CHAPTER 32

LIMITING CONDITION FOR STANDING WAVE THEORIES BY PERTURBATION METHOD

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ABSTRACT

The purpose of this paper is to make clear the validity and limiting condition for the application of the finite amplitude standing wave theories by the perturbation method In a numerical example, the errors of each order solution of these theories for two non-linear free surface conditions are computed for various kinds of wave characteristics and compared with each other

Some experiments on the wave pressure on a vertical wall by standing waves were carried out and a plot of the limiting condition for the application of these theories is proposed based on the comparison with theoretical curves

In addition, as an example of the application of these theories, the change of characteristics of wave pressure of standing waves accompanying the overtopping wave on a vertical wall is discussed

INTRODUCTION

Up to the present, the finite amplitude standing wave theories have been studied by many researchers such as Penny and Price, Kishi and etc In 1960, Tadjbaksh and Keller derived the third order solution of finite amplitude standing waves in shallow water by means of the perturbation method with two additional conditions Although the calculation for the higher approximation is very complicated, this method has often been used to solve such non-linear differential equations, because approximate solutions of arbitrary orders can be derived systematically Recently the fourth order solution of the finite amplitude standing wave theories in shallow water was obtained by Goda in 1966

The perturbation method is a formal one and no mathematical proof for the convergence of the solutions by this method is given Solutions by this method are only approximate ones consequently, the validity and limiting condition for the application of these theories to physical phenomena are subject to question To clarify these problems, we have carried out some investigations by the numerical consideration of the validity for two non-linear free surface boundary conditions using these theories and by the comparison between the theoretical curves based on these theories and experimental results for the wave pressure of standing waves

Moreover, as an example of the application of these theories, the change of the characteristics of the wave pressure of standing waves accompanying the overtopping wave is discussed in comparison with the results of standing waves described already

NUMERICAL CONSIDERATION FOR THE BOUNDARY CONDITIONS AT FREE SURFACE

(1) Two Non-Linear Boundary Conditions at Free Surface

Since these solutions are only approximate ones as mentioned above, the two non-linear boundary conditions at the unknown free surface, namely the kimematic condition prescribing the particle motion at the free surface and the dynamic condition describing the pressure at the free surface to be constant on the basis of Bernoulli's theorem are not satisfied rigorously by these solutions and errors for these conditions always exist So, we investigate to what extent the perturbation solution of each order satisfies these conditions by computing numerically the errors of each order solution for these non-dimensionalized conditions for various kinds of wave characteristics expressed by dimensionless parameters

The kinematic and dynamic conditions are given respectively as

$$v=\partial\eta/\partial t+u(\partial\eta/\partial x) \quad \text{at } y=\eta \tag{1}$$

$$\partial\phi/\partial t+(1/2)(u^2+v^2)+g\eta=0 \quad \text{at } y=\eta \tag{2}$$

Let ε_1 and ε_2 be the errors of kinematic and dynamic conditions for the solutions These errors can be written in dimensionless form respectively as

$$\varepsilon_1=[\sqrt{k/g}\{v-\partial\eta/\partial t-u(\partial\eta/\partial x)\}/\sqrt{gh}]_{y=\eta} \tag{3}$$

$$\varepsilon_2=[\{k\eta+(k/g)(\partial\phi/\partial t)+(k/2g)(u^2+v^2)\}/kh]_{y=\eta} \tag{4}$$

in which x is the horizontal coordinate at the still water level, y the vertical coordinate, directed upward positive, t the time, ϕ the velocity potential, u and v the horizontal and vertical particle velocities respectively, η the free surface elevation, h the depth of water, g the acceleration of gravity, $k=2\pi/L$ and L the wave length

If these conditions are rigorously satisfied by the solutions, ε_1 and ε_2 would be identically zero However, as the solutions are approximate, the equality is not satisfied Accordingly, by substituting these solutions into Eqs (3) and (4) and comparing the magnitude of these errors with each of them, we can investigate which solution has the best fitness for these conditions

We introduce the following criteria to evaluate the boundary condition errors

$$(E_1)_R=(\varepsilon_1)_{max}-(\varepsilon_1)_{min} \tag{5}$$

$$(E_2)_R=(\varepsilon_2)_{max}-(\varepsilon_2)_{min} \tag{6}$$

and

$$(E_3)_R=\sqrt{\overline{\varepsilon_1^2}}=\sqrt{[\sum_{n=1,3,5}^{M-1}\{(\varepsilon_1)^2_{n-1}+4(\varepsilon_1)^2_n+(\varepsilon_1)^2_{n+1}\}]/\{3(M-1)\}} \tag{7}$$

$$(E_4)_R=\sqrt{\overline{\varepsilon_2^2}}=\sqrt{[\sum_{n=1,3,5}^{M-1}\{(\varepsilon_2)^2_{n-1}+4(\varepsilon_2)^2_n+(\varepsilon_2)^2_{n+1}\}]/\{3(M-1)\}} \tag{8}$$

which are according to Dean's criteria, where (ε_i)max and (ε_i)min show the largest and smallest values of ε_1 and ε_2 calculated at each point respectively, M is the number of calculation point and bar indicates the average, and the validity of the wave theories can be investigated with the aid of numerical calculations

