CHAPTER 9

Equilibrium Range Spectra in Shoaling Water

by

Takeshi Ijima, Takahiro Matsuo and Kazutami Koga

Abstract

In shoaling water on sloping beach, waves break by hydraulic instability due to the finiteness of water depth, so that frequency spectra of waves in surf zone must have any limiting form similar to the equilibrium spectrum given by Phillips (1958). In this paper, authors have derived an equilibrium form of spectra for surf waves from the limiting wave condition at constant water depth by Miche (1944) and from breaking wave experiments on sloped bottom by Iversen (1952). The results are compared with surf wave spectra obtained from field observations by means of stereo-type wave meter devised by the authors (1968).

By means of this spectrum and by deep water wave spectra for various wind conditions, significant wave heights and optimum periods of limiting waves in surf zone are calculated.

1 Introduction

By dimensional considerations, Phillips (1958) has shown that in the frequency spectrum of deep water wind waves the energy spectral density \( \Phi(\sigma) \) of saturated high frequency component is proportional to \(-5\) powers of frequency as shown by the following equation

\[
\Phi(\sigma) = \beta \cdot \sigma^{-5}
\]

where \( g \) is gravity acceleration and \( \beta \) is non-dimensional constant, for which Phillips has given numerically \( 0.0117 \) from observed data and Pierson (1964) has obtained \( 0.0050 \) from observed spectra on the North Atlantic Ocean.

\( \Phi(\sigma) \) is independent of wind conditions and is interpreted as representing a limiting spectrum for waves breaking in deep water in the state of hydraulic instability.

As for waves breaking in shoaling water, the limiting wave height is determined by instability depending on water depth \( h \) and the similar limiting spectrum should exist, which, of course, is independent on wind conditions but depends only on water depth and gravity acceleration in the range of gravity waves.

(1) Professor of Kyushu University, Fukuoka, Japan
(2) Lecturer, Oita Technical College, Oita, Japan
(3) Master Course Student, Kyushu University

137
Here, we assume straight shore line with constant slope and waves incident normally, so that one-dimensional waves without refraction effect.

2 Derivation of equilibrium range spectrum

From observed wave record of a train of waves at a point of depth \( h \) in surf zone, apparent individual wave height \( \widetilde{H} \) and wave period \( \widetilde{T} \) of all the waves are measured in spite of breaking and non-breaking and then wave length \( \widetilde{L} \) of the wave at water depth \( h \) are calculated by small amplitude wave relation. Then, we obtain a scatter diagram of \( \widetilde{H} / \widetilde{T} \) related to \( h / L \) as shown in Fig. 1, which was obtained from wave record of 17 minutes long at depth of 5.7 meters on the Coast of Miyazaki (Pacific Coast of Kyushu).

The envelope curve through the upper limit of scattered points in the figure is interpreted to show the limiting wave condition for the depth \( h \).

For a train of single sinusoidal waves, the limiting condition at constant depth for \( h / L \) larger than about 1/20 is given by Mache (1944) as follows:

\[
\frac{\widetilde{H}}{L} = \frac{1}{7} \tanh \frac{2\pi h}{L} \tag{2}
\]

For deep water \( (h/L \rightarrow \infty) \), \( H/L \) is 1/7 and for shallow water long waves \( (h/L \rightarrow 0) \), eq 2 gives \( H/h \approx 0.9 \), which is somewhat larger than 0.78 by solitary wave theory. For the bottom of constant slope, we have not yet theoretical relation but Iversen (1952) has shown experimental relations of \( H/h' \) and \( H/l' \) to \( H/L_0 \) for bottom slopes \( \alpha = 1/10, 1/20, 1/30 \) and \( 1/50 \), where \( H', l', H_0, L_0 \) are deep water wave height and length and \( H, l \) are wave height and water depth of breakers. From these experimental relations, an empirical relation of \( H/L \) and \( h/L \) similar to eq 2 is obtained as follows:

\[
\frac{H}{L} = \frac{1}{7} \tanh \left[ f(d) \frac{2\pi h}{L} \right], \quad f(d) = \cosh 3.5 \alpha \frac{h}{L} \tag{3}
\]

where \( \alpha \) is bottom slope. Above relations are shown in Fig. 1 for \( d = 0 \) and \( 1/10 \).

