CHAPTER 22

THE ROLE OF SURFACE TENSION IN BREAKING WAVES

Robert L. Miller

Professor, Department of Geophysics
University of Chicago

Abstract.

Breaking criteria in the vicinity of the crest, such as limit crest angle and limit form, and larger dimensions such as limit height (H/L) and breaker height (H_b/d_b), are found experimentally to be significantly affected by change in surface tension. A number of wave types were examined, including periodic waves, solitary waves, and standing waves, over both constant depth and uniform slopes. Variations in natural waters in some cases were found to be of equivalent magnitude to those induced for the experiments. The conclusion is drawn that surface tension should be taken into account in development of a satisfactory theory of breakers. It is also an important factor in experimental studies, particularly engineering model studies involving breaking waves.

Introduction.

Experimental studies made several years ago, of run-up and transitions in bores brought out an interesting sequence of events during the breaking process. The first definite breaking at the crest was preceeded by a surface pattern of ridges or streaks normal to the crest, on the back side of the wave. These later coalesced, to form a pattern of cell-like indentations. Following this, the surface ruptured just at the crest and the wave broke. Miller 1968a and also Witting 1964.

It thus appeared that the breaking process in its earliest stages, may be a surface phenomenon, at the least in part. Accordingly, a preliminary series of experiments, Miller 1968b, were made with undular bores first at normal surface tension; and then repeated with the surface tension reduced by adding a detergent. Figure 1 gives a summary of the results. The ordinate represents the distance down-channel at which initial breaking of the lead undulation occurs, as it progresses over a flat bottom. D is the distance down-channel from the generating piston and d is the undisturbed water depth. The abscissa is in units of crest velocity converted in a second scale to Froude numbers. For a given Froude number, comparison is made of $D/_d$ for surface tension at a "normal" 73 dynes/cm vs surface tension reduced to 39 dynes/cm;

Froude no. at initial breaking	$D/_d$	
	T_{73}	T_{39}
1.256	(not breaking)	11.0
1.261	(not breaking)	14.75
1.267	9.75	9.50
1.271	5.0	4.5
1.278	12.3	10.4

Several conclusions were drawn. 1) For a given Froude number, the normal surface tension wave remains stable for a greater distance down-channel. 2) Breaking is observed for lower Froude numbers at surface tension T_{39} than at normal surface tension T_{73}. These results indicated that a more general study should be made.

Accordingly, a survey of the published literature was made, with respect to limit form, and breaking criteria in the vicinity of the crest. In addition, a search was made for papers on the role of surface tension at the crest particularly with respect to breaking waves. Table 1 gives a summary of the limit form and breaking criteria at the crest. The summary is not intended to be a complete reference list, but hopefully

Figure 1 Undular Bore: Average Distance From Piston To Initial Breaking, vs. Celerity

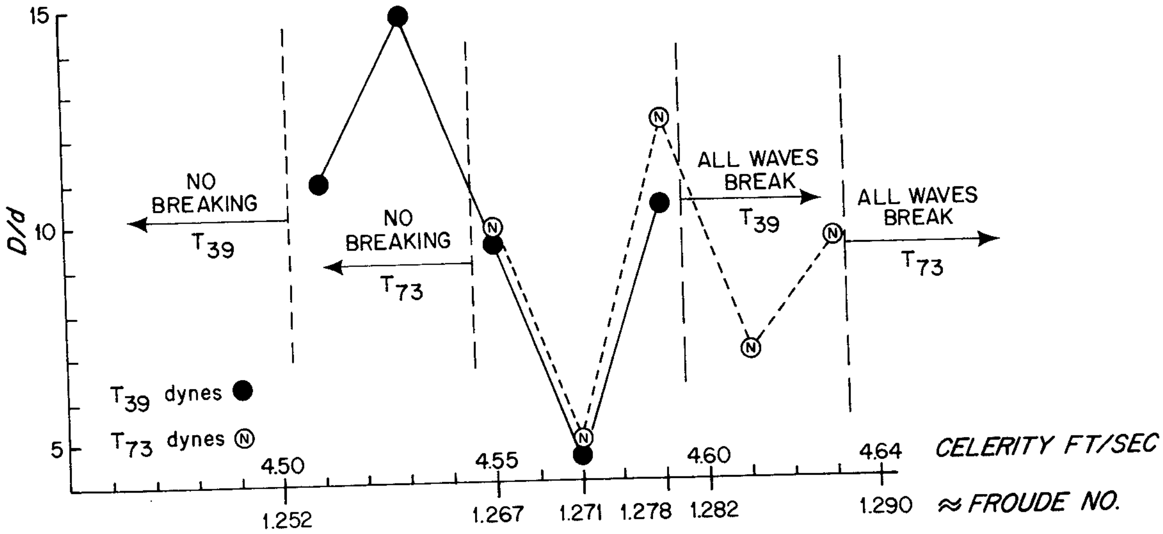


Table 1 Summary of limit form and breaking criteria, at the crest.