(2) The Fitness for Boundary Conditions of the Theories

Fig 1 shows one of the time variations of ε_1 and ε_2 at a vertical wall calculated for given values of $T\sqrt{g/h}$ and h/H where T is the wave period and H the amplitude of standing waves Notations 1, 2, 3 and 4 in Fig 1 indicate the first, the second, the third and the fourth order solutions respectively It is found from the figure that the fourth order solution has the best fitness for these conditions of the four theoretical curves

The relation between $(E_1)_R$ and $(E_2)_R$ of each order solution at the wall and $T\sqrt{g/h}$ in the case where the value of h/H is constant, where black points in these figures designate the breaking point of standing waves of each order solution calculated according to the criterion derived by Penny and Price are shown in Fig 2 It is found from Fig 2 that the higher approximate solutions certainly have better fitness for these boundary conditions within a limited range of wave characteristics It is also found that the errors for these boundary conditions do not always decrease for the higher approximation, when the value of $T\sqrt{g/h}$ becomes larger

Graphs expressing the areas where the boundary condition errors decrease for the higher approximation are given in Fig 3 according to each criterion Fig 3 shows that the perturbation solution of each order has a limited area of wave characteristics where the errors for the boundary conditions do not decrease by higher approximation, but there is a small difference between the areas for kinematic and dynamic conditions

Fig 4 is the same as Fig 3 but is evaluated according to Dean's criterion, Eqs (7) and (8) for all phases of wave motion

EXPERIMENT ON THE WAVE PRESSURE OF STANDING WAVES

(1) Experimental Equipment and Procedure

The wave tank used in this experiment is 63 m long, 0 5 m wide and 0 65 m deep and belongs to the Ujigawa Hydraulic Laboratory It has a piston type wave generator at the end and a caisson equipped with five pressure gages at about 39 5 m from the end In the experiment, the time variations in water level and wave pressure along the wall due to standing waves formed by wave reflection at the vertical wall were measured Since waves of sufficiently large height which the value of h/H is smaller than 2 8 could not be generated in the case of the experiment with uniform depth, a slope composed of 1/60 (3 m long) and 1/400 (20 m long) was constructed on the bottom of wave tank as shown in Fig 5, so that the waves of very large height near wave breaking were generated Experiments were carried out by changing the wave height, while the wave period and the water depth were determined by keeping the value of $T\sqrt{g/h}$ constant The wave characteristics used in the experiment are tabulated in Table 1

(2) Experimental Results and Considerations

Fig 6 shows the comparison between the theoretical curves already discussed and the experimental results for the wave crest height above the still water level η_0/H The notations are the same as those in Fig 1 except for the notation SHUTO which indicates the second approximation to stationary long waves of finite amplitude derived by Shuto as the interaction problem of cnoidal waves The theoretical curves agree comparatively well with the experimental results, but the theoretical curve of the second order approximate solution tends to deviate from the experimental results, as the value of h/H becomes smaller and the value of $T\sqrt{g/h}$ larger

Fig 7 is an example of the comparison between the theoretical curves and the experimental results for the wave pressure distribution along the wall at phase of wave crest, in which the theoretical wave pressure above the still water level is assumed to be triangular which connects the maximum point of elevation of water level with a point of wave pressure at the still water level, where p is the wave pressure and ρ the density of fluid The fourth order solution agrees well with the experimental values in the case of considerably large amplitude waves when the value of $T\sqrt{g/h}$ is relatively small, but the other solutions give excessive values because of the insufficiency of approximation for calculation

Fig 8 shows the comparisons between the theoretical curves and the experimental results for wave pressure at a point on the wall at phase of wave crest, where the notation KISHI is the second order approximate solution derived by Kishi using the Penny and Price method When the value of $T\sqrt{g/h}$ is relatively small as seen in Fig 8(a), the fourth order solution agrees well with the experimental values, but the lower order solutions deviate from the experimental results as the value of h/H becomes small In the case where the value of $T\sqrt{g/h}$ becomes larger as shown in Fig 8(e) and (h), the fourth order solution also tends to deviate from the experimental results in the case of small h/H The larger the value of $T\sqrt{g/h}$ becomes, the larger the value of h/H, which the experimental values deviate from the theoretical curves corresponding to each order approximate solution except for the stationary long wave theory becomes The theoretical curve for the stationary long wave theory agrees relatively well with the experimental results for large h/H as the value of $T\sqrt{g/h}$ becomes larger, but in the case of small h/H, it deviates rapidly This may be due to insufficiency of approximation order for the calculation

Fig 9 indicates the comparison between the theoretical curves and the experimental results for the time variations of water level and wave pressure on the wall It is found that the experimental results approach the theoretical curves of the higher order approximate solutions in Fig 9(a), that they do not agree with the experimental results in Fig 9(b) due to the distortion of theoretical curves of the third and the fourth order solutions, and that the theoretical curve of stationary long wave agrees well with the experimental results only for the wave form

Using the results obtained from the detailed comparison for the wave pressure of standing waves, a plot of the limiting conditions for the application of the theories is proposed in Fig 10 This figure shows that the limiting condition of the second order solution is confined to the area in which the value of $T\sqrt{g/h}$ is relatively small and the value of h/H large Also this shows that the area of poor correspondence between the theoretical results and the experimental ones exists, although the third and the fourth order solutions have relatively wide areas of applicability and that the area of correspondence for the stationary long wave theory is restricted to the area of relatively large value of $T\sqrt{g/h}$

