CHAPTER 99

FLUID FORCE ON ACCELERATING BODIES

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Abstract

The force exerted by a liquid on a moving body always depends on the preceding velocity and acceleration of the body.

The Basset-Lai equation, derived from the linearized equation of motion, gives the force on spheroids when convective acceleration of the water particles may be ignored. Examples prove that the history integral it contains accounts for a large portion (even all) of the force. Measured forces on a cylinder anchored in accelerating water show that history is equally important when convective accelerations are large. An important unanswered question is whether history terms that will fit a range of motions can be invented for simple non-linear problems.

For non-linear repetitive motion, such as the force exerted on piles by regular waves, no explicit history term is needed. The usual division of force into inertia and velocity portions is possibly less sound than a suggested alternative form from dimensional analysis.

One cannot expect to unravel the hydrodynamics of irregular wave forces, but he may use similarity principles to predict their probability distribution from measurements made elsewhere. Irregular waves will be statistically similar, altho mean heights may differ greatly, if the probability distributions of suggested characteristics of the gage records are alike. Given similar waves and structures scaled to the waves, the probability distributions of dimensionless wave forces also will be alike, and the forces at one place can be predicted from measured forces at another.

Introduction

Coastal engineering problems often involve the force exerted by water on a body when the water and/or the body are accelerating. Examples are wave forces on piles, the fluid force on a dam or underwater structure during an earthquake, and the traction on particles during bed-load movement.

If a body moves unsteadily thru a liquid, as when one stirs coffee with a spoon, the velocity pattern in the fluid at a given instant evidently depends upon the prior motion of the body. If, at the instant of interest, the velocity vector and its time rate of change were known at all points in the flow field, and if the shear and normal stress components at one point also were known, one could integrate the Navier-Stokes equations numerically to find the instantaneous stresses, and

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consequently the force, exerted by the fluid on the body. Since the velocity pattern was generated by a sequence of body motions, the instantaneous force also will depend upon this sequence, i.e., upon the history of the acceleration of the body.

Similarly, if irregular waves move past a stationary pile, the velocity and acceleration of the water particles near the body at a chosen time will depend partly upon the eddies in the wakes generated by antecedent waves. Consequently the instantaneous force on the pile will depend somewhat on the recent local wave history. Thus the maximum force exerted on a pile by a wave of given height and length is not a fixed number but a variable that depends upon the sizes of the two or three waves immediately preceding it.

The above reasoning and example suggest that a comprehensive equation for the force on a body accelerating in a fluid should contain the usual instantaneous velocity and acceleration terms plus corrections based on the acceleration that took place before the instant of interest--history terms. A review of the literature, however, shows that most authors have avoided using history terms in the force equation, even tho they recognized a general need, by choosing types of motion that do not require them. For example, References [1] thru [10] show that there is no lack of analyses of pendulums that oscillate harmonically in a fluid or of forces exerted by uniform oscillatory waves on cylinders and other objects. In such uniform periodic motion, the fluid velocity pattern changes continually during a cycle, but the succession of patterns is repeated during the next cycle and the next. Hence the force becomes a periodic function of time, altho not necessarily sinusoidal, and may be expressed in terms of phase or instantaneous velocity and acceleration.

Another group of authors [11] thru [14] has tried to avoid using history terms by treating other special problems such as (a) suddenly started, (b) constantly accelerated, or (c) freely falling objects.

While the need to study different types of accelerated motion case by case is genuine, the analyst who always thinks about the influence of antecedent motion on the present force will have a better grasp of the physical problem than one who does not. In some instances, moreover, he will be able to present the results of both mathematical analysis and experiment in best form if he uses a history term in the force equation.

The purposes of this paper are: (1) to present examples that show how important history may be; (2) to divide unsteady force problems into three categories according to the motion history, namely, (a) simple non-repetitive motion, (b) simple repetitive motion, and (c) irregular and random motion; and (3) to suggest a suitable way of expressing the fluid force for each category.

Definition

Both linear and non-linear flow problems will be considered in the paragraphs that follow. If the convective acceleration term $(\vec{u}\cdot\vec{\nabla})\vec{u}$[2] in the Navier-Stokes equation is small enough, compared to the other terms,

[2] $\vec{u}$ is the fluid velocity vector, a function of space and time.

to be ignored, the problem is linear. Physically this means that separation and turbulence will be absent altho the maximum Reynolds number may be quite large. Conversely, of course, convective accelerations are not negligible in non-linear problems, and wakes behind blunt bodies are to be expected.