Wave type	Limit Crest Angle	Symmetry[1]	Vertical Acceleration of fluid particle[2]	Initial Breaking Process[3]
Periodic Progressive (Deepwater)	120° Stokes 1880	Symmetrical almost a vertical plane.	$\frac{1}{2}g$ in vicinity of crest, directed away from crest. Directly beneath crest = $\frac{1}{2}g$, directed downward. Just at crest, acceleration is indeterminate. Longuet-Higgens 1963	? May be related to the parasitic capillary waves which form on the forward face. Longuet-Higgens 1963
Periodic Progressive (Constant depth)	120° ?	Wave front vertical Stoker 1948		Vertical front collapses and wave "curls over". Stoker 1948
Periodic Progressive (Uniform slope)	? May be less than 90°	Steep slope Vertical front face → plunging.	Mason 1951 Stoker 1948 Galvin 1968	Formation of downward curving jet at vertical face which strikes water surface in front of wave. Rapid collapse of wave.
		Gentle slope Vertical front face at crest? → Spilling.		A turbulent zone with air entertainment confined to vicinity of crest. Slow subsidence of wave.

[1] Analagous to "limiting shape" criterion which suggests that for plunging breaker for example, "breaking occurs when some part of shoreward face of wave becomes vertical" Ippen and Kulin 1954.

[2] In appropriate cases, such as progressive waves, the fluid particle may be subject to the "limiting velocity" criterion which states that "breaking occurs when the velocity of particles at some point along the wave usually at the crest, equals the celerity of the wave" Ippen and Kulin, 1954.

[3] The specific effect of wind in the keaking process is not known. For smaller waves, Cox 1958 has shown that capillaries will form in the absence of wind.

Wave type	Limit Crest Angle	Symmetry	Vertical Acceleration of fluid particle	Initial Breaking Process
Standing Wave	90° Penney-Price 1952 G. I. Taylor 1953	Symmetrical about vertical plane. Truly Periodic waves not established theoretically: thus perfect symmetry may not exist.	g directed downwards Penney-Price 1952 G. 1. Taylor 1953	Surface at crest bursts followed by a plume, then violent instability G. I. Taylor 1953 and present paper
Clapotis	Assumed same as standing wave			
Solitary: (const. depth)	120° McCowan 1891	Symmetrical about vertical plane.		?
Solitary: shoaling (uniform shape)	? May be less than 90° Front face vertical Back face sub-parallel to bottom.	For steep slopes: vertical forward face → "plunging breaker" For gentle slopes small vertical face at the crest? → spilling breaker.[1]		formation of a jet from the vertical face which strikes the water surface in front of the wave

[1] Note: under certain conditions, a shoaling wave may not "break" at all but simply subside into the run up phase. Caldwell 1949 Galvin 1969

Wave type	Limit Crest Angle	Symmetry	Vertical Acceleration of Fluid Particle	Initial Breaking Process
Undular bore First undulation (constant depth)	120° McCowan (assumed similar to solitary wave)	Symmetrical about vertical plane.		Turbulent zone with air entrainment, at vicinity of crest followed by collapse of lead undulation, and transformation to fully developed bore form; turbulence rapidly engulfs trailing undulations. Miller 1968
Undular bore First undulation shoaling over uniformly sloping bottom.	(assumed similar to solitary wave)	vertical face at crest on front side.		Analogous to plunging breaker for steep slope and to spilling breaker for gentle slope. Miller 1968
Reforming Solitary wave				
Very short crested waves superposition of to opositly progressing wave trains.	90° (Analogous to standing wave?) (present paper) Observation, 98°	varies	g directed downward? analogous to standing wave?	bursting upward at crest or overturning crest (present paper)

does include a reasonably complete list of breaking criteria and limit form, in the vicinity of the crest. Thus, breaker classifications such as that of Miller and Zeigler, 1964 and Galvin 1968, are not considered, in this study.

