CHAPTER 18

SHOALING OF CNOIDAL WAVES

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ABSTRACT

An equation is derived which governs the propagation of a cnoidal wave train over a gently sloping bottom. The equation is solved numerically, the solution being tabulated in terms of f_H (Eq. 47) as a function of $E_1 = (E_{tr}/\rho g)^{1/3}/gT^2$ and $h_1 = h/gT^2$. Results are compared with sinusoidal wave theory. Two numerical examples are included.

0. INTRODUCTION

Although discovered almost 80 years ago and having received increasing attention during the last 10 years cnoidal waves have still not achieved the position of a tool for engineers, which one would expect from the demand for a consistent and reasonably accurate long wave theory.

One of the reasons is of course their complexity. Unlike the sinusoidal wave theory the velocity of propagation and thus the wave length depends on the wave height already in the lowest order of approximation and further the shape of the surface profile is characterised by the much more complicated elliptic function cn.

Another reason may be related to the fact that it has not hitherto been possible to determine how the main parameters such as wave height and wave length change if the wave propagates over a slowly varying bottom.

The main object of this paper is to analyse this transformation process which for sinusoidal waves is usually termed "shoaling". It is throughout assumed that the propagation is without refraction.

It is well known, see e.g. [3], that on a horizontal bottom the first approximation of the cnoidal wave theory predicts the speed of wave propagation c as

$$c = (gh)^{1/2}\left\{1 + H/mh\ \left(2 - m - 3\ E(m)/K(m)\right)\right\}^{1/2} \qquad (1)$$

where h is water depth, H wave height, E and K are complete elliptic integrals with m as parameter. m satisfies the equation

$$H\ L^2/h^3 = 16/3\ m\ K(m)^2 \tag{2}$$

where L is the wave length.

Let us consider two-dimensional wave trains. The concept of shoaling is based on the assumption that for sufficiently gentle variations of the bottom, the reflexion is negligible and the local value of c is given by Eq. 1 with h the local depth.

To define the local cnoidal wave uniquely under such conditions two magnitudes have to be specified in addition to the depth. Therefore, to follow a wave as it propagates over the gently sloping bottom we must know the variation - or constancy - of two magnitudes. These are the wave period T and the energy transport E_{tr}, which together do define the wave. Both these magnitudes stay constant during the propagation. Thus

$$L = c\ T \tag{3}$$

defines a wave length L locally in the same sense as c, and Eq. 2 yields a local value of m.

1. ENERGY TRANSPORT

To carry on the development the energy transport in a cnoidal wave must be expressed in terms of the main parameters H and L (or T). We first determine the energy flux through a control section A, i.e. a section fixed in the x-direction (Fig. 1).

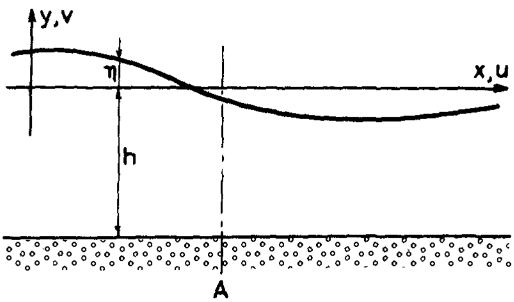

Fig. 1 Definition sketch

Using the definitions in the figure, the energy flux through A is

$$E_f = \int_{-h}^{\eta} \rho\, u \{ g\, y + p/\rho + \tfrac{1}{2}(u^2 + v^2) \}\, dy \tag{4}$$

where p is the pressure.

By introduction of the excess pressure p^+ defined as

$$p^+ \equiv p + \rho\, g\, y \tag{5}$$

Eq. 4 can be rewritten

$$E_f = \int_{-h}^{\eta} \rho\, u \{ p^+/\rho + \tfrac{1}{2}(u^2 + v^2) \}\, dy \qquad (6)$$

In cnoidal waves the first approximation to the horizontal velocity u is

$$u = c\ \eta/h \qquad (7)$$

The corresponding expression for p^+ is

$$p^+ = \rho\, g\, \eta \qquad (8)$$

(see e.g. [5]) and η is given by

$$\eta = H \left\{ \frac{1}{m}\left(1 - E(m)/K(m)\right) - 1 + \mathrm{cn}^2 \left(2\,K\,(m) \left(\frac{t}{T} - \frac{x}{L}\right) \middle| m\right)\right\} \qquad (9)$$

where cn is a Jacobian elliptic function, m its parameter.

Since we are only interested in the leading term in the expression for E_f, the relative magnitude of the terms in Eq. 9 must be considered.

The theory of cnoidal waves is based on the assumption that the dimensionless Ursell-parameter UR founded on a characteristic horizontal length λ is

$$UR \equiv H\,\lambda^2/h^3 = 0(1) \qquad (10)$$

From this we obtain, by introducing the small parameter $\varepsilon \equiv h/\lambda$ that

$$\eta/h = 0(H/h) = 0(h^2/\lambda^2) = 0(\varepsilon^2) \qquad (11)$$

so that Eq. 7 yields

$$\frac{u}{c} = 0(\varepsilon^2) \qquad (12)$$

Since in long waves v << u we need only consider the first two terms in Eq. 6. By Eqs. 7 and 8 we have

$$\frac{\rho u^2}{2p^+} = \frac{c^2 \eta}{2gh^2} \qquad (13)$$

and as $c^2 = 0(gh)$ we get

$$\frac{\rho u^2}{p^+} = 0\left(\frac{\eta}{h}\right) = 0(\varepsilon^2) \qquad (14)$$

In other words, we may write

$$E_f = \int_{-h}^{0} u\ p^+\, dy + 0(\varepsilon^2 E_f) = \rho\, g\, \eta^2\, c \qquad (15)$$

since also $\int_0^{\eta}$ represents a small term.

The energy transport E_{tr} is defined in the same way as for Stokes waves, viz. as the transport per wave period

$$E_{tr} = \int_0^T E_f\ dt = \rho\, g\, c \int_0^T \eta^2\ dt \qquad (16)$$

Introducing for η Eq. 9 and noting that

$$\int_0^{2K} \mathrm{cn}^2\ \theta\ d\theta = (2/m)\ \ (E - (1 - m)K) \qquad (17)$$

$$\int_0^{2K} \mathrm{cn}^4\ \theta\ d\theta = (2K/3m^2)\ \ \left(3m^2 - 5m + 2 + (4m - 2)E/K\right) \qquad (18)$$

the expression for E_{tr} finally becomes

$$E_{tr} = \rho g H^2 L/m^2 \left\{ \frac{1}{3} \left(3m^2 - 5m + 2 + (4m-2)\frac{E}{K} \right) - \left(1 - m - \frac{E}{K} \right)^2 \right\} \tag{19}$$

(Eq. 17 may be obtained from e.g. [1] and Eq. 18 can be determined by suitable substitutions but is actually given in [4] directly.)