Solution to a Linear Problem

By neglecting convective accelerations, Lai and Mockros [15] were able to formulate a mathematical solution to the problem of the force on a spheroid moving along its axis of symmetry. The resulting equation can be used to calculate the fluid force for linear problems of motion--repetitive, non-repetitive, and irregular. Further, it contains an explicit history integral and thus can be used to illustrate the importance of accelerations that take place before the instant at which the force is calculated. The Lai equation becomes identical to the Basset [16] equation when major and minor axes of the spheroid are equal, i.e., when the body is a sphere. The general form is

$$-F(t) = C_A m \frac{dv}{dt} + C_V 3\pi\mu D v + C_H \sqrt{\pi\mu\rho}\,\frac{D^2}{4}\int_0^t \frac{dv}{d\tau}(t-\tau)^{-\frac{1}{2}}\,d\tau \quad \ldots\ldots\ldots (1)$$

in which F(t) is the fluid force at time t, C_A, C_V, and C_H are the added mass, velocity, and history coefficients, m is the mass of displaced fluid, v is the velocity of the spheroid, and D its diameter normal to the axis of symmetry, μ and ρ are the viscosity and density of the fluid, and t and τ are time measured from the beginning of motion. For a sphere: $C_A = \frac{1}{2}$; $C_V = 1$; $C_H = 6$.

It is worth noting that the first term on the right of Eq. (1) is the resistance due to the irrotational added mass, and the second is the steady-state viscous drag.

Simple Non-Repetitive Motion

We introduce as an example a body that has been moving thru an incompressible fluid for a long time at constant velocity. An external force then stops the object quite rapidly. During deceleration and when it is stationary the body obstructs the forward flow it previously generated. Thus the fluid will exert a forward force on the body, i.e., a force opposite in direction to the original drag force. The magnitude of the forward force cannot be expressed as a function of the instantaneous velocity and acceleration of the object after the object has stopped; history terms are required.

Other examples of simple non-repetitive motion are: constant acceleration or deceleration from one steady state velocity to another; acceleration from rest by a constant external force such as gravity or buoyancy; and the deceleration that a body falling thru air may experience as it enters water. In each of these examples the force at a given instant depends on the antecedent motion as well as on the instantaneous velocity and its derivatives. By definition then, motion may be classified

as simple and non-repetitive if the fluid force on the body at any instant can be expressed by a manageable equation in which history terms need to appear.

Suppose the body in the first example is a small sphere moving in oil. Then values of diameter, viscosity and velocity may be chosen to satisfy linear problem requirements and make Eq. (1) valid. The writer chose as a sample problem a sphere 0.01 ft. in diameter traveling with an initial velocity of 0.1 ft./sec. thru SAE30 oil. The ball was brought to rest in 1/90 sec. by a suddenly applied external force that produced $\frac{dv}{dt}\Big|_0 = -9$. The resulting fluid force calculated from Eq. (1) is shown in dimensionless form in Fig. 1. Note that the time scale is made logarithmic to expand the deceleration period and that the actual time in seconds is one-tenth of the dimensionless time. The deceleration began arbitrarily at 10^{-4} on the dimensionless time scale.