EXPERIMENT ON THE WAVE PRESSURE WITH WAVE OVERTOPPING

(1) Experimental Equipment and Procedure

The experimental equipment used is same as in th previous section Experiments were carried out alternatively for the case of the wave pressure accompanying the overtopping wave and standing waves The characteristics of waves and the crest height of wall are presented in Table 2

(2) Experimental Results and Considerations

The relation between the theoretical curves and the experimental results for wave crest height above the still water level is shown in Fig 11, where the white circles indicate the case of standing waves, the other circles indicate the case of the wave pressure accompanying the overtopping wave and Hc is the crest height of wall Experimental results for standing waves accompanying the overtopping wave agree relatively well with theoretical curves on the basis of the theories within a range of the experiment, although in the case of $T\sqrt{g/h}$=10 they are a little less than the experimental values for the standing waves in spite of the scatter

Fig 12 shows the comparison between the theoretical curves of the fourth order solution calculated by taking into account the reduction of wave height due to the wave overtopping and the experimental results for the wave pressure distribution along the wall at phase of wave crest in the case of wave overtopping It is seen from Fig 12(a) that even if the wave height in comparison with the crest height of wall increases considerably, the change of the wave pressure due to the presence of wave overtopping may apparently disappear when the reduction of the wave height at the wall can be taken into account and that the theoretical curves agree well with the experimental results Fig 12(b) describes the effect of the crest height of the wall on the wave pressure distribution It is seen that the theoretical curve agrees well with the experimental results in spite of the difference in the crest height of wall

Fig 13 is the comparison between the theoretical curves of wave pressure at a point on the wall at phase of wave crest and the experimental results for the overtopping wave The theoretical curves agree fairly well with the experimental results in spite of the existence of wave overtopping within a range of the limiting area for the application of the theories corresponding to each order solution found out in the previous section if the reduction of wave height at the wall can be taken into account However, when the wave height increases considerably in comparison with the crest height of wall and becomes near the breaking wave height, the experimental results for the overtopping wave are a little less than those for standing waves This may be due to the change of the field of wave motion because of the increase of the rate of wave overtopping Accordingly, the value of H/Hc is an important parameter which dominates the change of wave pressure caused by wave overtopping as well as the phenomenon of wave overtopping itself

Fig 14 shows the time variations of the water level and wave pressure, where the theoretical curve is based on the fourth order solution The absolute values of experimental results for the overtopping wave decrease in comparison with those of standing waves with the same wave characteristics, while the values agree fairly well with theoretical ones in both the case, if the reduction of wave height is taken into account But the second peak of double humped wave pressure record disappears and then an unsymmetric wave pressure record appears

The relation between the relative amplitude in the water level at the wall H/Hc and the one for the overtopping wave H'/Hc is given in Fig 15 as evaluated by reducing of wave height due to the wave overtopping It is found that the rate of reduction of wave height is almost uniquely determined by the value of H/Hc, regardless of the wave period

CONCLUSIONS

The main results of this investigation are summarized as follows

1 As a result of detailed numerical and experimental considerations for the finite amplitude standing wave theories, the limiting conditions for the application of these theories are presented and it is verified that the approximate solutions for various orders of the theories are valid within a certain range of h/H and $T\sqrt{g/h}$ corresponding to each order of approximation

2 If the rate of reduction of wave height at the wall caused by wave overtopping can be taken into account, these theories are applicable for estimating the wave pressure of standing waves on a wall in the case where wave overtopping exists within a range of the limiting area of applicability of the theories except for relatively large values of H/Hc

ACKNOWLEDGEMENT

Part of this investigation was accomplished with the support of the Science Research Fund of the Ministry of Education, for which the authors express their appreciation Thanks are due to Dr Iwagaki, Professor, Department of Civil Engineering Kyoto University, for his cordial encouragement during this investigation

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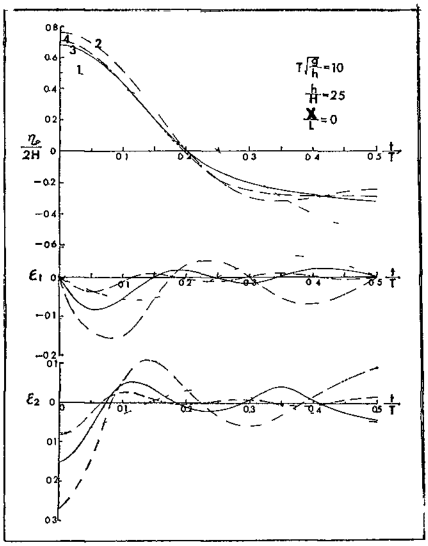
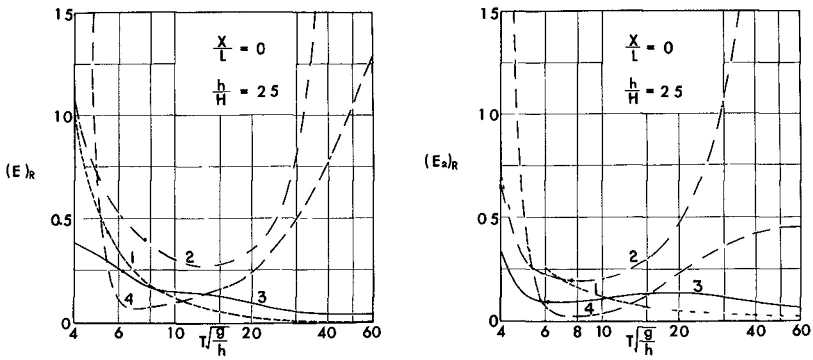