Eq 3 may give somewhat larger values for \( h/L < 1/20 \) for single sinusoidal waves but for actual random waves the form of eq 3 may be assumed to represent approximate relations of limiting apparent wave height \( \widetilde{H}_{\text{max}} \) and length \( \widetilde{L} \) derived from apparent wave period \( \widetilde{T} \) for all \( h/T \). Accordingly, we assume next relation for limiting apparent waves:

\[
\frac{\widetilde{H}_{\text{max}}}{L} = C_1 \tanh \left[ f(d) \frac{2\pi h}{L} \right] \tag{4}
\]
Where $C_1$ is assumed to be constant.

As for actual waves with continuous spectrum, the limiting wave height $\bar{H}_{\text{max}}$ is interpreted as the result that all the phases of component waves whose periods are within narrow range of period band between $\bar{T} - \Delta \bar{T}/2$ and $\bar{T} + \Delta \bar{T}/2$ centered at period $\bar{T}$ happened to coincide and their spectral wave heights were summed up to attain the limiting height $\bar{H}_{\text{max}}$. And for sufficiently narrow width of $\Delta \bar{T}$, the component waves within the period band are considered to have nearly equal spectral wave height $K_T$ proportional to the limiting wave height $\bar{H}_{\text{max}}$, that is,

$$H_T \propto \bar{H}_{\text{max}}$$

Therefore, from eq 4 we have

$$H_T = \text{const} \left[ \frac{1}{L} \text{tanh} \left( \frac{f(\alpha)2\pi h}{L} \right) \right]$$

Assuming that the number of component waves within the period band is $N$ and they are mutually independent, the total energy density within the period band of $\bar{T} - \Delta \bar{T}/2$ and $\bar{T} + \Delta \bar{T}/2$ is

$$N \cdot \frac{\pi^2}{8} \text{const} \left[ \frac{1}{L} \text{tanh} \left( \frac{f(\alpha)2\pi h}{L} \right) \right]$$

Accordingly, the total frequency spectral energy density $\Phi(\omega) \Delta \omega$ between $\omega - \Delta \omega/2$ and $\omega + \Delta \omega/2$ is given by the following equation.

$$\Phi(\omega) \Delta \omega = \text{const} \, N \left[ \frac{1}{L} \text{tanh} \left( \frac{f(\alpha)2\pi h}{L} \right) \right]$$

The number of component waves $N$ in above equation is considered to become large when the period band $\Delta \bar{T}$ becomes wide, but to become small when the period $\bar{T}$ becomes large for limited length of wave record. Accordingly, we may assume the following relation

$$\text{const} \, N = \text{const} \, \bar{T} \Delta \alpha = \text{const} \, \Delta \omega/\alpha, \quad (\alpha = 2\pi/\bar{T})$$

Now, eq 7 is written as follows

$$\Phi(\omega) \Delta \omega = \text{const} \left[ \frac{1}{L} \text{tanh} \left( \frac{f(\alpha)2\pi h}{L} \right) \right]$$

$$= \text{const} \left( \frac{2\pi^2}{\bar{T}} \frac{h^{1/2}}{g^{1/2}} \frac{\text{tanh} \left[ f(\alpha) \frac{2\pi h}{L} \right]}{(2\pi h)^2} \Delta \omega \right)$$

$$= \text{const} \left( \frac{2\pi^2}{\bar{T}} \frac{h^{1/2}}{g^{1/2}} \int_{-\alpha}^{+\alpha} \frac{f(\alpha, \frac{2\pi h}{L})}{\Delta \omega} \right)$$

(9)
where

$$F(\alpha, \frac{\sigma^2 h}{g}) = \frac{\tanh^2 \left(\frac{2\pi h}{\sigma^2 h/g} \right)}{(\frac{2\pi h}{\sigma^2 h/g})^2 (\frac{\sigma^2 h}{g})^2}$$