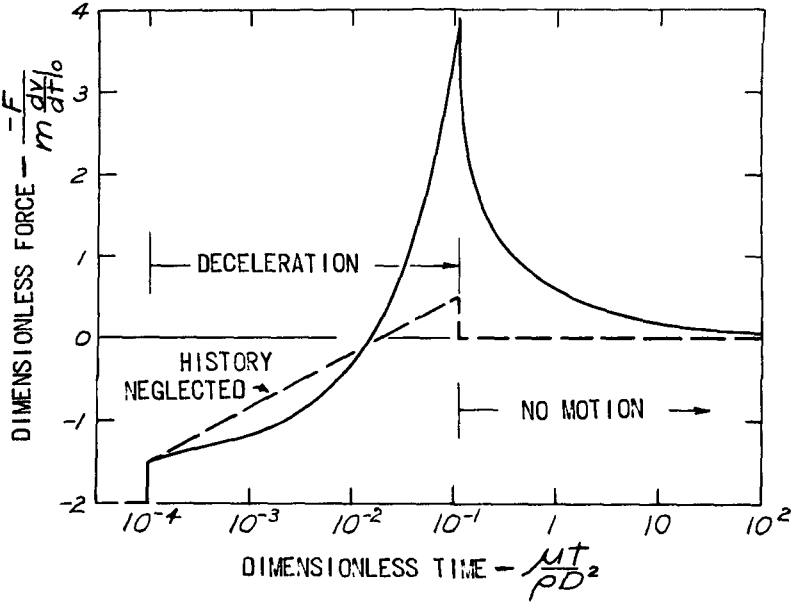

Fig. 1 Influence of history on fluid force acting on small sphere stopped in oil

The dimensionless force on the sphere changed from a steady state drag of -2 to -1.5 as soon as the deceleration began, rose to a maximum value of 3.88 (in the forward direction) just as the body stopped, and then decayed toward zero during a comparatively long period. The large effect of the history integral is apparent, since the maximum forward force is 0.5 if this term is omitted and no force occurs after the motion stops.

Other authors have used the fluid force given by Eq. (1) to study such matters as spheres falling thru a fluid and the response of a submerged sphere mounted on a vertical cantilever spring. Hjelmfelt and Mockros [17], for example, published the curves shown in Fig. 2, which show that a spherical sand grain falling from rest in water (density ratio 2.65) will take about three times as long to reach 0.6 of its terminal velocity as it would if there were no history force. Mockros and Lai [18] compared the same linearized falling-sphere theory with experiments. They found better agreement between the two when the history term was included than when it was not.

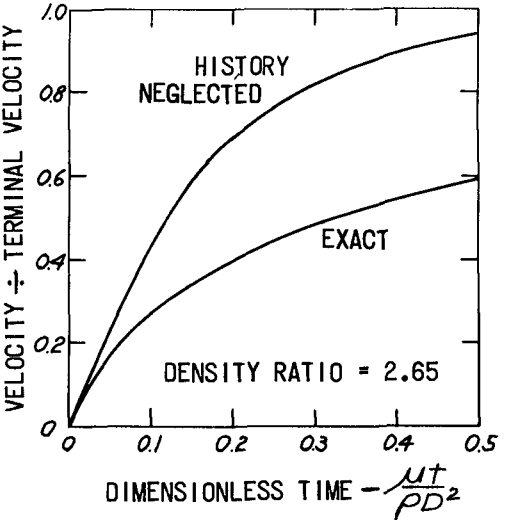

Fig. 2 Influence of history on velocity of sand grain falling from rest in water

For a submerged spherical mass on a spring Hjelmfelt et al., [19], discovered that the history term prevents critical damping if the mass is displaced from its equilibrium position and released. The moving fluid always pushes the mass past the equilibrium point.

We now turn to non-linear problems in the simple non-repetitive motion category. In the first place no general mathematical solution like Eq. (1) is available for any shape of body. Second, one must suppose that prior motion influences the present force altho the Basset history integral may have little resemblance to the kind of history terms required. Third, an empirical equation should give the proper force when the body has been moving with constant velocity for a long time and when it begins to accelerate from rest or from a long period of constant velocity.

Suppose we choose that the history terms, whatever their form, amount to zero after a long period of rest or constant velocity and hypothesize that the correct force is given by

$$-F(t) = C_A m\frac{dv}{dt} + C_V\frac{\rho A|v|v}{2} + \text{History terms} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

in which A is the appropriate cross-sectional or wetted area. To satisfy the constant velocity condition, C_V must be the steady state drag coefficient, and to give the proper force at the instant acceleration from rest begins, C_A must be the irrotational[3] added mass coefficient. Consequently if the hypothesis is to be correct, the irrotational C_A must apply at the instant a body begins to accelerate from constant velocity. Hamilton and Lindell [20] have shown, using spheres towed in water, that it does apply for low velocities (Reynolds numbers less than 35,000). More evidence is desirable.