Fig 1 Time variations of ε_1 and ε_2 at vertical wall



(a) (b)

Fig 2 Relation between $(E_1)_R$ and $(E_2)_R$ of each order solution at wall and $T\sqrt{g/h}$

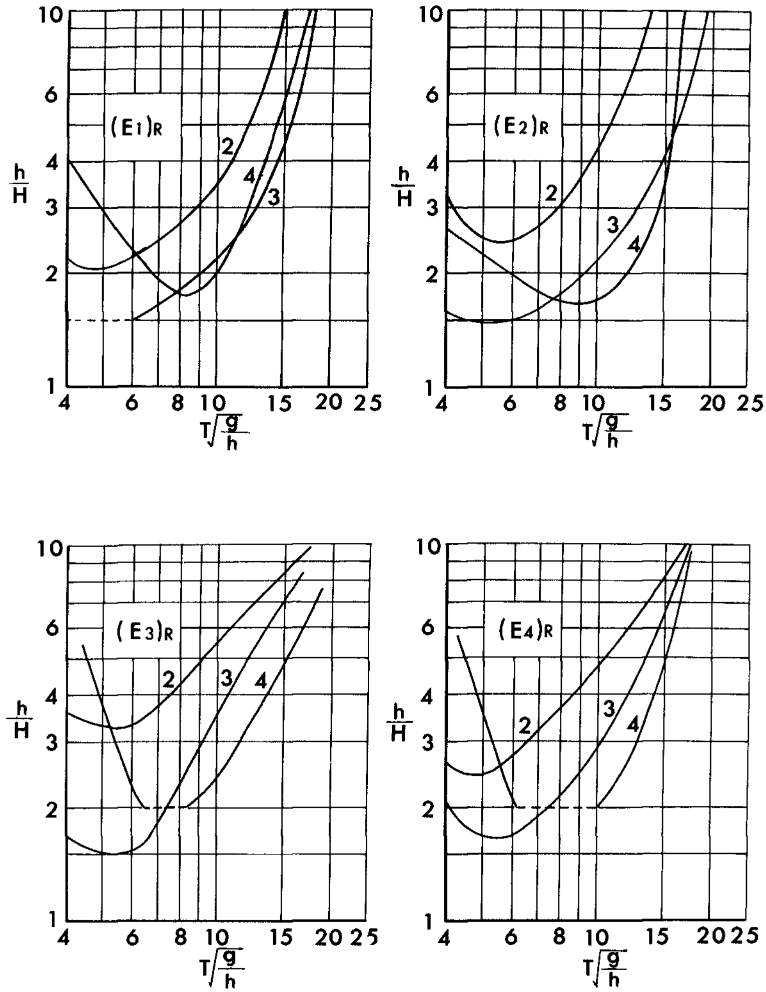


Fig 3 Decreasing areas, estimated approximately, of free surface boundary condition errors for validity of finite amplitude standing wave theories by higher approximation

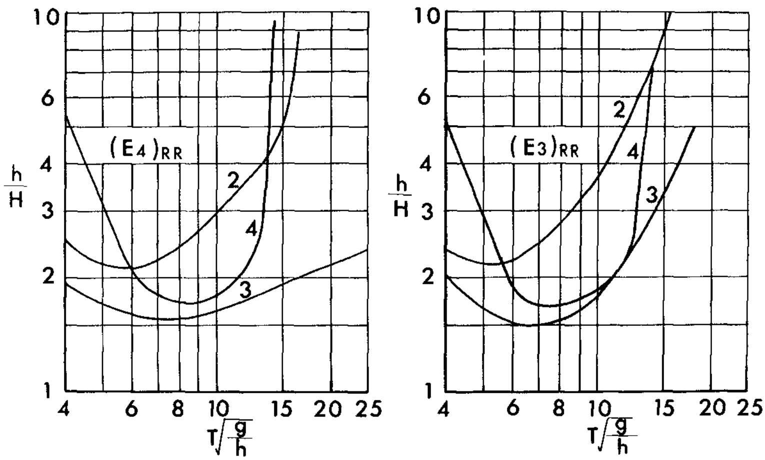


Fig 4 Decreasing areas, estimated approximately, of free surface boundary condition errors for validity of finite amplitude standing wave theories by higher approximation

Table 1 Wave characteristics used in the experiment on standing waves

$T\sqrt{g/h}$	h(cm)	T(sec)	H(cm)
6	25.0 20 0	0 958 0 857	10 70～2 05 6 72～3 18
8	20 0 25 0 17 5	1 143 1 278 1 069	10 51～1 45 7 11～2 97 5 80～4 30
10	20 0 25 0 17 5	1 427 1 597 1 336	10 00～1 42 8 36～3 05 4 97～3 00
12	17 5 20 0 15 0	1 603 1 714 1 485	9 50～1 01 5 93～2 98 4 34～1 52
14	15 0 20 0	1 732 2 000	9 21～1 23 6 08～1 84
16	15 0 12 5	1 979 1 807	9 20～1 01 2 61～2 44
18	12 5 10 0	2 033 1 818	6 78～1 03 2 31～0 73
20	10 0	2 020	5 04 ～1 15