2. THE "SHOALING EQUATION"

Since the main parameters of the wave at each depth must satisfy Eqs. 1, 2 and 3 with h as local depth and since the wave period T and energy transport E_{tr} stay constant during the propagation, the four equations governing the shoaling process can be written

$$\frac{c^2}{gh} = 1 + \frac{H}{h} A \tag{20}$$

$$U = 16/3 \, m K^2 \tag{21}$$

$$L = c \, T \tag{22}$$

$$E_{tr}/\rho g = H_r^2 L_r B_r = H^2 L B \tag{23}$$

where we for convenience have introduced the following definitions

$$A = A(m) \equiv 2/m - 1 - 3E/(mK) \tag{24}$$

$$U = U(m) \equiv H L^2/h^3 \tag{25}$$

$$B = B(m) = \frac{1}{m^2} \left\{ \frac{1}{3} \left(3m^2 - 5m + 2 + (4m-2)\frac{E}{K} \right) - \left(1 - m - \frac{E}{K} \right)^2 \right\} \tag{26}$$

The variation of A is shown in Fig. 2.. The variation of B is shown in Fig. 3. Numerical results for both A and B are given in Table 1. We see that B is a dimensionless measure for the energy transport. In the formulas index $_r$ means "reference", indicating values corresponding to the water depth h_r where the wave is initially specified.

This is a system of four simultaneous transcendent equations. The four unknowns are H, L, c, and m, all values corresponding to the water depth h, and to solve these equations means finding these wave data at the water depth h provided T and E_{tr} are specified in some way by data at h_r (say H_r and L_r or H_r and T).

However, it is obvious that only one parameter, i.e. m, appears in transcendental form in the equations, namely as the independent variable in the elliptic functions E and K.

Thus in principle it is possible to reduce the four equations to one transcendental equation in m by eliminating the other three unknowns. As the manipulations are fairly trivial the result is presented directly as the following equation

$$M f_1(m) + N f_2(m) = 1 \tag{27}$$

where

$$M \equiv (H_r^2 L_r B_r)^{2/3}$$

$$N \equiv M^{-1} h/gT^2$$

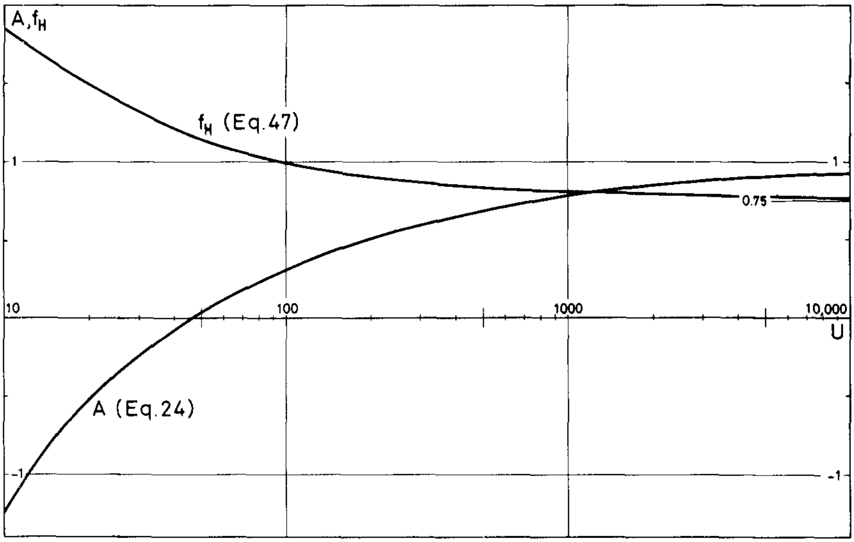

Fig. 2 A and f_H versus U

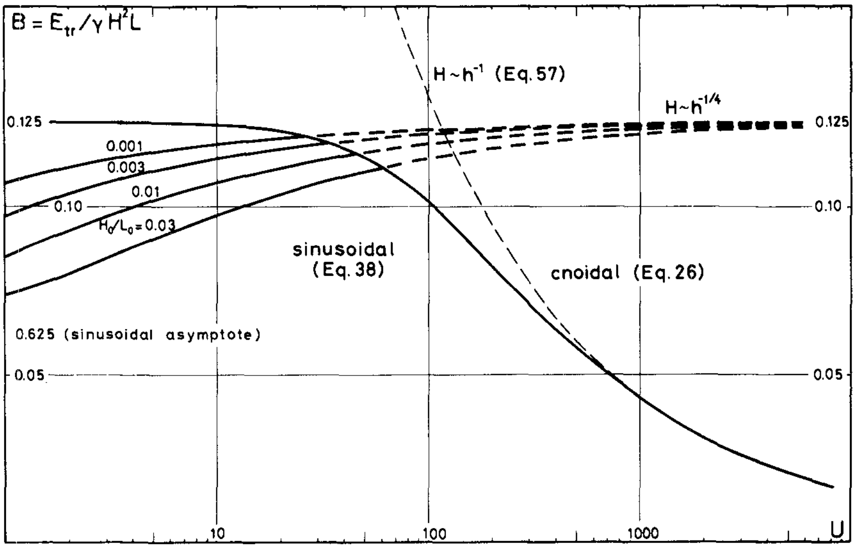

Fig. 3 B versus U

$$f_1 \equiv - U^{-1/3} B^{-2/3} A \tag{30}$$

$$f_2 \equiv U^{4/3} B^{2/3} \tag{31}$$

A closer investigation of Eq. 21 would reveal the (well-known) fact that U is a monotonous function of m. In the following we will choose to consider the problem in terms of U as independent variable. By this change we bring in the main parameters H, L and h for the wave directly and at the same time avoid the inconveniencies caused by the singularity in the K(m)-function for m = 1. Hence we may write

$$\boxed{M f_1(U) + N f_2(U) = 1} \tag{32}$$

where in Eqs. 30 and 31 A and B are to be regarded as functions of U.

Eq. 32 may be called the "Shoaling Equation". From the solution U of this equation H, c and L can be determined by the original equations 20 - 23.

3. SOLUTION OF THE SHOALING EQUATION

As mentioned above the wave may be specified at water depth h by, say, H and L. Thus three lengths are necessary to define the problem. From this we conclude that the problem has two (independent) dimensionless parameters, a result verified by Eq. 32 which requires the two parameters M and N to be defined.

The solution of Eq. 32 could of course be presented as a function of M and N. However, it proves more useful to note that M and N can be expressed as

$$M = E_1^2 h_1^{-2} \tag{33}$$

$$N = E_1^{-2} h_1^3 \tag{34}$$

where

$$E_1 = \frac{(H_r^2 L_r B_r)^{1/3}}{g T^2} = \frac{(E_{tr}/\rho g)^{1/3}}{g T^2} \tag{35}$$

$$h_1 = \frac{h}{g T^2} \tag{36}$$

In the following E_1 and h_1 will be used as parameters. This has the advantage - among others - that for a wave of a given period and height at a certain depth E_1 is independent of h, i.e. the shoaling process is represented by a variation of h_1 only. Variations in E_1 represent changes in the initial wave data.

The shoaling equation has been solved numerically on a digital computer. It appears that the equation has two roots. One of them, however, is readily shown to be false as it does not assume the value U_r when h is h_r. Hence if this argument of "continuity" is included there is one root U for each set of parameters E_1 and h_1 in the cnoidal region. For sufficiently large values of h_1 and small values of E_1 (corresponding to the sinusoidal region) the negative values of A dominates and the shoaling equation has no roots. The variation of U is shown in Fig. 4.