The history terms of Eq. (2) remain to be invented by studying experimental data. As an example of the job that these terms need to do, the writer has used Eq. (2) and data published by Sarpkaya [21] to prepare Fig. 3. Sarpkaya measured the fluid force on a 2.75-inch circular cylinder fixed in a water tunnel. Its axis was normal to the flow. The water, initially at rest, was given an essentially constant acceleration for about 0.12 sec., producing a velocity which thereafter remained essentially constant at 3.1 ft./sec.

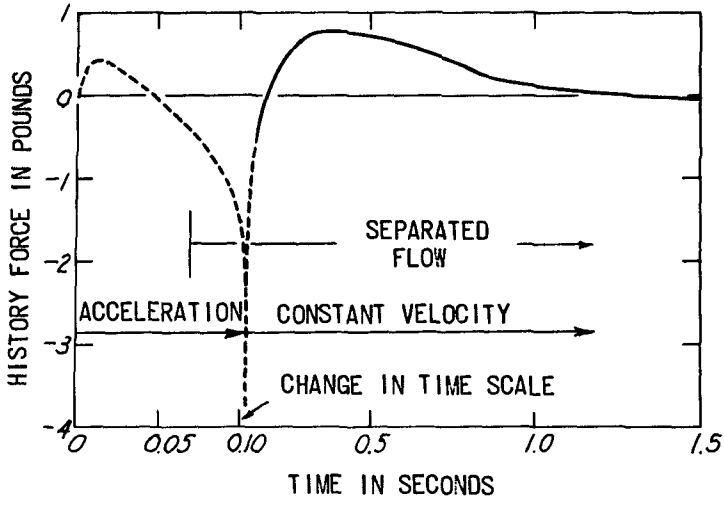

Fig. 3 History force on a cylinder fixed in rapidly accelerated water

[3]For supporting evidence see Mellsen et al. [14] and Hamilton and Lindell [20].

For Fig. 3 the history force was calculated by subtracting the calculated added mass force and the calculated steady state drag from the measured force. The irrotational coefficient C_A for fluid accelerating past a fixed cylinder is 2.00 and the steady state drag coefficient C_V was taken from a standard plot. The instantaneous velocity was used to compute the Reynolds number.

Referring now to Fig. 3, we see that for a short time after acceleration began the history force was positive, meaning that the added mass and drag terms were not quite large enough to make up the total force. This is compatible with linear theory, which may be expected to apply for a very short time after the beginning of motion. But after about 0.02 sec. the required history force begins to depart from what the linear history integral would indicate, and by the end of the acceleration period it reaches a poorly defined negative value. This simply means that, although separation has begun, the wake is much smaller than the steady state wake at the instantaneous velocity. Thus the drag term is much too large and a negative correction is needed. On the other hand, the wake overexpands by time = 0.2 sec. and the steady state drag is inadequate from then until about time = 1.3 sec. In this region a positive history correction is required.

This example shows that adopting Eq. (2) for non-linear force problems puts the burden of correcting the drag for transient wakes onto the history terms. They would also need to compensate for the existence of a laminar boundary layer when the steady state drag at the instantaneous Reynolds number presumes a turbulent one--and vice versa.

In the more general problem we have a change from one speed to another. The magnitude and duration of the corrections depend upon the initial and final velocity, the magnitude of the acceleration (or deceleration), the fluid properties, and the size and shape of the object. And the essence of the force problem is to discover whether or not there are similarities in flow behavior that permit unique history terms or any other explicit expression to apply to a range of problems rather than to one special case.

Simple Repetitive Motion

An example of this kind of motion is a steady state oscillation, not necessarily harmonic, but with an easily identified period. Uniform waves are repetitive; a damped oscillation is not. The basic reason for introducing the repetitive category is that no history terms or history integrals are needed in a fluid force expression. The force is a function of phase.

To illustrate the linear case let us consider a sphere oscillating in a liquid according to

$$v = b\sigma \sin \sigma t \qquad (3)$$

in which b and σ are the amplitude and frequency.