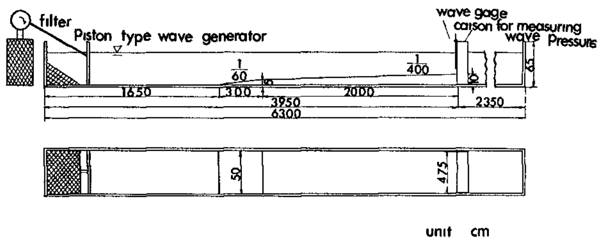
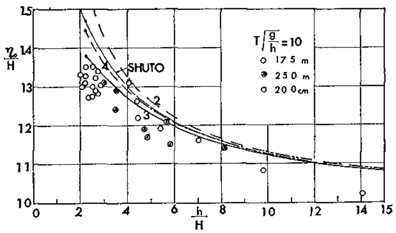
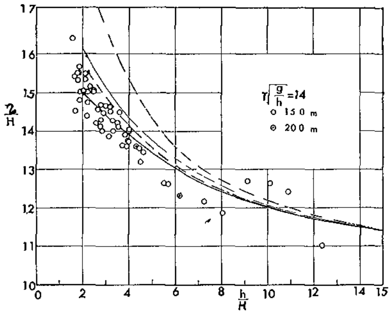


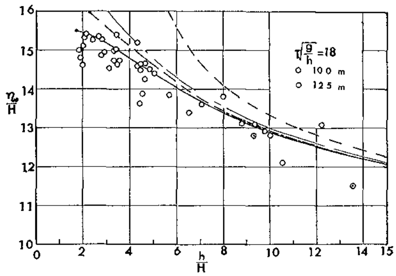
Fig 5 Schematic sketch of wave tank used



(a)



(b)



(c)

Fig 6 Comparison between theoretical curves and experimental results for wave crest height above still water level

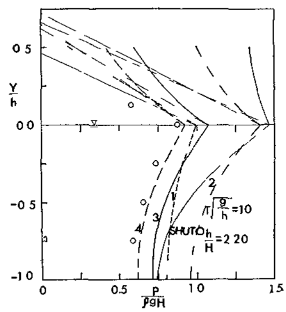


Fig 7 Comparison between theoretical curves and experimental results for wave pressure distribution along wall at phase of wave crest

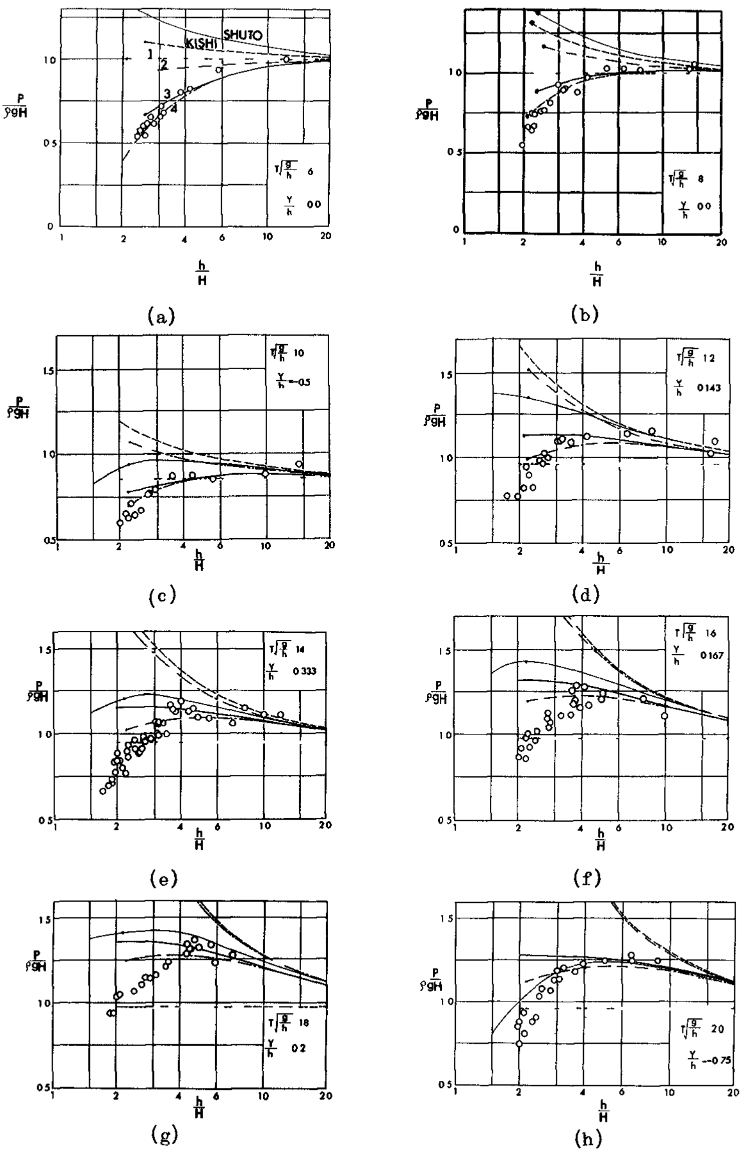
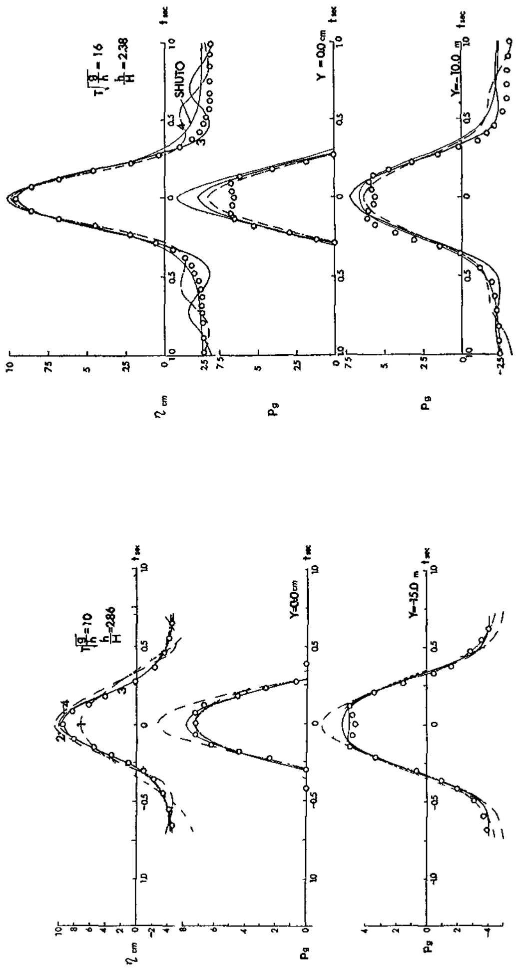