Experimental procedure.

Several wave types exhibit a cusped profile, prior to breaking. In the vicinity of the crest, however, the profile limbs are approximately straight as shown in Figure 2, except at the apex, where the radius of curvature R, becomes an important consideration. Pressure difference across the air-water interface may be expressed as $\Delta P = 2T/R$ where T is the surface tension and R is the radius of curvature. The highest pressure is on the concave side of the interface.

For progressive waves, α is defined as the angle the front face makes with the vertical line to the apex. β is the angle formed by the trailing face of the wave. As in Figure 2, $\alpha + \beta$ = the crest angle, and symmetry is expressed by comparison of α with β.

A primary aim of the study is to assess the effect of surface tension on crest angle, on symmetry, and where appropriate, on larger scale breaker characteristics as secondary effects. The experimental plan is to measure the above properties for surface tension normal, and reduced to several lower values.

A number of additives intended to reduce surface tension were tried. The most satisfactory of these were Triton X-45 and Triton CF32. It was determined that a solution of 0.01% reduces surface tension of pure water from normal 72 dynes/cm at 20°C to about half at 36 dynes/cm. An anti-foaming agent, G. E. AF-60, was also added at a strength of about 25 parts/million.

Test for change in viscosity indicated a drop of about 2% when the surface tension was reduced to 35 dynes/cm using the above solution. The surface tension gradient, surface to bottom, was checked by withdrawing fluid at various levels, taking care to withdraw slowly thru glass pipette to insure a local sample. The results showed a slow decrease at the surface, indicating a concentration of the Triton at the surface with time. However, when the fluid was thoroughly mixed by agitation, the surface tension remained essentially constant throughout the tank for a period of several hours. It is well known that surface films which tend to increase surface tension will form on open water with time, even in the cleanest of laboratories. It was necessary in this study to insure the absence of surface films, and the absence of surface concentration of the detergent. Thus, a procedure was established whereby the tank was drained and refilled with fresh water, before each set of experiments. The Triton plus AF-60 was then immediately added in the desired proportion and thoroughly mixed with the tank water, followed by a check of the resulting surface tension on the ring tensiometer. The tank was again agitated by running the wave generator. Elapsed time for the above procedure after the tank is refilled, is about 20 minutes. Experiments were then run immediately. I feel that the surface film and surface concentration of detergent were eliminated satisfactorily.

Several methods were tried for measuring surface tension. It was found that the ring tensiometer method was most satisfactory in terms of repeatability, ease of operation, and time required to obtain a measurement. It was, however, necessary to establish a careful procedure including cleaning the ring by flaming and rinsing after each measurement. Calibration of the ring tensiometer was checked repeatedly against standards during the course of this study.

Most of the experiments were conducted in a wave tank approximately 90 feet long. The waves were of the order of $L \simeq 2.25$ ft. and $H \simeq 0.20$ ft. for the progressive waves and $H \simeq 0.30$ ft. for the solitary wave and undular bore. The standing wave experiments were conducted in a small aquarium sized tank with waves of the order of $L = 35$ cm. in an undisturbed depth of $\simeq 18$ cm.

Attempts to measure crest angles from oscillograph records proved arduous and inaccurate. The following method proved satisfactory. 16 mm motion picture films were taken through a reference grid on the glass wall of the tank, at 64 frames/sec.

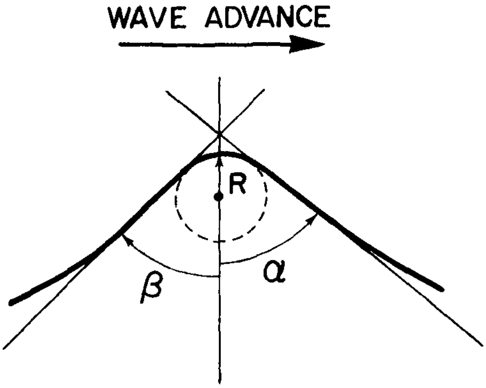


Figure 2 Detail of Crest Angle and Symmetry.