When Eq. (3) is differentiated and substituted into Eq. (1) and the integral evaluated from $\tau = -\infty$, when the motion began, to $\tau = t$, the present time, the result may be put into the alternative forms

$$-F(t) = m\left(\frac{1}{2} + \frac{9}{2}\alpha\right)\frac{dv}{dt} + 3\pi\mu D\left(1 + \frac{1}{2\alpha}\right)v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

or

$$-F(t) = C(\alpha)mb\sigma^2 \sin\left[\sigma t + \varphi(\alpha)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

in which $C(\alpha)$ is a force coefficient, $\varphi(\alpha)$ is a phase shift, and

$$\alpha = \left(\frac{2\mu}{\rho\sigma D^2}\right)^{\frac{1}{2}}, \qquad \tan\varphi(\alpha) = \frac{1 + 9\alpha}{9\alpha + 18\alpha^2},$$

$$C(\alpha) = \frac{1}{2}\left[(1 + 9\alpha)^2 + (9\alpha + 18\alpha^2)^2\right]^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Form (4) is due to Stokes [1]. Since the coefficient of dv/dt contains α and thus depends on viscosity, this form is sometimes used to support the argument that added mass coefficients depend on viscosity. But whether they do or not is entirely a matter of definition since there is no a priori rule that says all multipliers of the instantaneous acceleration must be included as part of the added mass. The writer prefers to regard m/2 as the added mass, which is the value for a sphere from irrotational theory, and recognize $m\left(\frac{9}{2}\alpha\right)\frac{dv}{dt}$ as part of the history integral.

In form (5) the force is a harmonic function with a known amplitude. It leads the velocity function by a known phase angle. By deriving a similar expression with the history integral omitted and comparing it with Eq. (5), one may show that the amplitude of the force always is increased by the history integral. This is consistent with the result of Hjelmfelt and Mockros [22] who determined how well sand particles would follow straight line oscillations of a liquid in which the particles were suspended. They found that including the history integral increased the ratio of the amplitude of the particle motion to the amplitude of the water motion. Their result applies to sand transport by waves.

Stokes' [1] form for expressing the fluid force on an oscillating sphere in the linear case, Eq. (4), has been carried over to non-linear oscillation problems by numerous authors[4] who like to use

$$-F(t) = C_M m\frac{dv}{dt} + C_D\frac{\rho A|v|v}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

in which C_M and C_D are experimental inertia and velocity coefficients which may or may not vary during a cycle.

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[4] For example, see References [5] thru [10] and [23].

This form has become a standard one for expressing the wave force on piles. The horizontal components of acceleration and velocity of the approaching water particles are calculated from one of the oscillatory wave theories, and the coefficients, assumed constant thruout a cycle, usually are based on tests. Values often are chosen to produce the proper maximum and minimum forces.

It is instructive to compare Eq. (7) with Eq. (2) Eq. (2) has been recommended for non-repetitive motion, but is valid for periodic motion.[5] The equations are similar except that (2) contains explicit history terms and (7) does not. Since the wave force on a pile at any chosen value of the phase depends on the prior motion, C_M and C_D in Eq. (7) must absorb the influence of history. Hence, at best, their values can equal C_A and C_V in Eq. (2) only under particular circumstances. C_M and C_D depend on phase and wave height divided by pile diameter, whereas C_A and C_V are irrotational added mass and steady state drag coefficients respectively.

Eq. (7) cannot be put into the harmonic form of Eq. (5) because the drag force in (7) is proportional to the velocity squared. Nevertheless, the fluid force in any repetitive motion is a repeating function of the phase, which may be expressed non-dimensionally as t/T, where T is the period. The dimensionless displacement of the body (or of the fluid in the absence of the body if the fluid moves), say S/D, is presumed to be a known function of t/T. (Here D is a suitable body dimension, usually transverse.) It produces a measured fluid force pattern F(t/T) which may be written in dimensionless form as

$$\frac{F}{\rho D^2 b^2/T^2}\left(\frac{t}{T}\right) = \frac{F_{max}}{\rho D^2 b^2/T^2} \times f\left(\frac{t}{T}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

where $f\left(\frac{t}{T}\right)$ = the measured $F\left(\frac{t}{T}\right)$ divided by the measured F_{max}, and b is a pertinent amplitude such as the maximum displacement.