Fig 8 Comparison between theoretical curves and experimental results for wave pressure at a point on wall at phase of wave crest



(a) (b)

Fig 9 Comparison between theoretical curves and experimental results for time variations of water level and wave pressure on wall

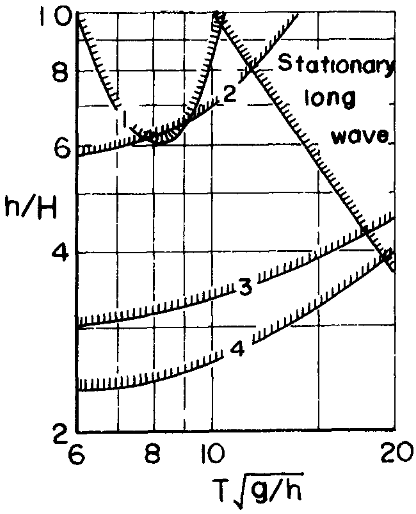


Fig 10 Limiting conditions for application of finite amplitude standing wave theories

Table 2 Wave characteristics used in the experiment on standing waves with wave overtopping.

$T\sqrt{g/h}$	T(sec)	h(cm)	Hc(cm)	H(cm)
8	1 143	20 0	5 0	10 50~4 91
10	1 429 1 336 1 336 1 336 1 237	20 0 17 5 17 5 17 5 15 0	5 0 2 5 5 0 7 5 5 0	12 38~4 40 9 87~2 89 9 70~4 60 8 53~5 26 8 58~4 50
12	1 604	17.5	5 0	9 22~3 70
14	1 732	15 0	5 0	9 58~4 01
16	1 979	15 0	5 0	9.12~3.39
18	2 033	12 5	5 0	8 28~3 04