(10)

In above equations, $2\pi h/L$ is determined as a function of $\sigma^2 h/g$ by the following equation

$$\sigma^2 h/g = \frac{2\pi h}{L} \tanh \frac{2\pi h}{L}$$

(11)

In eq 10, when $\sigma^2 h/g$ becomes large, $2\pi h/L$ tends to $2\pi h/L_0$ and $F$ tends to $(\sigma^2 h/g)^{-\frac{1}{2}}$, and when $\sigma^2 h/g$ becomes small, $F$ tends to $f_0(\sigma^2 h/g)^{-\frac{1}{2}}$.

Accordingly, from eq 9 it is seen that

$$\Phi(\sigma) = \text{const} \cdot (2\pi)^2 \sigma^2 \sigma^{-5}$$

for deep water waves

(12)

$$\Phi(\sigma) = \text{const} \cdot (2\pi)^2 f_0(\sigma) h^2 \sigma^{-1}$$

for shallow water waves

(13)

Comparing eq 12 with eq 1,

$$\text{const} \cdot (2\pi)^2 = \beta$$

Thus, eq 9 is determined as follows

$$\Phi(\sigma) = \beta \frac{h^{\frac{5}{2}}}{\sigma^2} F(\alpha, \frac{\sigma^2 h}{g})$$

(14)

Above equation is considered to represent the equilibrium spectrum of surf waves.

3 Comparisons with observations

The observation of surf waves is difficult because of various troubles in setting and maintaining wave meters. The authors (1968) have devised a stereo-type wave meter, with which field observations at Miyazaki Coast and Hata Coast of Kyushu were carried out. Fig. 2 is an example of observed frequency spectra at Hata Coast. Fig. 3 and 4 are non-dimensional plot of frequency spectra at Miyazaki- and Hata Coast, respectively, in which the curves are equilibrium spectra by eq 14 for $\alpha = 0$ and $1/10$ with $\beta = 0.00610$.

In Fig. 3, measured spectral densities for large $\sigma^2 h/g$ attain to the equilibrium values but for small $\sigma^2 h/g$ measured densities are lower than the latter.
This seems to be due to the fact that few waves are breaking by the effect of finite depth and only high frequency waves are saturated by local wind. In Fig. 4, measured energy densities almost attain to the equilibrium values, which mean that almost waves are breaking by the effect of finite depth.

4 Estimation of limiting significant waves in surf zone

Assuming that eq 14 is the limiting spectral energy density at water depth $h$, the limiting spectrum for given wind conditions at the depth may be estimated by the following considerations.

In Fig 5, suppose that various spectra are drawn in linear scale and the curve MW is equilibrium spectrum at given water depth. When the spectrum of offsea wind waves is given by the curve MWQ, the spectrum to be observed at the point of depth $h$ should be given by the curve MQ, which is the limiting spectrum for the given wind condition. When the spectrum of offsea swell is given by the curve ASQ, the spectrum at depth $h$ should be the curve ABCQ, which is the limiting spectrum for given swell. Fig 6 shows an example of offsea wind wave spectrum for wind speed $U = 15$ m/s, fetch length $F = 400$ km, which is proposed by the authors as shown in Appendix, and equilibrium spectra at water depth $h = 8$ meters for bottom slope $\alpha = 0$ and $1/10$.

Fig 7(a)(b) are significant wave height and its optimum period of limiting waves calculated by above-derived limiting spectrum at various water depth for fetch length 100 km and 400 km with wind speed 15 m/s and 30 m/s.

5 Conclusions and remarks

Omitting the effect of wave refraction and directional distribution of wave spectrum, the equilibrium spectrum of surf waves at water depth $h$ is represented by eq 14, by means of which and a proposed fetch- or duration spectrum of offsea wind waves the limiting spectrum at any depth is estimated as shown in Fig 5.