Further, for a particular repetitive motion $\frac{F_{max}}{\rho D^2 b^2/T^2}$ will depend on $\frac{b}{D}$ and $\frac{\rho D b}{\mu T}$. Consequently

$$\frac{F}{\rho D^2 b^2/T^2}\left(\frac{t}{T}\right) = \frac{F_{max}}{\rho D^2 b^2/T^2}\left(\frac{b}{D}, \frac{\rho D b}{\mu T}\right) \times f\left(\frac{t}{T}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

The form of Eq. (9) makes the maximum force an easily described function but provides the option of expressing the force detail thruout the cycle by plotting $f\left(\frac{t}{T}\right)$.

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[5] Odar and Hamilton [24] used Eq. (2) for the force on a sphere oscillating in oil and invented a history term based on the Basset history integral.

For the force exerted on objects by regular waves the counterpart of Eq. (9) is

$$\frac{F}{\rho D^2 H^2/T^2}\left(\frac{t}{T}\right) = \frac{F_{max}}{\rho D^2 H^2/T^2}\left(\frac{h}{D}, \frac{H}{D}, \frac{H}{gT^2}, \frac{\rho DH}{\mu T}\right) \times f\left(\frac{t}{T}\right) \quad \text{.....(10)}$$

in which H is the wave height, h is the water depth, and g is the acceleration of gravity. Eq. (10) assumes that surface tension may be neglected and that the wavelength will be fixed by the choice of h, H, and T. The viscous parameter may be unimportant. Also for objects that are large compared to the water particle displacements, acceleration forces will dominate, and a more suitable dimensionless force is $\frac{F}{mH/T^2}$. Garrison and Perkinson [25] have used a set of dimensionless parameters somewhat like those suggested to express the maximum wave force on horizontal cylinders.

Obviously the writer prefers Eq. (10) to Eq. (7). Eq. (7) uses inertia and drag coefficients that are too easily confused with irrotational added mass and steady state drag coefficients.

Irregular and Random Motion

In this class belong vibrations generated by earthquakes, turbulence, and the particle motions caused by irregular waves. The dispersion or settling of particulate matter in turbulent flow is a linear force problem involving random motion. A non-linear problem, of course, is the force exerted by irregular waves on piles.

As in previous examples, the fluid force at a particular time is conditioned by eddies or velocity patterns set up by prior motion. Thus, since irregular motion is not repetitive, an attempt to express force details in terms of instantaneous velocity and acceleration is fundamentally unsound. But if the motion is truly irregular or random, the task of unraveling the contributions of innumerable possible histories, except in special linear problems to which Eq. (1) applies, is overwhelming. Therefore non-linear problems, at least, should be handled statistically. For an example, let us focus on forces exerted by irregular waves.

Grace [10] found that seemingly identical ocean swells produced quite different forces on a submerged sphere. He attributed the discrepancies to probable differences in water particle motions under waves of equal height and period. But suppose waves (a) and (b) have identical particle motions, yet wave (a) is preceded by a smaller wave and wave (b) by a larger one. Then the forces exerted by (a) and (b) would differ because the motion history is different.

The data published by Wiegel [7] for forces exerted by ocean waves on piles show practically no correlation between wave height and force. Nevertheless, according to Borgman [26] an analysis of some of these data showed that the spectral density of force was similar to the spectral density of height except in the high-frequency off-peak region. Borgman used constant coefficients in Eq. (7), first-order wave theory to get particle motions, and a Gaussian wave-height distribution to prove that this similarity should exist. He set up a procedure for transferring from one spectrum to the other.

Aware of Borgman's work Paape [27] measured the forces exerted on piles subjected to irregular waves in the laboratory. Spectral densities of force and wave height calculated from his data were not similar and he failed to find a satisfactory transfer function. Thus one must conclude that the spectra may or may not be similar and that Borgman's assumptions oversimplified matters.

It is unfortunate that the designer cannot rely upon similarity between force and wave height spectral densities. For resonance problems he needs to know the frequency at which the force "energy" peaks. Perhaps Paape's tests and others may be used to find how much the frequency at the peak of the force spectrum may differ from the frequency at the peak of the wave height spectrum.

Because wave forces and heights do not correlate, Bretschneider [28] recommends ranking the heights and maximum forces[6] in order of magnitude and working with cumulative probability plots of wave height and force. His idea makes sense because it avoids any inference that the present wave alone is responsible for the present force. Bretschneider developed the idea into a method for analyzing measurements and predicting magnitudes of design forces for statistically similar wave sequences. An alternative method follows.