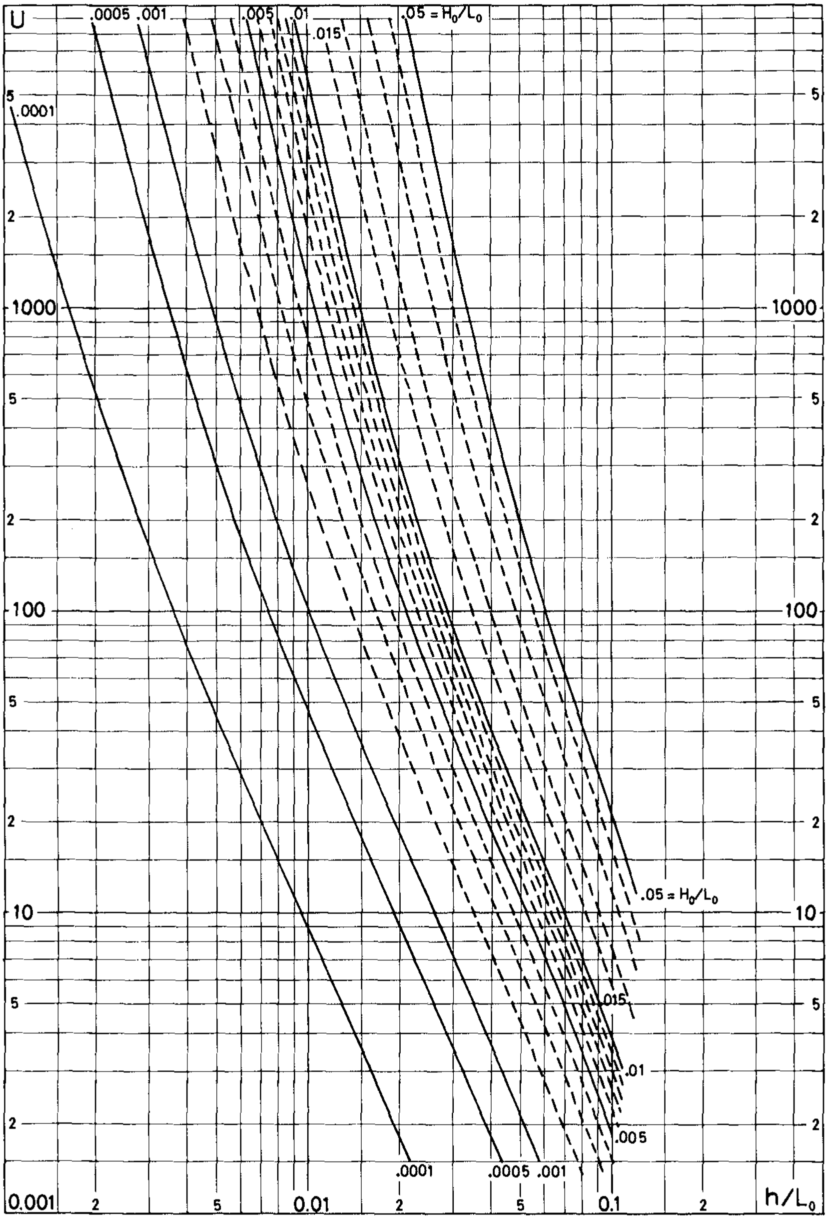

Fig. 4 Solution of shoaling equation

4. RELATION TO SINUSOIDAL WAVE THEORY

The basic equations 20 - 23 show that the only way in which the initial wave enters the data is through the wave period in Eq. 22 and the specification of the energy transport in Eq. 23.

This means that in the case, where the initial wave is specified at a depth so great that sinusoidal wave theory must be applied instead of cnoidal, we instead of Eqs. 23 and 26 have to determine the energy transport from

$$E_{tr}/\rho g = H_r^2 L_r B_r \tag{37}$$

where we for B_r use the well-known expression

$$B_r = \frac{1}{16}\left(1 + \frac{2k_r h_r}{\sinh 2k_r h_r}\right) \tag{38}$$

$k_r = 2\pi/L_r$ being the wave number.

This result is of particular interest because it makes it possible at any water depth to compare the results of the two theories for a wave which is characterised by data at any other water depth including deep water. For a closer discussion on this point the reader is referred to section 7.

Fig. 3 shows the variation of B according to Eq. 38. Since the abscissa U does not specify the kh parameter uniquely the value of B in this plot depends on another parameter too, which is for convenience chosen as the deep water steepness H_o/L_o. Comparison between the curves representing Eqs. 26 (cnoidal theory) and 38 (sinusoidal theory) shows that the amount of energy which according to the two theories is transported by a wave of a certain height and length varies in a very different way. Thus we can expect considerable differences in the wave heights and wave lengths predicted for the same period and energy transport.

Another important aspect appears by noting that the deep water wave length is

$$L_o = gT^2/2\pi \tag{39}$$

Hence the parameter h_1 may be rewritten

$$h_1 = \frac{1}{2\pi}\frac{h}{L_o} \tag{40}$$

and E_1 as

$$E_1 = (H_r^2 L_r B_r)^{1/3}/2\pi L_o \tag{41}$$

which by virtue of Eqs. 23 and 38 becomes

$$E_1 = (H_o^2 L_o B_o)^{1/3}/2\pi L_o = (2\pi \sqrt[3]{16})^{-1} (H_o/L_o)^{2/3} \quad \left(= 0.0632\, (H_o/L_o)^{2/3}\right) \tag{42}$$

In other words, the parameters E_1 and h_1 are proportional to h/L_o and $(H_o/L_o)^{2/3}$, index $_o$ referring to deep water values.

TABLE 1

U	A	B	f_H
2	-6.565	0.124	3.175
4	-3.261	0.124	2.521
6	-2.150	0.124	2.203
8	-1.588	0.124	2.003
10	-1.245	0.124	1.862
12	-1.012	0.124	1.754
14	-0.842	0.123	1.669
16	-0.711	0.123	1.599
18	-0.607	0.123	1.540
20	-0.521	0.122	1.490
22	-0.449	0.122	1.447
24	-0.387	0.122	1.408
26	-0.333	0.121	1.375
28	-0.285	0.121	1.345
30	-0.242	0.120	1.318
32	-0.204	0.120	1.294
34	-0.169	0.119	1.272
36	-0.137	0.118	1.251
38	-0.108	0.118	1.233
40	-0.081	0.117	1.216
42	-0.056	0.117	1.200
44	-0.033	0.116	1.186
46	-0.011	0.116	1.172
48	0.009	0.115	1.160