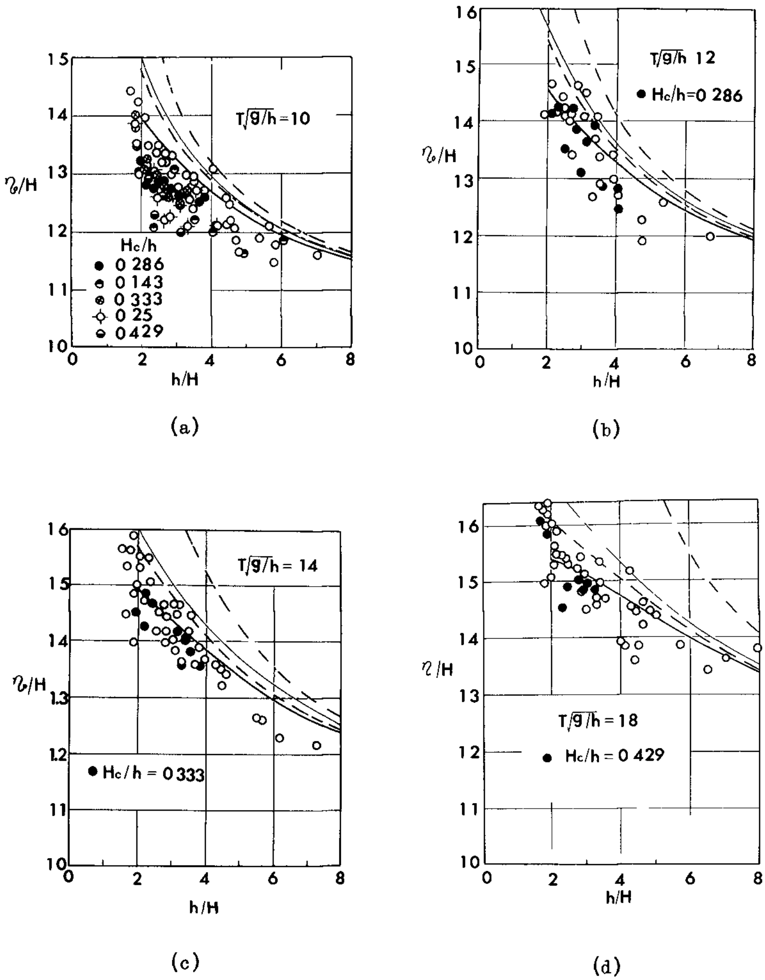
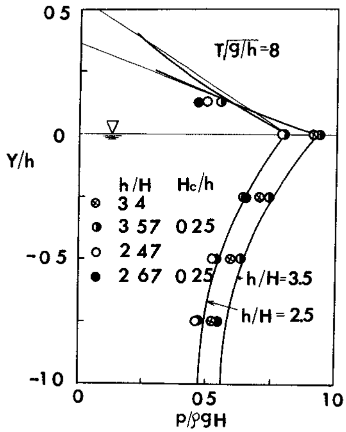
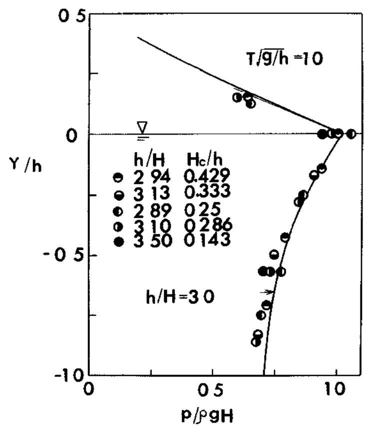


Fig 11 Comparison between theoretical curves and experimental results for wave crest height above still water level in the case where wave overtopping exists



(a)



(b)

Fig 12 Comparison between theoretical curves and experimental results for wave pressure distribution along wall at phase of wave crest in the case where wave overtopping exists