Suppose we have force and wave height measurements for a particular body shape in a particular irregular sea. How can one arrive at the force that, say, would be equaled or exceeded five percent of the time if a similar sea of greater magnitude acted upon a similar structure? Note first of all that this is a model-prototype kind of problem and one must be able to identify a similar sea. Criteria will appear presently.

We attribute the force at a given instant to the unbalanced shear and normal stresses exerted by the fluid on the body. They are caused by the instantaneous velocities and accelerations of the fluid particles in a rather large region surrounding the object. The motion pattern in the region, of course, is caused by the waves and the presence of the object, but the only index of velocity available is H/T and of acceleration H/T^2. H may be defined as the height of a crest above the surface depressions before and after it and T as the time interval between the two depressions. Both are often a matter of judgment.

Altho these indices represent the particle motion quite inadequately, the dimensionless acceleration H/gT^2, where g is the acceleration of gravity, is certainly a variable on which the force depends. Using γ as specific weight, we may write for a particular wave

$$\frac{F_{max}}{\gamma D^3} = \frac{F_{max}}{\gamma D^3}\left(\frac{h}{D}, \frac{\gamma DH}{g\mu T}, \frac{H}{D}, \frac{H}{gT^2}, \text{History}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

The history term is a reminder that a range of values of $F_{max}/\gamma D^3$ may accompany each set of values of the other four variables.

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[6] He separated drag and inertia forces by picking values at wave peaks and zero crossings respectively.

For a given sea state and a particular structure the variables

$$\frac{\gamma DH}{g\mu T}, \frac{H}{D}, \frac{H}{gT^2}, \text{ as well as } \frac{F_{max}}{\gamma D^3},$$

will have permanent probability distributions. That is, if the gage height record may be taken as a stationary time series, the statistical properties of the force record also will be permanent. Hence we may avoid history and hydrodynamics by writing

$$\left\{\frac{F_{max}}{\gamma D^3}\right\} = \left\{\frac{F_{max}}{\gamma D^3}\left(\frac{h}{D}, \left\{\frac{\gamma DH}{g\mu T}\right\}, \left\{\frac{H}{D}\right\}, \left\{\frac{H}{gT^2}\right\}\right)\right\} \quad \ldots\ldots(12)$$

in which the braces are used (unconventionally) to mean the probability distribution of the quantity enclosed.

Let us compare these variables for two cases, model m and prototype p. If the model is built to the proper length scale ratio and the model sea is similar to the prototype sea,

$$\frac{h}{D}\Big|_m = \frac{h}{D}\Big|_p \quad \ldots\ldots(13)$$

$$\left\{\frac{H}{D}\right\}_m = \left\{\frac{H}{D}\right\}_p \quad \ldots\ldots(14)$$

$$\left\{\frac{H}{gT^2}\right\}_m = \left\{\frac{H}{gT^2}\right\}_p \quad \ldots\ldots(15)$$

$$\left\{\frac{\gamma DH}{g\mu T}\right\}_m \neq \left\{\frac{\gamma DH}{g\mu T}\right\}_p \quad \ldots\ldots(16)$$

Because of Ineq. (16) $\left\{F_{max}/\gamma D^3\right\}_m$ will not be exactly equal to $\left\{F_{max}/\gamma D^3\right\}_p$. In what follows we shall assume that all the Reynolds numbers are large enough and the structure blunt enough to make the inequality unimportant. Then

$$\left\{\frac{F_{max}}{\gamma D^3}\right\}_p \simeq \left\{\frac{F_{max}}{\gamma D^3}\right\}_m \quad \ldots\ldots(17)$$

and if the right hand probability distribution is known, the probability distribution of $F_{max}|_p$ is obtained simply by multiplying the numbers on the abscissa scale by the constant $\gamma D^3|_p$ as indicated in Fig. 4. (In the figure the measurements have been grouped into eight class intervals.) Then the distribution of $F_{max}|_p$ may be summed to get the cumulative probability of the prototype force and answer the question of what force will be equaled or exceeded five percent of the time.