U	A	B	f_H
50	0.028	0.114	1.148
55	0.072	0.113	1.122
60	0.110	0.111	1.099
65	0.144	0.110	1.080
70	0.175	0.108	1.063
75	0.202	0.107	1.048
80	0.227	0.106	1.035
85	0.250	0.104	1.023
90	0.271	0.103	1.012
95	0.290	0.102	1.003
100	0.308	0.100	0.994
125	0.380	0.095	0.959
150	0.434	0.090	0.935
175	0.476	0.085	0.918
200	0.510	0.082	0.904
225	0.538	0.078	0.893
250	0.561	0.076	0.884
275	0.582	0.073	0.876
300	0.600	0.071	0.870
325	0.615	0.068	0.864
350	0.629	0.067	0.859
375	0.642	0.065	0.855
400	0.653	0.063	0.851

U	A	B	f_H
425	0.663	0.062	0.847
450	0.673	0.060	0.844
475	0.682	0.059	0.841
500	0.690	0.058	0.839
550	0.704	0.055	0.834
600	0.717	0.053	0.830
650	0.728	0.052	0.826
700	0.738	0.050	0.823
750	0.747	0.049	0.820
800	0.755	0.047	0.818
850	0.762	0.046	0.815
900	0.769	0.045	0.813
950	0.775	0.044	0.812
1000	0.780	0.043	0.810
2000	0.845	0.031	0.791
3000	0.873	0.026	0.783
4000	0.890	0.023	0.778
5000	0.902	0.020	0.775
6000	0.910	0.018	0.773
7000	0.917	0.017	0.771
8000	0.922	0.016	0.770
9000	0.926	0.015	0.768
10000	0.930	0.014	0.767

5. DETERMINATION OF WAVE HEIGHT H

When the Shoaling Equation has been solved H can be found in several ways.

The most natural way is to consider the expression Eq. 23 for the energy transport and eliminate L by means of Eq. 25. We have

$$(L/h)^2 = U\,(H/h)^{-1} \tag{43}$$

Eq. 23 can be rewritten

$$\frac{H}{h} = \frac{H_r}{h}\left(\frac{L_r}{h}\right)^{1/2}\left(\frac{L}{h}\right)^{-1/2}\left(\frac{B}{B_r}\right)^{-1/2} \tag{44}$$

which after introduction of Eq. 43 becomes

$$\frac{H}{h} = \left(\frac{H_r}{h}\right)^{4/3}\left(\frac{L_r}{h}\right)^{2/3}\left(\frac{B}{B_r}\right)^{-2/3} U^{-1/3} \tag{45}$$

or

$$\frac{H}{H_r} = \frac{h_r}{h}\,\frac{f_H\,(U)}{f_H\,(U_r)} \tag{46}$$

where we have defined

$$f_H\,(U) \equiv U^{-1/3}\,B^{-2/3} \tag{47}$$

and

$$U_r \equiv H_r L_r^2 / h_r^3 \tag{48}$$

The relation between f_H and U is given in Table 1 and Fig. 2. However, since f_H is a monotonous function of U, f_H has been tabulated directly as a function of E_1 and h_1 in Table 2, which will facilitate the practical applications. Thus we can either determine U from Fig. 4 and f_H from Table 1 or f_H directly from Table 2.

If the reference depth h_r where we initially have data for the wave train is infinite we must consider $h_o/f_H(U_o)$ instead of $f_H(U_o)$. We see that

$$h_o/f_H(U_o) = H_o^{1/3} L_o^{2/3} B_o^{2/3} \tag{49}$$

so that Eq. 46 becomes (B_o being 1/16 according to Eq. 38)

$$\frac{H}{H_o} = 0.157 \left(\frac{H_o}{L_o}\right)^{1/3} \left(\frac{h}{L_o}\right)^{-1} f_H \tag{50}$$

Eqs. 46 and 50 are of course only valid for a wave height H in the cnoidal region.

H can also be determined by eliminating L and c from Eqs. 20, 22 and 25.

Rewriting Eq. 20 as

$$\frac{c^2}{g h} = \frac{L^2}{g T^2 h} = \frac{h^2}{g T^2} \frac{U}{H} = 1 + \frac{H}{h} A \tag{51}$$

we see that when U and T are known and hence A, Eq. 51 represents an equation for H/h which can be rearranged into the following

$$A \left(\frac{H}{h}\right)^2 + \frac{H}{h} - \frac{h U}{g T^2} = 0 \tag{52}$$

From this we obtain

$$\frac{H}{h} = \left((1 + 4 A h U/gT^2)^{1/2} - 1\right)/2 A \tag{53}$$

However, for the true cnoidal wave H/h << 1 the first term in Eq. 52 will be small, so that the exact calculation of H/h from Eq. 53 will be represented numerically by a small difference between two almost equal numbers. Thus it can only be recommended using Eq. 53 if $4 A h U/g T^2$ is larger than, say, unity.

For $4 A h U/g T^2 \leqq 1$ Eq. 52 can be solved by iteration. We write

$$\frac{H}{h} = \frac{h U}{g T^2} - A \left(\frac{H}{h}\right)^2 \tag{54}$$

and start with the approximation $H/h \sim h U/g T^2$ which is used in the last term on the right hand side etc.

The asymptotic behaviour of H for vanishing h can be derived from Eq. 46. If H/h is assumed small enough to avoid breaking, decreasing h means rapidly increasing U, whereby the elliptic parameter m approaches unity. Hence Eq. 26 yields

$$B \rightarrow \frac{2}{3 K} \qquad \text{for} \quad U \rightarrow \infty \tag{55}$$

which substituted into Eq. 47 together with U from Eq. 21 yields

$$f_H(U) \rightarrow \left(\frac{16 K^2}{3}\right)^{-1/3} \left(\frac{2}{3 K}\right)^{-2/3} = \frac{3}{4} \qquad \text{for } U \rightarrow \infty \tag{56}$$

Thus for large values of U, f_H approaches the fixed value 3/4. Substituting into Eq. 46 we get

$$\frac{H}{H_o} = \text{const}\ h^{-1} \qquad \text{for} \quad U \to \infty \tag{57}$$

This variation is illlustrated by the dotted curve in Fig. 3. It shows that the wave height in very shallow water increases considerably faster with decreasing h than the variation as $h^{-1/4}$ predicted by the classical linear long wave theory (see e.g. [6] § 185). For large values of U each of the crests in the cnoidal wave train more and more resembles a solitary wave in profile. Thus for large U the wave will show all the features of the solitary wave. It is therefore relevant to recall that the same result was obtained directly for a solitary wave by Grimshaw [2].

Another of the solitary wave characteristics can be demonstrated if we consider the energy transport. Substitute Eq. 55 for B and $\frac{3}{16} H L^2/h^3$ for K (from Eqs. 21 and 25) into Eq. 23. We then get

$$E_{tr}/\rho g = \frac{2}{3} H^{3/2} h^{3/2} \tag{58}$$

which shows that the energy transport does not depend on the wave length L.

Although in most practical cases this limit corresponds to H/h values which are far in excess of any breaking point the tendency that the wave height grows faster than $h^{-1/4}$ is felt even at moderate values of U (section 7).

6. DETERMINATION OF WAVE LENGTH L

The wave length is determined from Eqs. 20 and 22. Once f_H or U are known the value of A can be obtained from Table 1 so that Eq. 20 gives c when H has been calculated, and then L is readily determined from Eq. 22.