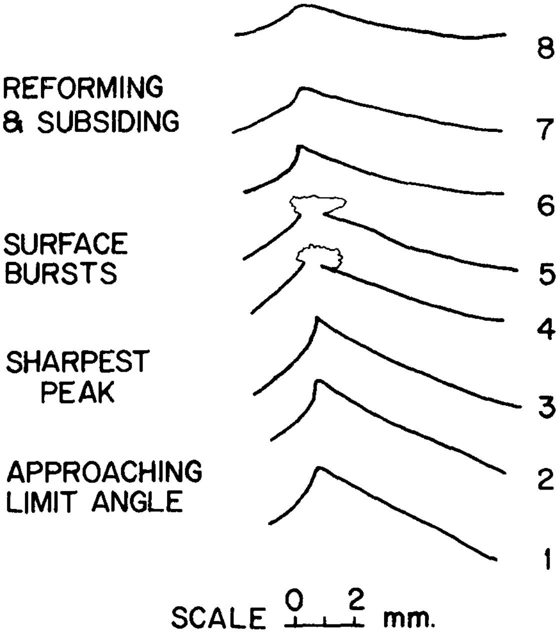


Figure 3 Breaking and Reforming Sequence. Standing Wave, at 64 Frames/Sec., Motion Picture Sequence. The Possibility of Calculating Bursting Pressure is Suggested.

A "stop-action" projector was utilized against a white background wall to locate a satisfactory breaking sequence and then the film was advanced one frame at a time to obtain the maximum unbroken profile. By projecting upon a smooth white surface at a suitable distance, sufficient image enlargement was acquired to aid in accurate measurements. Sharp contrast in the color film was gained by using rhodamine dye in the water. Tests indicated that the slight addition of rhodamine dye did not affect either the surface tension or viscosity. Larger scale dimensions of the waves were measured in the same way.

Experimental results.

A summary of experimental results is given in table 2. Several of the notable results are indicated in the following discussion.

1) Limit crest angle

It seems clear that the limit crest angle of 120° is significantly exceeded when surface tension is reduced, for progressive periodic waves in deep water or over constant depth bottom. This latter case also includes solitary waves and the lead undulation in the undular bore. A similar conclusion is drawn for the 90° limit crest angle for standing waves, including clapotis.

The limit crest angle concept, does not appear to apply at all to waves shoaling on a plane beach, dispite statements to the contrary, i.e. Kinsman 1965 p. 273. Rather than 120°, the limit angle by implication is approximately 90° with a vertical front face, and a back face sub-parallel to the bottom slope. (Figure 5). It seems reasonable that the subsequent stage is effected by surface tension whether collapse of the vertical face (waves shoaling over low angle slopes) or jet-like overturn (waves shoaling over steep slope). This merits careful experimental study.

2) Symmetry

Periodic progressive waves in deep water do not preserve symmetry up to the point of breaking, but rather the front face is notably steeper than the trailing face. However, the solitary wave over constant depth, and the standing wave, do preserve symmetry to initial breaking i.e. α approximately equal to β. The observations of clapotis indicate that α (the wave face toward the reflecting plate) is considerably smaller than β for the wave face away from the reflecting plate. However, the waves recorded in these observations may not have reached the stable standing wave configuration. This could account for the asymmetry. It is also interesting to note that symmetry is reversed in the reforming solitary wave, with the rear face being the steepest.

3) Limit steepness (H/L)

If the breaking mechanism at the crest is affected by changes in surface tension, it is reasonable to expect secondary effects on larger scale breaker properties. The next three properties are of this nature, and as can be seen, several of these do show the effect of change in surface tension. Periodic progressive waves showed a progressive increase in limit steepness for decreasing surface tension, although the steepest waves observed were less steep than the theoretical prediction of Michell, 1893.

In the case of standing waves the limit steepness for surface tension normal was very close to that observed by G. I. Taylor, and slightly higher for surface tension reduced.

4) Breaker height (H_b/d_b) over sloping bottom.

For periodic progressive waves and solitary waves over a low slope, breaker height is slightly higher for surface tension reduced. For solitary waves over steep slope breaker height is significantly lower! The magnitude of H_b/d_b, however, is much greater for solitary than for periodic waves. It is possible that the backwash of the previous wave with its "tripping" effect may be involved in some manner, since it is present in the periodic case and absent in the solitary wave case.

5) Limit height (H/d) constant-depth bottom.

Table 2 Summary of experimental results.