Similar seas have been assumed. Now we must face the question of how to identify statistically similar seas. The definition itself can vary, depending on why they are defined. We are interested in making Eq. (17) true when surface tension and viscosity are unimportant.

The first requirement is that Eq. (15) be satisfied, but it says nothing about the sequences of changes in wave height. From a history standpoint these sequences have an important influence on the force

distribution. Hence $\left\{\frac{H}{gT^2}\right\}$ is a necessary but insufficient index for similar seas and an equation involving differentials such as

$$\left\{\frac{(H_j-H_i)^2/\overline{H^2}}{(t_j-t_i)^2/\overline{T^2}}\right\}_m = \left\{\frac{(H_j-H_i)^2/\overline{H^2}}{(t_j-t_i)^2/\overline{T^2}}\right\}_p \quad \text{.....(18)}$$

is indicated. Wave j follows wave i.

Moreover, since Eq. (14) scales the structure to the waves it does not pertain to identifying wave similarity only. Hence for sea similarity an equation such as

$$\left\{H^2/\overline{H^2}\right\}_m = \left\{H^2/\overline{H^2}\right\}_p \quad \text{.....(19)}$$

is more appropriate. Thus for the present purpose, similar seas probably exist if Eqs. (15), (18), and (19) are satisfied. Experimental data are required to find if they are adequate or not. For example, altho the orientation of the structure with respect to the dominant wave direction presumably is part of the modeling, no account has been taken of the directions of component waves in the description of the sea.

In addition to dimensional analysis and hopefully some physical insight in the choice of dimensionless variables, the writer has used two rather self-evident requirements if force and wave height records from one situation are to be used to find forces in another:

(a) The wave records at the two locations must have certain similarities.

(b) The structure size must be scaled to the wave size, Eq. (14).

These conditions are no different from the ones required in the case of regular waves, Eq. (10), for which wave similarity is implied by a choice of h/D, H/D, and H/gT^2.

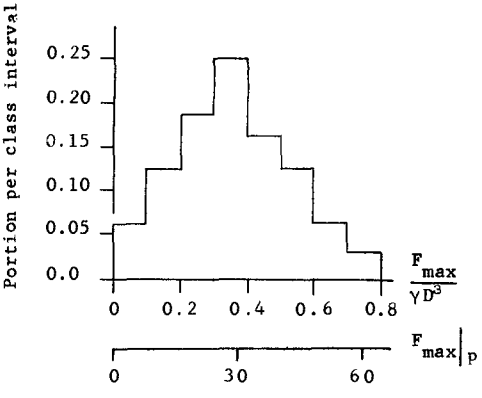

Fig. 4

For regular wave problems engineers use data from tests containing a range of dissimilarities, get a range of coefficients of some sort, and use average coefficient values to calculate approximate forces. Sometimes they calculate extremes to find how much error may be involved.

If similarity requirements are ignored, corresponding errors will occur when statistical quantities, such as probability distributions, are used for analyzing irregular wave and force measurements. To isolate variations due to scale differences, one might well test several sizes of pile, for example, as Wiegel [7] did, but test them simultaneously in the same wave environment.

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Notation

A = cross-sectional or wetted area
b = amplitude of a harmonic or periodic motion
C = dimensionless force coefficient
C_A = added mass coefficient--irrotational motion
C_D = velocity coefficient--unsteady motion
C_H = history coefficient
C_M = inertia coefficient
C_V = steady state drag coefficient
D = diameter or other significant length
F = force exerted by fluid
F_{max} = maximum fluid force in repetitive motion or when a wave passes
f = force-variation coefficient
g = acceleration of gravity
H = wave height
h = water depth
i,j = subscripts to indicate one wave and the next
m = mass of fluid displaced by a body; also subscript indicating model
p = subscript indicating prototype
S = displacement during repetitive motion
T = period of a repetitive motion or time interval between waves
t = time at which force is calculated or measured
$\vec{u}$ = velocity of fluid particle
v = speed of body

$\alpha = (2\mu / \rho \sigma D^2)^{\frac{1}{2}}$
γ = specific weight of fluid
μ = viscosity of fluid
ρ = density of fluid
σ = frequency, radians/second
τ = time during history of motion--an integration variable
$\vec{\nabla}$ = space derivative operator

$\{\}$ indicate probability distribution of the quantity enclosed