Up to date, for design purposes of surf zone structures the limiting wave height by such an equation 2 or 3 is frequently used as design wave height. But in some cases, it seems to be more reasonable to use such a limiting spectrum as above. The effect of directional distribution of wave spectrum should be considered in future.
Appendix

Derivation of Duration and Fetch Spectrum

We assume the general form of frequency spectrum as follows

\[ \Phi(\omega) = \alpha_\omega \frac{g^2}{U_0^2} \omega^{-5} \exp \left\{ -\beta_\omega \left( \frac{\omega}{U_0} \right)^P \right\} \]  

(A1)

where \( g \) is gravity acceleration, \( U_0 \) is friction velocity of wind and \( \alpha_\omega, \beta_\omega \) are non-dimensional constants.

Above spectrum has the maximum energy density \( \Phi(\omega_0) \) at frequency \( \omega_0 \) as follows

\[ \frac{\rho_0 U_0}{g} = \left( \frac{4}{5} \beta_\omega \right)^{\frac{3}{4}} \]  

(A2)

from which we obtain

\[ \beta_\omega = \frac{5}{4} \left( \frac{\rho_0 U_0}{g} \right)^{\frac{4}{3}} \]  

(A3)

\[ \alpha_\omega = \frac{g^{\frac{2}{5}}}{\rho_0^{\frac{1}{5}}} \Phi(\omega_0) \omega_0^5 \]  

(A4)

The time development of the maximum energy density of wind generated-wave spectrum is given by Phillips (1966) as follows

\[ \Phi(\sigma_t) = \bar{\Phi}_0 \frac{\sigma_t}{g^2} \frac{\mu h_1(\mu \sigma_t)}{\mu} \]  

(A5)

where \( \bar{\Phi}_0, \sigma_t, h_1, \mu \) are densities of air and water and \( \mu \) is coupling coefficient of air-sea interaction.

Substituting eq 44, 45 and 46 into eq 41, we obtain as duration spectrum

\[ \Phi(\sigma_t) = \bar{\Phi}_0 \frac{\sigma_t^{\frac{5}{2}}}{h_1} \left( \frac{\sigma_0}{\sigma} \right)^5 \frac{\mu h_1(\mu \sigma_t)}{\mu} \exp \left\{ -\frac{5}{4} \left( \frac{\sigma_0}{\sigma} \right)^4 \right\} \]  

(A7)

Meanwhile, Pierson (1964) has proposed the following spectrum for fully-developed wind waves

\[ \Phi(\sigma) = \alpha_\sigma \frac{g^2}{U^2} \sigma^{-5} \exp \left\{ -\beta_\sigma \left( \frac{\sigma}{U} \right)^4 \right\} \]  

(A8)

where \( \alpha_\sigma = 0.0081, \beta_\sigma = 0.74 \) and \( U \) is mean wind speed at 19.5 meters high above mean sea surface, which is related to \( U_0 \) and \( U_{10} \) (wind speed at 10 meters above mean sea level) by the following empirical relationships

\[ \frac{U}{U_*} = \frac{K}{U_*} \]  

where \( K = 1.60 + \frac{1}{\sqrt{C_{10}}} \) 

(A9)
\[ \Xi_{10} = \frac{\Xi^*}{\sqrt{C_{10}}} \quad C_{10} = (0.80 + 0.114 \Xi_{10}) \times 10^{-3} \quad (A10) \]

For any given wind speed, the maximum energy density \( \Phi(\sigma) \) of duration spectrum should lie always on the saturated spectrum curve of that wind speed given by eq A8, so that equating eq A6 to eq A8, we obtain next relation for \( \sigma^* \) at any duration time \( t \)

\[ \frac{A}{a} \Xi^* \left( \frac{\sigma^*}{\sigma_0} \right)^5 \sinh \left( \frac{2\mu a t}{\mu} \right) = \exp \left\{ -\frac{3}{\Xi^*} \left( \frac{\sigma^*}{\sigma_0} \right)^4 \right\} \quad (A11) \]