Another method which is less accurate consists in using Eq. 25. When U and H/h are known this equation gives a value for L/h. However, as in many cases U does not have to be determined very accurately for the purpose of getting accurate results for H by means of f_H, this method cannot be recommended except as a guidance.

If T is known the following method can also be used.

Initial Wave Data Supplied as H_r and T

A problem related to the determination of L arises if the initial wave data are in terms of wave height and wave period. This will for example usually be the case when they originate from a wave recorder. In the cnoidal wave theory L_r cannot be determined explicitly from H_r and T, and neither can U_r, A_r or B_r.

For this case Table 3 gives values of L/h as a function of H/h and $T\sqrt{g/h}$. The values in the table represent the solution of Eq. 52 with respect to U and determination of L/h by Eqs. 20 and 22.

TABLE 2(a)

E_1 \ h_1	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.010	0.011	0.012	0.013	0.014	0.015
.0002	1.168	1.908	2.591	3.226	3.826										
4	0.929	1.363	1.835	2.283	2.708	3.118	3.514	3.901							
6	0.856	1.140	1.505	1.866	2.212	2.546	2.871	3.188	3.499	3.806					
8	0.822	1.023	1.313	1.619	1.917	2.206	2.487	2.761	3.031	3.298	3.562	3.825			
.0010	0.802	0.954	1.188	1.452	1.716	1.973	2.224	2.470	2.711	2.950	3.187	3.423	3.660	3.899	
12	0.789	0.909	1.101	1.331	1.569	1.802	2.031	2.255	2.475	2.693	2.910	3.126	3.342	3.561	3.790
14	0.780	0.879	1.038	1.240	1.455	1.670	1.881	2.088	2.292	2.493	2.694	2.894	3.095	3.297	3.504
16	0.774	0.856	0.990	1.168	1.364	1.563	1.759	1.953	2.144	2.332	2.520	2.707	2.895	3.085	3.278
18	0.770	0.839	0.954	1.111	1.290	1.475	1.659	1.841	2.021	2.199	2.375	2.552	2.729	2.908	3.090
.0020		0.826	0.926	1.065	1.229	1.401	1.575	1.747	1.917	2.086	2.253	2.421	2.589	2.759	2.932
22		0.815	0.903	1.027	1.177	1.338	1.502	1.666	1.828	1.988	2.148	2.308	2.468	2.630	2.795
24		0.807	0.885	0.995	1.133	1.284	1.439	1.595	1.750	1.903	2.056	2.209	2.363	2.518	2.676
26		0.800	0.870	0.969	1.095	1.236	1.384	1.533	1.681	1.829	1.975	2.122	2.270	2.419	2.570
28		0.794	0.857	0.946	1.062	1.195	1.335	1.477	1.620	1.762	1.903	2.045	2.187	2.330	2.477
.0030		0.789	0.846	0.927	1.033	1.158	1.291	1.428	1.565	1.702	1.838	1.975	2.112	2.251	2.392
32		0.785	0.836	0.911	1.008	1.125	1.252	1.383	1.515	1.647	1.779	1.912	2.044	2.179	2.315
34		0.781	0.828	0.897	0.987	1.096	1.216	1.342	1.470	1.598	1.726	1.854	1.983	2.113	2.246
36		0.778	0.821	0.884	0.967	1.069	1.184	1.305	1.428	1.552	1.676	1.801	1.926	2.053	2.182
38		0.776	0.815	0.873	0.950	1.046	1.155	1.271	1.390	1.510	1.631	1.752	1.874	1.997	2.123
.0040		0.773	0.810	0.863	0.935	1.025	1.128	1.240	1.355	1.472	1.589	1.707	1.826	1.946	2.068
45		0.769	0.799	0.843	0.904	0.980	1.071	1.172	1.278	1.387	1.497	1.607	1.719	1.832	1.948
50			0.790	0.828	0.880	0.945	1.025	1.115	1.213	1.315	1.418	1.523	1.629	1.736	1.845
55			0.784	0.816	0.861	0.918	0.987	1.069	1.158	1.253	1.350	1.450	1.550	1.652	1.756
60			0.779	0.807	0.846	0.895	0.957	1.030	1.111	1.199	1.291	1.385	1.481	1.579	1.678
65			0.775	0.799	0.833	0.877	0.931	0.997	1.071	1.153	1.239	1.329	1.420	1.513	1.609
.0070			0.772	0.793	0.823	0.862	0.910	0.969	1.037	1.112	1.193	1.278	1.366	1.455	1.546
75			0.769	0.788	0.815	0.849	0.893	0.945	1.007	1.077	1.153	1.233	1.316	1.402	1.490
80				0.784	0.808	0.839	0.877	0.925	0.981	1.045	1.116	1.192	1.272	1.354	1.439
85				0.780	0.802	0.830	0.865	0.907	0.959	1.018	1.084	1.155	1.231	1.310	1.391
90				0.777	0.796	0.822	0.854	0.893	0.939	0.993	1.055	1.122	1.194	1.270	1.348
95				0.775	0.792	0.815	0.844	0.880	0.922	0.972	1.029	1.092	1.161	1.233	1.308
.0100				0.772	0.788	0.809	0.836	0.868	0.907	0.953	1.006	1.065	1.130	1.199	1.271
120					0.777	0.792	0.811	0.835	0.863	0.897	0.936	0.981	1.031	1.088	1.148
140					0.770	0.781	0.796	0.814	0.835	0.860	0.890	0.924	0.964	1.008	1.058
160						0.774	0.786	0.799	0.816	0.836	0.859	0.886	0.916	0.951	0.991
180						0.769	0.778	0.789	0.803	0.818	0.837	0.858	0.883	0.911	0.943
.0200							0.773	0.782	0.793	0.806	0.821	0.838	0.858	0.881	0.907
220							0.769	0.777	0.786	0.796	0.809	0.823	0.840	0.859	0.880
240								0.772	0.780	0.789	0.800	0.812	0.826	0.842	0.860
260								0.769	0.776	0.784	0.793	0.803	0.815	0.828	0.844
280									0.772	0.779	0.787	0.796	0.806	0.818	0.831
.0300									0.769	0.775	0.782	0.790	0.799	0.809	0.820
320										0.772	0.778	0.785	0.793	0.802	0.812
340										0.770	0.775	0.781	0.788	0.796	0.805
360											0.772	0.778	0.784	0.791	0.799
380											0.770	0.775	0.781	0.787	0.794
.0400											0.768	0.773	0.778	0.783	0.790
450												0.768	0.772	0.776	0.781
500														0.771	0.775
550															0.771
600															
650															
700															
750															
800															
850															
900															
950															
.1000							$f_H \to 0.750$								
.1200															

f_H as a function of E_1 and h_1

TABLE 2(b)