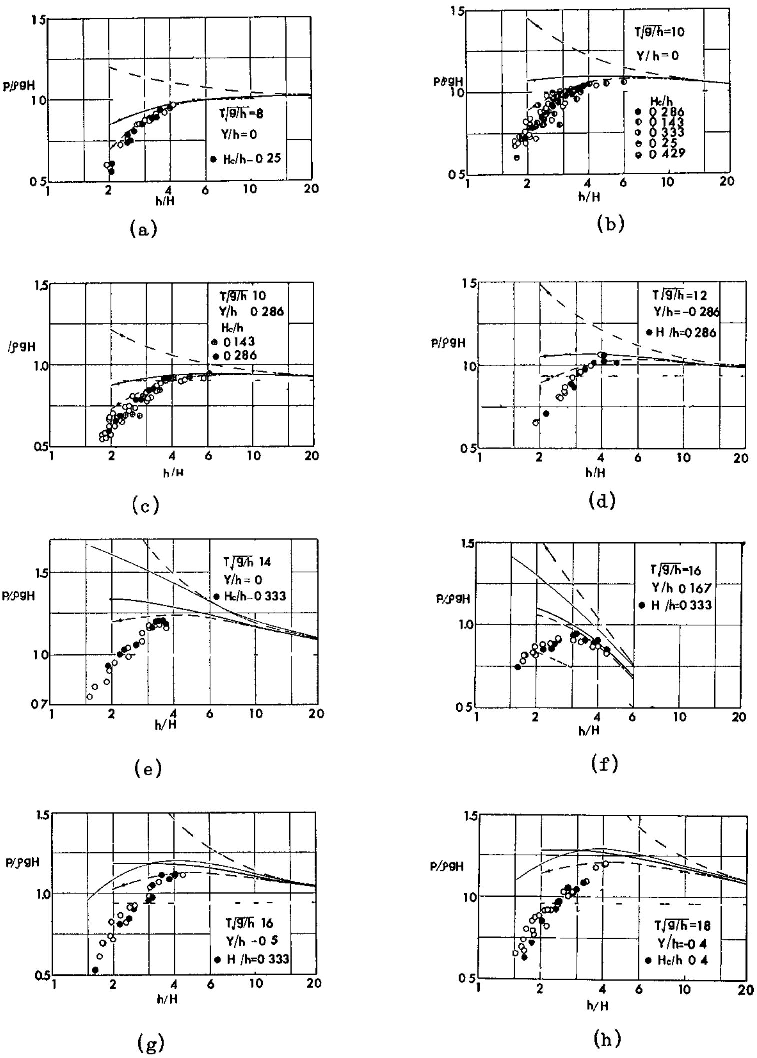
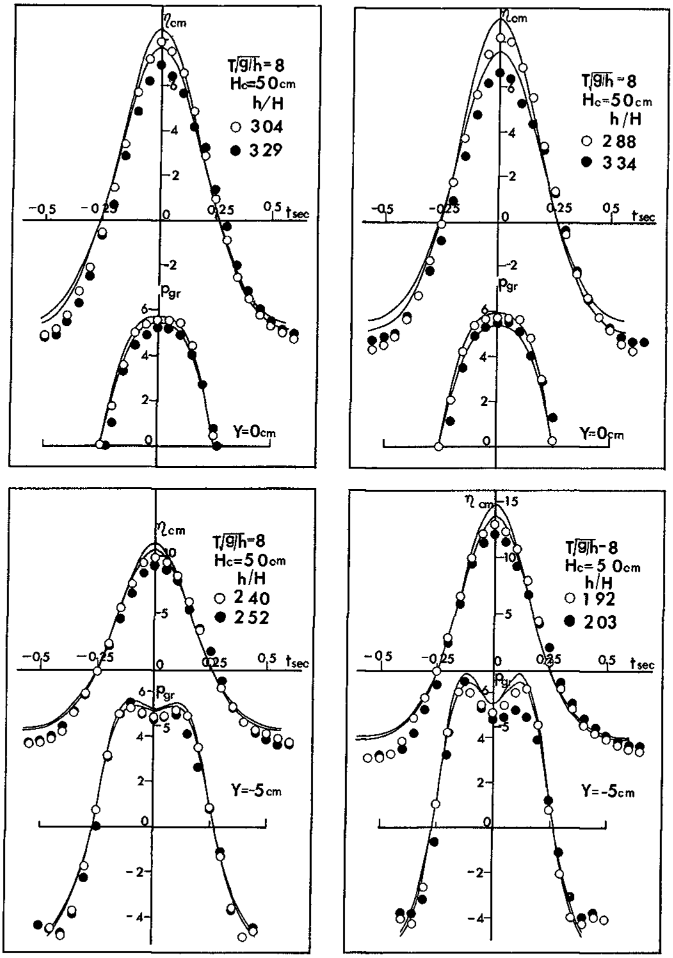
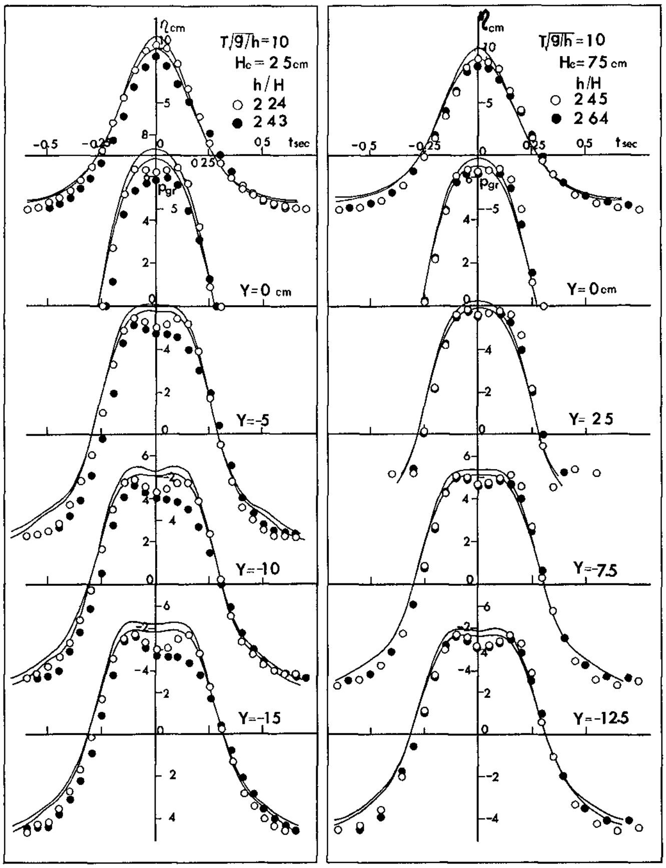


Fig 13 Comparison between theoretical curves of wave pressure at a point on wall at phase of wave crest and experimental results in the case where wave overtopping exists.



(a)

Fig 14 Comparison between theoretical curves and experimental results for time variations of water level and wave pressure on wall in the case where wave overtopping exists



(b)

Fig 14 Comparison between theoretical curves and experimental results for time variations of water level and wave pressure on wall in the case where wave overtopping exists.

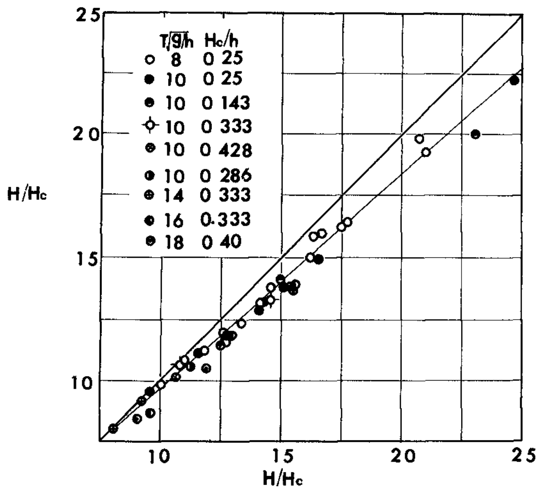


Fig 15 Relation between relative amplitude in water level at wall H/Hc and one for overtopping wave H′/Hc