Eq A7 together with eq A11 gives duration spectrum

Fetch length \( \mathcal{X} \) is related to duration time \( t \) by the following equation

\[ \mathcal{X} = \frac{1}{2} C_0 t \quad (A12) \]

where \( C_0 \) is the phase velocity of waves with frequency \( \sigma^* \)

Thus, fetch spectrum is obtained from duration spectrum as follows

\[ \Phi(\sigma, x) = \frac{5}{6} \Xi^* \left( \frac{\sigma^*}{\sigma_0} \right)^5 \sinh \left( \frac{2\mu a t}{\mu} \right) \exp \left\{ -\frac{5}{4} \left( \frac{\sigma^*}{\sigma_0} \right)^4 \right\} \quad (A13) \]

By the relation \( \sigma^* = \frac{3}{C_0} \), duration and fetch spectrum is written as follows

\[ \Phi(\sigma, t) = \frac{5}{6} \Xi^* \left( \frac{\sigma^*}{\sigma_0} \right)^5 \sinh \left( \frac{2\mu a t}{\mu} \right) \exp \left\{ -1.25 \left( \frac{\sigma^*}{\sigma_0} \right)^4 \right\} \quad (A14) \]

\[ \Phi(\sigma, x) = \frac{5}{6} \Xi^* \left( \frac{\sigma^*}{\sigma_0} \right)^5 \sinh \left( \frac{2\mu a x}{\mu} \right) \exp \left\{ -1.25 \left( \frac{\sigma^*}{\sigma_0} \right)^4 \right\} \quad (A15) \]

In above equations, coupling coefficient \( \mu \) is given by Phillips(1966) as a function of \( C/\Xi^* \). But when fetch spectrum is given by eq A15, the significant wave height \( H_{\text{ys}} \) is shown by following equation

\[ \frac{H_{\text{ys}}}{\Xi^*} = 4.00 \sqrt{\frac{A}{\Xi^*}} \left( \exp \left( \frac{2\mu a x}{\mu} \right) \right)^{1/2} \quad (A16) \]

And also from observation data, significant wave height is empirically related to wind speed by the following equation by Wilson(1965)

\[ \frac{H_{\text{ys}}}{\Xi^*} = \frac{K_3}{C_{10}} \left[ 1 - \frac{1}{\sqrt{\Xi_{10} K_d(\Xi_{10} C_{10})^2}} \right], \quad K_3 = 0.30, \quad K_d = 0.004 \quad (A17) \]
Equating above two equations, coupling coefficient $\mu$ is obtained for $C_0/U_*$ with parameter $U_*$ as shown in Fig A1. Fig A2 is an example of fetch spectrum.

References


Miche, A. (1944) "Mouvements ondulatoires de la mer en profondeur constante ou décroissante" Annales des ponts et chaussées, Vol 114

Iversen, H. W. (1952) "Laboratory Study of Breakers" Gravity waves Nat Bur Standards Cir No 521


Fig 1: AN EXAMPLE OF SCATTER DIAGRAM

Fig 2: EXAMPLES OF FREQUENCY SPECTRA
Fig. 3 Non-dimensional Spectra at Miyazaki Coast

Fig. 4 Non-dimensional Spectra at Nata Coast
EQUILIBRIUM RANGE

Fig -5, EXPLANATION SKETCH

Fig -6 Limiting Spectra at Depth of 8 meters for an Offsea Wind Wave Spectrum
Fig. 7(a) LIMITING SIGNIFICANT WAVES FOR FETCH LENGTH 100Km

Fig. 7(b) LIMITING SIGNIFICANT WAVES FOR FETCH LENGTH 400Km
EQUILIBRIUM RANGE

**Fig-A1** Relations on $\mu$ and $C/U_*$ and $U_*$

**Fig-A2** Fetch Spectrum

- $U_*=31 \text{ m/s}$
- $U_*=41.6 \text{ m/s}$

- $8 \times 10^3 \text{ Km}$
- $4 \times 10^2$
- $2 \times 10^3$
- $8 \times 10^1$