E_1 \ h_1	0.016	0.017	0.018	0.019	0.020	0.021	0.022	0.023	0.024	0.025	0.026	0.027	0.028	0.029	0.030
.0002															
4															
6															
8															
.0010															
12															
14	3.716	3.940													
16	3.477	3.687	3.919							sinusoidal region					
18	3.278	3.476	3.696							(no roots)					
.0020	3.112	3.298	3.507	3.820											
22	2.965	3.145	3.343	3.640											
24	2.839	3.011	3.201	3.486											
26	2.727	2.893	3.075	3.342											
28	2.627	2.787	2.962	3.220											
.0030	2.538	2.691	2.861	3.105											
32	2.457	2.605	2.770	3.003											
34	2.383	2.527	2.686	2.909											
36	2.315	2.455	2.610	2.823											
38	2.252	2.389	2.539	2.745											
.0040	2.194	2.327	2.473	2.672											
45	2.066	2.191	2.329	2.510											
50	1.958	2.076	2.205	2.372											
55	1.863	1.976	2.099	2.253											
60	1.781	1.888	2.005	2.148											
65	1.707	1.810	1.921	2.055											
.0070	1.641	1.739	1.845	1.971											
75	1.581	1.675	1.777	1.895											
80	1.526	1.617	1.715	1.826											
85	1.476	1.564	1.657	1.763											
90	1.429	1.514	1.604	1.705	1.863										
95	1.387	1.468	1.555	1.651	1.779										
.0100	1.347	1.426	1.509	1.601	1.716										
120	1.213	1.281	1.354	1.432	1.519										
140	1.112	1.170	1.232	1.299	1.372	1.456									
160	1.035	1.084	1.137	1.195	1.257	1.325	1.405								
180	0.979	1.019	1.063	1.112	1.165	1.222	1.287	1.363							
.0200	0.936	0.969	1.006	1.047	1.092	1.141	1.195	1.255	1.327						
220	0.904	0.932	0.962	0.996	1.034	1.076	1.121	1.172	1.229	1.297					
240	0.880	0.903	0.928	0.957	0.988	1.024	1.062	1.105	1.153	1.207	1.271				
260	0.861	0.880	0.902	0.926	0.952	0.982	1.015	1.052	1.092	1.137	1.188	1.249			
280	0.846	0.862	0.880	0.901	0.924	0.949	0.977	1.009	1.043	1.081	1.124	1.172	1.231		
.0300	0.833	0.847	0.863	0.881	0.901	0.922	0.947	0.973	1.003	1.036	1.072	1.113	1.159	1.214	
320	0.823	0.836	0.849	0.865	0.882	0.901	0.922	0.945	0.970	0.999	1.030	1.064	1.103	1.147	1.200
340	0.815	0.826	0.838	0.852	0.866	0.883	0.901	0.921	0.943	0.968	0.995	1.025	1.058	1.095	1.137
360	0.808	0.818	0.828	0.840	0.854	0.868	0.884	0.902	0.921	0.943	0.966	0.992	1.021	1.052	1.088
380	0.802	0.811	0.820	0.831	0.843	0.856	0.870	0.885	0.902	0.921	0.942	0.965	0.990	1.017	1.048
.0400	0.797	0.805	0.813	0.823	0.833	0.845	0.858	0.872	0.887	0.903	0.922	0.942	0.964	0.988	1.015
450	0.787	0.793	0.800	0.807	0.816	0.825	0.834	0.845	0.857	0.869	0.883	0.898	0.915	0.932	0.952
500	0.780	0.785	0.790	0.796	0.803	0.810	0.818	0.826	0.836	0.846	0.856	0.868	0.881	0.895	0.910
550	0.775	0.779	0.783	0.788	0.794	0.800	0.806	0.813	0.820	0.828	0.837	0.846	0.857	0.868	0.879
600	0.771	0.774	0.778	0.782	0.787	0.792	0.797	0.803	0.809	0.815	0.823	0.830	0.839	0.848	0.857
650		0.771	0.774	0.777	0.781	0.785	0.790	0.795	0.800	0.806	0.812	0.818	0.825	0.832	0.840
.700		0.768	0.771	0.774	0.777	0.780	0.784	0.788	0.793	0.798	0.803	0.808	0.814	0.821	0.827
750			0.768	0.771	0.773	0.776	0.780	0.783	0.787	0.791	0.796	0.801	0.806	0.811	0.817
800				0.768	0.771	0.773	0.776	0.779	0.783	0.786	0.790	0.794	0.799	0.803	0.808
850					0.768	0.771	0.773	0.776	0.779	0.782	0.786	0.789	0.793	0.797	0.802
900						0.768	0.771	0.773	0.776	0.779	0.782	0.785	0.788	0.792	0.796
950							0.769	0.771	0.773	0.776	0.778	0.781	0.784	0.788	0.791
.1000		$f_H \rightarrow 0.750$						0.769	0.771	0.773	0.776	0.778	0.781	0.784	0.787
.1200												0.769	0.771	0.773	0.775