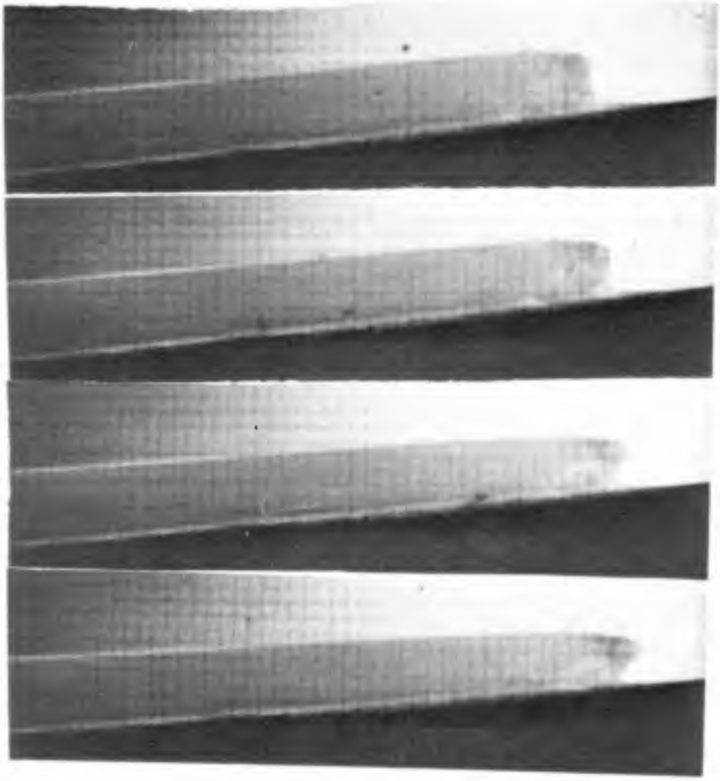
Wave	Limit Crest Angle	Symmetry	Limit Steepness H/L	Breaker Height (H_b/d_b) or Limit Ht(H/d)	Breaker Distance (x_b) from shoreline. (actual scale)
Periodic Progressive d/L = 0.311 ("Intermediate depth" wave)	Theory: 120° Observed: T_{74} = 129° T_{53} = 123° T_{40} = 116°	Theory: $\alpha = \beta = 60°$ Observed: T_{74}: $\alpha = 57°$ $\beta = 72°$ T_{53}: $\alpha = 35°$ $\beta = 88°$ T_{40}: $\alpha = 44°$ $\beta = 72°$	Theory: 0.142 (Michell 1893) Observed: T_{74}: 0.088 T_{53}: 0.093 T_{40}: 0.120		
Periodic Progressive (Shealing over uniform slope) 9°	Theory $\leq 90°$ Observed: T_{73} : 81° T_{40} : 85°	Theory: vertical front Observed: T_{73} : $\alpha = 0$ $\beta = 81$ T_{40} : $\alpha = 5$ $\beta = 80$		(H_b/d_b) Theory: ? Observed: T_{73} : 0.74 T_{40} : 0.80	T_{73} = 0.73 ft. T_{40} = 0.60 ft.
Solitary (constant depth)	Theory: 120° Observed: T_{73} = 129° T_{74} = 113°	Theory: $\alpha = \beta = 60°$ Observed: T_{73} : $\alpha = 63°$ β 66° T_{40} : $\alpha = 54°$ $\beta = 59°$		(H/d) Theory: $\simeq$ .78 McCowan 1894 Observed: T_{73} = 0.73 T_{40} = 0.80	

