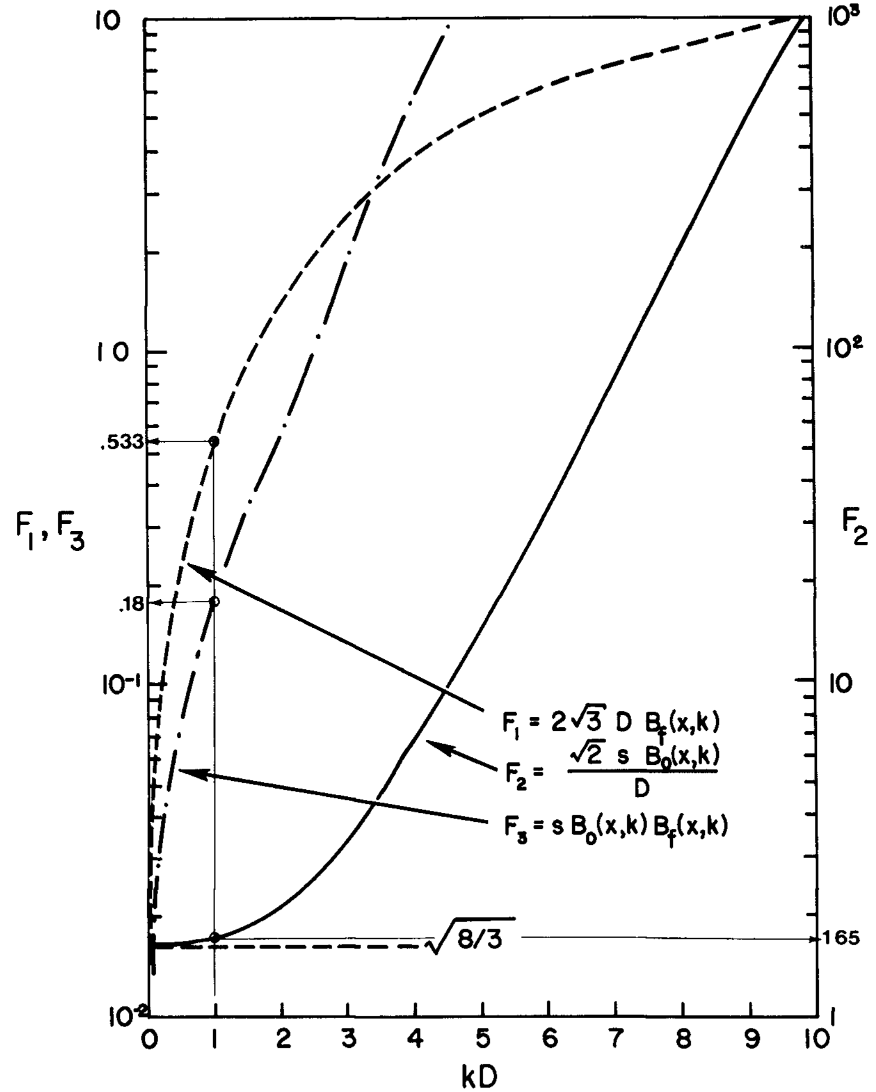

FIG. 2. Spectral Bandwidths in a Unidirectional Shoaling Wave Field

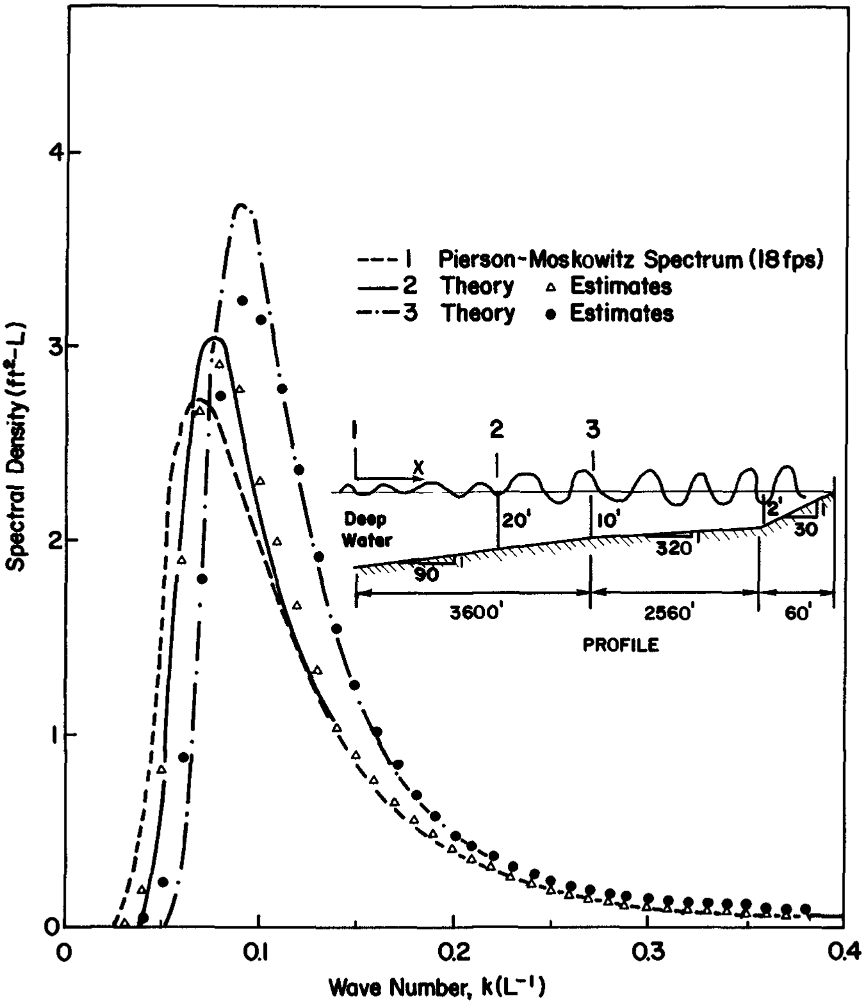

FIG. 3. Theoretical Spectra and Spectral Estimates in a Unidirectional Shoaling Wave Field