f_H as a function of E_1 and h_1

TABLE 3

$T\sqrt{g/h}$ \ H/h	0.01	0.02	0.03	0.04	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.60	0.70	0.80
6.0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
7.0	-	-	-	-	-	-	-	-	-	-	-	-	-	-	5.8	6.1	6.4
8.0	-	-	-	-	-	-	-	-	6.9	6.9	7.0	7.1	7.2	7.3	7.6	7.8	8.1
9.0	-	-	-	-	-	-	-	8.1	8.2	8.3	8.4	8.5	8.6	8.7	9.0	9.3	9.7
10.0	-	-	-	-	-	-	9.2	9.3	9.4	9.5	9.6	9.7	9.9	10.0	10.4	10.7	11.1
11.0	-	-	-	-	-	10.3	10.4	10.4	10.6	10.7	10.8	11.0	11.2	11.3	11.7	12.1	12.5
12.0	-	-	-	-	-	11.4	11.5	11.6	11.7	11.9	12.0	12.2	12.4	12.6	13.0	13.5	13.9
13.0	-	-	-	-	-	12.5	12.6	12.7	12.9	13.0	13.2	13.4	13.7	13.9	14.4	14.8	15.3
14.0	-	-	-	-	-	13.5	13.7	13.8	14.0	14.2	14.4	14.7	14.9	15.1	15.7	16.2	16.7
15.0	-	-	-	-	14.5	14.6	14.7	14.9	15.1	15.4	15.6	15.9	16.1	16.4	16.9	17.5	18.1
16.0	-	-	-	-	15.5	15.7	15.8	16.0	16.3	16.5	16.8	17.1	17.4	17.6	18.2	18.8	19.4
17.0	-	-	-	16.6	16.6	16.7	16.9	17.1	17.4	17.7	18.0	18.3	18.6	18.9	19.5	20.2	20.8
18.0	-	-	-	17.6	17.6	17.8	18.0	18.2	18.5	18.8	19.1	19.5	19.8	20.1	20.8	21.5	22.1
19.0	-	-	18.6	18.6	18.6	18.8	19.0	19.3	19.6	20.0	20.3	20.7	21.0	21.4	22.1	22.8	23.5
20.0	-	-	19.6	19.7	19.7	19.9	20.1	20.4	20.8	21.1	21.5	21.9	22.2	22.6	23.4	24.1	24.9
21.0	-	-	20.7	20.7	20.7	20.9	21.2	21.5	21.9	22.3	22.7	23.1	23.4	23.8	24.6	25.4	26.2
22.0	-	-	21.7	21.7	21.7	22.0	22.3	22.6	23.0	23.4	23.8	24.2	24.7	25.1	25.9	26.7	27.6
23.0	-	22.7	22.7	22.7	22.8	23.0	23.4	23.7	24.1	24.6	25.0	25.4	25.9	26.3	27.2	28.1	28.9
24.0	-	23.7	23.7	23.7	23.8	24.1	24.4	24.8	25.3	25.7	26.2	26.6	27.1	27.5	28.5	29.4	30.3
25.0	-	24.7	24.7	24.8	24.8	25.1	25.5	25.9	26.4	26.9	27.3	27.8	28.3	28.8	29.7	30.7	31.6
26.0	-	25.7	25.7	25.8	25.8	26.2	26.6	27.0	27.5	28.0	28.5	29.0	29.5	30.0	31.0	32.0	33.0
27.0	-	26.7	26.8	26.8	26.9	27.2	27.7	28.1	28.6	29.2	29.7	30.2	30.7	31.2	32.3	33.3	34.3
28.0	-	27.7	27.8	27.8	27.9	28.3	28.7	29.2	29.8	30.3	30.8	31.4	31.9	32.5	33.6	34.6	35.7
29.0	-	28.8	28.8	28.8	28.9	29.3	29.8	30.3	30.9	31.4	32.0	32.6	33.1	33.7	34.8	35.9	37.0
30.0	-	29.8	29.8	29.9	29.9	30.4	30.9	31.4	32.0	32.6	33.2	33.8	34.4	34.9	36.1	37.2	38.4
31.0	-	30.8	30.8	30.9	31.0	31.4	32.0	32.5	33.1	33.7	34.3	35.0	35.6	36.2	37.4	38.5	39.7
32.0	31.8	31.8	31.8	31.9	32.0	32.5	33.0	33.6	34.3	34.9	35.5	36.1	36.8	37.4	38.6	39.9	41.0
33.0	32.8	32.8	32.9	32.9	33.0	33.5	34.1	34.7	35.4	36.0	36.7	37.3	38.0	38.6	39.9	41.2	42.4
34.0	33.8	33.8	33.9	33.9	34.0	34.6	35.2	35.8	36.5	37.2	37.8	38.5	39.2	39.9	41.2	42.5	43.7
35.0	34.8	34.8	34.9	35.0	35.1	35.6	36.3	36.9	37.6	38.3	39.0	39.7	40.4	41.1	42.4	43.8	45.1
36.0	35.8	35.8	35.9	36.0	36.1	36.7	37.3	38.0	38.7	39.5	40.2	40.9	41.6	42.3	43.7	45.1	46.4
37.0	36.8	36.8	36.9	37.0	37.1	37.7	38.4	39.1	39.9	40.6	41.3	42.1	42.8	43.5	45.0	46.4	47.8
38.0	37.8	37.8	37.9	38.0	38.1	38.8	39.5	40.2	41.0	41.7	42.5	43.3	44.0	44.8	46.2	47.7	49.1
39.0	38.8	38.9	38.9	39.0	39.2	39.9	40.6	41.3	42.1	42.9	43.6	44.4	45.2	46.0	47.5	49.0	50.5
40.0	39.8	39.9	40.0	40.1	40.2	40.9	41.6	42.4	43.2	44.0	44.8	45.6	46.4	47.2	48.8	50.3	51.8

L/h as a function of H/h and $T\sqrt{g/h}$

7. NUMERICAL COMPARISON BETWEEN SINUSOIDAL AND CNOIDAL WAVE THEORY

As mentioned in section 4 it is possible to trace a wave train as it propagates from deep water into shallower water and determine the wave heights, lengths etc. according to both the sinusoidal as well as the cnoidal theory.

Fig. 5 shows the variation of the wave height. The abscissa is h/L_o so the values for the cnoidal theory split up into curves, one for each value of H_o/L_o. The significant feature is that the cnoidal wave height grows faster with decreasing depth although at intermediate water depths its value is up to 10% less than that predicted by sinusoidal wave theory. Waves with considerable deep water steepness (2-3%) will break at a depth where the cnoidal wave height is only slightly larger than that of sinusoidal waves. Waves with small deep water steepness, however, such as swells pass to much smaller depth before they break and consequently a major part of their shoaling process is governed by the cnoidal theory. For these waves the two theories will yield results for the wave height differing by factors of up to 2

For the wave length L the opposite applies, as can be seen from Fig. 6. In sufficiently shallow water in particular, the length of swell type waves is almost independent of the water depth. This is caused by the increase in H influencing c through the second term on the right hand side in Eq. 20.

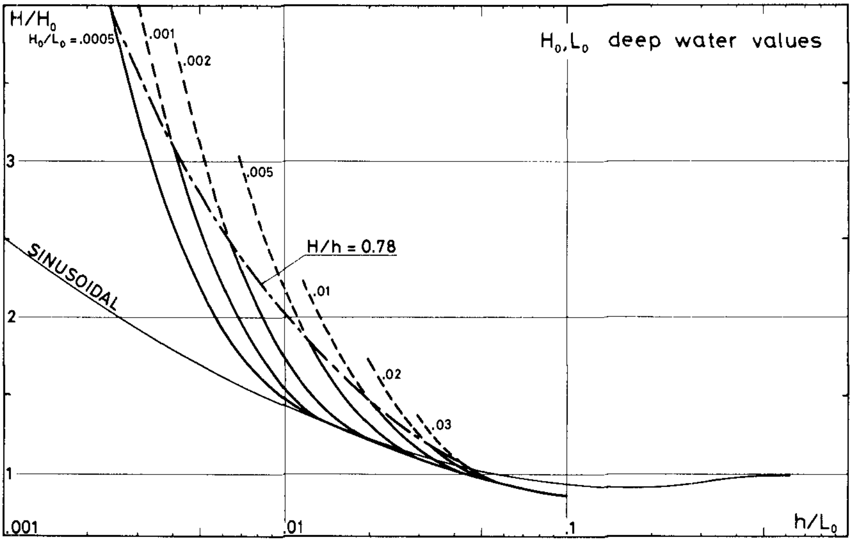

Fig. 5 H/H_o versus h/L_o for various H_o/L_o

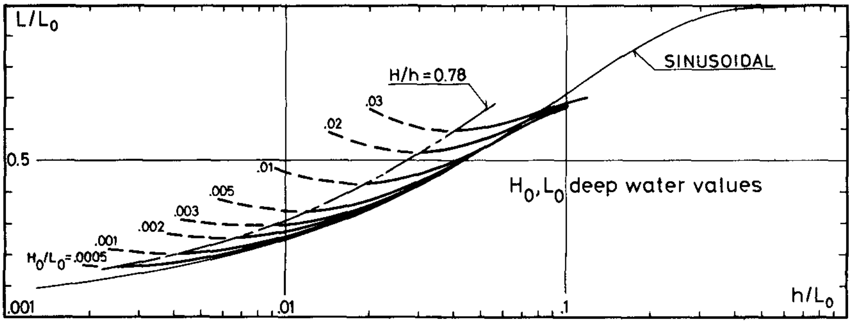

Fig. 6 L/L_o versus h/L_o for various H_o/L_o

8. NUMERICAL EXAMPLES

1. Consider wave forecast predicting a deep water wave height H_o = 3 m and a corresponding wave period T = 10 s.

We want to determine the wave height and wave length at h = 5 m.