Wave	Limit Crest Angle	Symmetry	Limit Steepness H/L	Breaker Height H_b/d_b (slope) or Limit Height H/d (flat)	Breaker Distance (x_b) from shoreline (actual scale)
Solitary (uniform slope)	6° Theory ? Observed: $T_{73} = 90°$ $T_{40} = 90°$ front verticle back horizontal	Theory ? Observed: T_{73} $\alpha = 90°$ $\beta = 0$ T_{40} $\alpha = 90°$ $\beta = 0$		(H_b/d_b) Theory ? Observed: $T_{73} = 4.28$ $T_{40} = 3.78$	$T_{73} = 0.16$ ft. $T_{40} = 0.30$ ft. averages over 18 runs each.
	2°30' Observed: $T_{74} = 90°$ $T_{39} = 90°$	Observed: T_{74} $\alpha = 90°$ $\beta = 0$ T_{40} $\alpha = 90°$ $\beta = 0$		(H_b/d_b) Observed: $T_{74} = 1.55$ $T_{39} = 1.67$	$T_{74} = 1.55$ ft. $T_{39} = 1.67$ ft. averages over 18 runs each.
Solitary Reforming Wave (constant depth)	Theory ? Observed: $T_{73} = 149°$ $T_{40} = 140°$	Theory ? Observed: T_{73} : $\alpha = 77°$ $\beta = 74°$ T_{40} : $\alpha = 72°$ $\beta = 68°$		(H/d) Theory ? Observed: $T_{73} = 0.65$ $T_{40} = 0.65$	
Standing Wave	Theory: 90° Observed: $T_{74} = 93°$ $T_{35} = 80°$	Theory: $\alpha = \beta = 45°$ Observed: T_{74}: $\alpha = 46°$ $\beta = 47°$ T_{35}: $\alpha = 41°$ $\beta = 39°$	Theory 0.141 (Penney and Price 1952) Observed: $T_{74} = 0.122$ (0.125, Taylor 1953) $T_{35} = 0.131$		

Wave	Limit Crest Angle	Symmetry	Limit Steepness H/L	Breaker Height H_b/d_b (slope) or Limit Height H/d (flat)	Breaker Distance (x_b) from shoreline. (actual size)
"clapotis"	Theory: 90° Observed: $T_{74} = 101°$ $T_{55} = 99°$ $T_{40} = 80°$	Theory: ? Observed: T_{74} $\alpha = 47°$ $\beta = 54°$ T_{55} $\alpha = 57°$ β 42° T_{40} $\alpha = 37°$ β 43°			
Very short Crested Superposition of solitary progressing opp. direction to wave train	Theory ? Observed: $T_{73} = 99°$ $T_{40} = 66°$				
Undular Bore; lead undulation "constant depth"	Theory ? Observed: $T_{73} = 130°$ $T_{35} = 111°$	Theory ? Observed: T_{73} $\alpha = 62°$ $\beta = 68°$ T_{35} $\alpha = 43°$ $\beta = 68°$		(H/d) $T_{73} = 0.78$ $T_{35} = 0.85$	

All "Observed" values are averages of three lowest measurements, except where noted otherwise.

Figure 5. Solitary wave breaking sequence on 6° slope. Images at 1/64 sec. from motion picture camera study.



When surface tension is reduced, the limit height is significantly greater for both solitary wave and lead undulation of undular bore. The "reforming" solitary wave, as should be expected, has lower height that the limit height noted just before breaking.

6) Breaker distance from shoreline

In the case of the periodic progressive wave, the initial breaking occured consistently closer to the shore when surface tension was reduced. On the other hand, the solitary wave initial breaking was consistently further from the shore for surface tension reduced, the magnitude being greater for lower slope, as would be expected. As in the case of breaker height, it is possible that the presence of backwash in the case of the periodic waves accounts for the differences noted.

Ippen and Kulin 1954 show a series of graphs of breaking amplitude to breaking depth ratio. Figure 4 shows a similar graph for data acquired during the present study. Although the bottom slope is steeper than the steepest given by Ippen and Kulin, the same general trend is noted in the data points. However, for a second quite distinct curve results, for data points obtained when surface tension is reduced! It would appear that for a given initial amplitude, the breaking amplitude to breaking depth ratio is significantly greater.

Variation in surface tension in natural waters.

In view of the results described previously for surface tension adjusted under experimental conditions, it is of interest to consider the variation which may be observed in natural waters. A series of samples were taken over a period of several years. The procedure involved taking the water sample in carefully cleaned containers at a standard depth of 30 cm. below the surface. In this manner, surface films which can yield considerable variation in surface tension (i.e. Jarvis 1967, Lumby and Folkard 1956), were excluded from the sample. Measurements were made on a ring tensiometer in the same manner as for the laboratory experiments. The results, intended only to give some indication of natural variation, are given in table 3. A rather surprising range of variation was noted, particularly in the vicinity of organic concentrations.

Conclusions

The most useful conclusion that can be drawn from this study is that surface tension should not be neglected as a significant factor in the breaking process. It seems reasonable to suggest that it should be included in any realistic theoretical study.

In view of the relatively small size of the experimental waves, some consideration of scaling is needed, perhaps similar to the studies by Diephuis, 1957, and Maxwell, Hall, and Weggel, 1969 on related topics. It is evident that surface tension is a significant factor when breaking waves are included in engineering design scale models. Some extension of the present study should be made, including consideration of Froude and Weber numbers.