We have directly

$$L_o = g/2\pi \cdot T^2 = 156 \text{ m}$$

$$h/L_o = 5/156 = 0.032$$

$$H_o/L_o = 3/156 = 0.0192$$

and by virtue of Eqs. 40 and 42

$$E_1 = 0.0632 \quad 0.0192^{2/3} = 0.00454$$

$$h_1 = 0.032/2\pi = 0.0051$$

Table 2 then yields

$$f_H = 0.910$$

so that by virtue of Eq. 50

$$\frac{H}{H_o} = 0.157 \cdot 0.0192^{1/3} \cdot 0.032^{-1} \cdot 0.910 = 1.20$$

or

$$H = 3.60 \text{ m}$$

We observe that since U = 190 (Table 1) we are within the cnoidal region.

If the sinusoidal wave theory is used we get

$$\frac{H}{H_o} = 1.11 \quad \text{or} \quad H = 3.33 \text{ m}$$

i.e. 7.5% less.

From H/H_o we get H/h as

$$\frac{H}{h} = \frac{H}{H_o}\frac{H_o}{L_o}\frac{L_o}{h} = 1.20 \cdot 0.0192/0.032 = 0.72$$

and since the other parameter of Table 3 is $T\sqrt{g/h} = 14$ we get from that table $L/h = 16.3$.

As was mentioned in section 6 L/h can also be determined by virtue of Eqs. 20 and 22. Table 1 yields

$$A = 0.50$$

and Eqs. 20 and 22

$$\frac{L}{h} = T\sqrt{\frac{g}{h}}\sqrt{1+\frac{H}{h}A}$$

or

$$\frac{L}{h} = 14\sqrt{1+0.72 \cdot 0.50} = 16.3$$

2. Data from a wave recorder at $h_r = 10$ m show $H_r = 1.20$ m, $T = 18$ s.

Looking for the wave height at $h = 3.5$ m we have

$$H_r/h_r = 1.20/10 = 0.12$$
$$T\sqrt{g/h_r} = 18\sqrt{9.81/10} = 17.8$$

Table 3 yields directly $L_r/h_r = 17.7$ so that $U_r = H_rL_r^2/h_r^3 = 0.12 \cdot 17.7^2 = 37.6$ which is within the cnoidal region. Table 1 supplies the values $B_r = 0.118$ and $f_H(U_r) = 1.237$.

Hence (Eqs. 35 and 36)

$$h_1 = h/gT^2 = 3.5/9.81 \cdot 18^2 = \underline{0.00110}$$

$$E_1 = (H_r^2L_rB_r)^{1/3}/gT^2 = (H_r/h_r)^{2/3}(L_r/h_r)^{1/3}B_r^{1/3}(T\sqrt{g/h_r})^{-2}$$
$$= 0.12^{2/3} \cdot 17.7^{1/3} \cdot 0.118^{1/3} \cdot 17.8^{-2} = \underline{0.00099}$$

We then get from table 2

$$f_H = 0.818$$

so that Eq. 46 yields

$$\frac{H}{H_r} = \frac{10}{3.5} \cdot \frac{0.818}{1.237} = 1.89$$

or

$$\underline{H = 2.27 \text{ m}}$$

If sinusoidal theory is used we get, using wave tables [7]

$$\frac{H}{H_r} = \frac{H}{H_o}\frac{H_o}{H_r} = 1.565/1.229 = 1.27$$

or

$$\underline{H = 1.53 \text{ m}}$$

i.e. 33% smaller wave height.

9. CONCLUSION

A method has been developed by which the cnoidal wave theory can be used to calculate changes in wave height and wave length due to shoaling of waves.

The basis is a calculation of the energy transport in cnoidal waves presented in section 1.

It is shown that the problem has two independent parameters which are chosen as $E_1 = (E_{tr}/\rho g)^{1/3}/gT^2$ and $h_1 = h/gT^2$ which define the parameters in the shoaling equation (32). The solution U of this equation is shown in Fig. 4. Numerical results for the function $f_H(U)$ (Eq..47) which is used for determination of the wave height H (Eq. 46) are given in Table 2.

The relation to sinusoidal wave theory is established. Thus E_1 and h_1 can be related to deep water wave data (Eqs. 40 and 42). Hence it can be shown (sections 5 and 6) how wave height and wave length can be determined if sufficient data is specified for the wave either at a point in the region where the cnoidal theory is valid or at a point where sinusoidal theory applies, including deep water.

Results for wave height and wave length predicted by the two theories are compared in Figs. 5 and 6. They show that in particular swell type wave motions will undergo a much more rapid increase in wave height according to the cnoidal wave theory.

The limits of validity of theory have not been investigated.

10. NOMENCLATURE

c	propagation velocity
$f_1(U)$	function of U (Eq. 30)
$f_2(U)$	function of U (Eq. 31)
f_H	dimensionless wave height factor (Eq. 47)
g	acceleration of gravity
h	water depth
h_1	parameter (Eq. 36)
k	$= 2\pi/L$ wave number
m	elliptic parameter
p	pressure
p^+	excess pressure (Eq. 5)
t	time
u,v	particel velocities (Fig. 1)
x,y	rectangular coordinates (Fig. 1)
A	function of m (Eq. 24)

B function of m (Eq. 26)

E complete elliptic integral of second kind

E_1 parameter (Eq. 35)

E_f energy flux

E_{tr} energy transport per wave period

H wave height

K complete elliptic integral of first kind

L wave length

M,N parameters (Eqs. 28 and 29)

T wave period

U parameter (Eq. 25)

UR Ursell-parameter (Eq. 10)

η surface elevation (Fig. 1)

ρ specific density

index $_r$: reference i.e. initial wave data

index $_o$: deep water data

11. REFERENCES

The following references have been used in the paper.

[1] Abramowitz, M. and Segun, I.A. (Ed.)
Handbook of Mathematical Functions.
Dover Publications, New York 1965.

[2] Grimshaw, R.
The solitary wave in water of variable depth.
J. Fl. Mech. (1970), 42, part 3, pp. 639-656.

[3] Keulegan, G.H. and Patterson, G.W.
Mathematical theory of irrotational translation waves.
J. Res. Nat. Bur. Stand. (1940), 24, pp. 48-101.

[4] Laitone, E.V.
Limiting conditions for cnoidal and Stokes' waves.
J. Geophys. Res. (1962), 67, No. 4.

[5] Laitone, E.V.
Higher approximation to non-linear water waves and the limiting heights of cnoidal, solitary, and Stokes' waves.
Tech. Mem. 133, (1963), Beach Erosion Board, Corps of Engineers.

[6] Lamb, H.
Hydrodynamics
Cambridge Univ. Press, 1932.

[7] Wiegel, R.L.
Gravity waves, tables of functions.
Council on Wave Res., The Engrg. Foundation, 1954.