Another interesting possibility is an extension by varying surface tension, of the experiments of Cox 1958 with attention focused on the parasitic capillary waves of Longuet-Higgens 1963. A study of bursting pressures (Figure 3) would also be of interest.

Acknowledgements.

Robert V. White, Associate Engineer, contributed importantly to all phases of the experimental program. Michael Penney and A. Hanson carried out the surface tension measurements. The study was supported by the Office of Naval Research, Geography Programs, N000 14-67-A-0285-0013, N. R. 388-074.

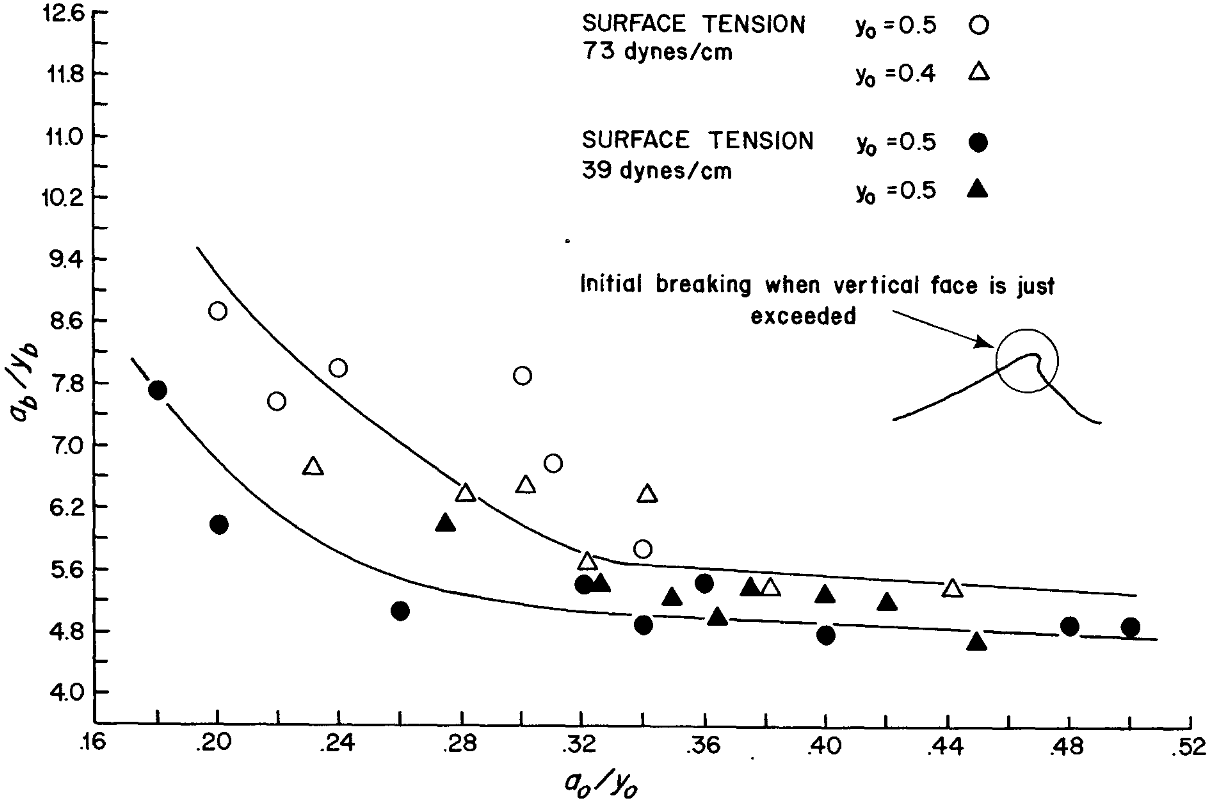


Figure 4 Solitary Wave 6° Slope: Initial Amplitude/S.W.L. vs. Breaking/Depth

Table 3 Variation in surface tension in natural waters.

Location	Description	Deviation from standard at given temperature.[1]
Atlantic Ocean		
Cape Cod, Mass.		
Great Harbor	Max. tide at 2.6 knots	+ 0.95 dynes/cm
Woods Hole Channel	rapid choppy flow	+ 0.25
Vineyard Sound	open ocean	-13.15
Great Harbor	WHOI dock	-15.15
	MBL club beach	-14.25
	Marsh creek mouth	- 9.05
	Sewage outfall	-14.65
Buzzards Bay	slick off Quisset	+ 0.35
	dumping grounds	-14.05
	mouth, Wild Harbor	- 3.65
Pacific Ocean		
Tomales Bay, Calif.		
open ocean	well mixed surf	+ 1.0
Great Rock	rapid channel flow	-16.1
Lawson's Pier	" " "	-15.9
Walker Creek		-12.9
Nicks Cove	Sewer outfall	-11.4
Millerton Creek	Brackish water	+ 0.1
Chicago		
Garfield Park	Fresh water	- 0.1
Lake Michigan	Near-shore	- 0.2

[1] Standard surface tension for seawater at S $^{0}/_{oo}$ = 30 and for fresh water at S $^{0}/_{oo}$ = 0, from Riley and Skirrow 1965, Deviation from standard value at temperature at which measurement was made.

References

Caldwell, J. M. 1949, Reflection of solitary waves. U.S. Army Corps. of Engineers, Beach Erosion Board Tech. Memo. No. 11.

Cox, D. S., 1958, Measurements of slopes of high-frequency waves. Jour. of Mar. Research. V 16, pp. 199-225.

Diephuis, J.G.H.R., 1957, Scale effects involving the breaking of waves. Proceedings 6th Conf. Coastal Engineering Florida Publ. by Council for Wave Res. Berkeley, California.

Galvin, C. J. 1968, Breaker type classification. Jour. Geophys. Res. V. 73:12, pp. 3651 ff.

Ippen, A. T. and G. Kulin, 1954, The shoaling and breaking of the solitary wave. Proceedings 5th Conf. Coastal Eng. Grenoble, France. Publ. by Council for Wave Res. Univ. Calif., Berkeley, California.

Jarvis, N. L. 1967, Adsorption of the surface active material at the sea-air interface. Limnology and Oceanography V12:2, pp. 213-221.

Kinsman, B., 1965, Wind Waves, Prentice-Hall, Inc. Englewood Cliffs, New Jersey, p. 273.

Longuet-Higgens, M. S., 1963, The generation of capillary waves by steep gravity waves. J. Fluid. Mech. 16, pp. 138-159.

Lumby and Folkard 1956 (quoted in Riley and Skirrow 1965 ibid, p. 117.)

Mason, M. A. 1951, Some observations of breaking waves. Paper 24, Gravity Waves, Circ. 521 Nat Bur. Standards, Washington, D. C.

McCowan, J., 1891, On the solitary wave. London, Edinburgh, Dublin, Phil. Mag. J. Sci. 32:5, pp. 45-58.

Michell, J. H., 1893, On the highest waves in water, Phil. Mag. Ser. 5, V 36, pp. 430-437.

Miller, R. L. and J. M. Ziegler, 1964. Internal velocity field in breaking waves. Proceed. 9th Conf. on Coastal Engineering, Lisbon, Portugal. Publ. by Council for Wave Res. Univ. of Calif. Berkeley, California.

Miller, R. L., 1968a, Experimental determination of run-up of Undular and fully developed bores. Jour. Geophys. Res. V73:14, pp. 4497-4510.

Miller, R. L. 1968b, The effect of surface tension on the limit form of waves. Trans. Am. Geophys. Union, V 49, pp. 209.

Penney, W. G. and A. T. Price, 1952. Finite periodic stationary gravity waves in a perfect liquid. Phil. Soc. Roy, Soc. London, Ser. V. 244, pp. 254-284.

Riley, J. Paul and G. Skirrow, 1965 (Editors) Chemical Oceanography (Table XIV p. 117). V. 1, Academic Press, New York.

Stokes, G. G., 1880, On the theory of oscillatory waves,(1847) Reprinted in Math. and Physical Papers V. 1, Cambridge, University Press.

Taylor, G. I., 1953, An experimental study of standing waves. Proc. Roy. Soc. London, Ser. A:218, pp. 44-59.

Witting, J. M. 1964, Dissipation Mechanisms in weak shock waves in collisionless plasmas. Ph.D. Dissertation Dept. Physics, Mass. Inst. of Technology.

Stoker, J. J. 1948, The formation of breakers and bores. Communications, Pure and Applied Math. Vol. 1, pp. 1-87.