CHAPTER 12

LONG WAVES IN CHANNELS OF ARBITRARY CROSS-SECTION

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This paper summarises some recent work on long gravity waves on still water in channels of arbitrary constant cross-section. Theoretical results have been obtained for both straight and curved channels. Some experimental work has been performed in straight trapezoidal channels and shows reasonable agreement with theory. For straight channels some details of the second approximation are given, and the cases where the approximation breaks down are indicated. For curved channels it is found that the effect of channel curvature is more pronounced when the cross-sectional shape of the channel is not symmetric with respect to its centre-line.

The waves considered are long gravity waves on the surface of water contained in a channel. If the waves are very long compared with both the breadth and depth of the channel, then it is reasonable to assume uniform conditions across the whole cross-sectional area of the channel. That is, a uniform height of water and a uniform velocity of water along the channel. If, further, it is assumed that the amplitude of the wave is very small compared with the depth of water, it is simple to show that the pressure is hydrostatic, and the wave motion in a straight channel is governed by the equations,

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \qquad B_0\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(A_0 u) = 0, \qquad (1)$$

where u is the velocity of water along the channel which is taken to be in the x direction, ζ is the height of the water surface above its undisturbed level, A_0 is the cross-sectional area of the channel and B_0 is the width of the free surface of the channel, where both A_0 and B_0 are evaluated for the undisturbed water surface. If attention is now confined to straight channels of uniform cross-section, A_0 and B_0 are constant and the wave velocity c_0 is given by

$$c_0^2 = g\frac{A_0}{B_0} = gh_0$$

where h_0 is the mean depth.

For the linearized equations (1) to be useful it is necessary for waves to have exceedingly small amplitudes, so it is desirable to find a better approximation. This can be done by retaining the assumption of uniform conditions over a cross-section, but such an approach leads to equations which are not uniformly valid. That is, the assumption of long waves breaks down, since the equations predict that the front of waves of elevation continually steepens. However, for two-dimensional motion it is known that this only happens for relatively large amplitudes, and that the appropriate uniformly valid approximation is that leading to solutions such as the solitary wave. This includes the effect of the vertical acceleration of the water on the pressure, as well as the next approximation in the amplitude. In addition to these we therefore also include here the effect of transverse water motions in channels which are not rectangular in section. This has been done and mathematical details may be consulted in a recently published paper (ref. 1).

It is found that the transverse and vertical velocities (v,w) are given by

$$\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial y}, \; -\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial z},$$

where $\psi(y,z)$ satisfies

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 1 \tag{2}$$

within the area A_0, with boundary conditions $\frac{\partial \psi}{\partial x} = 0$ on the channel walls and $\frac{\partial \psi}{\partial z} = h_0$ on the free surface. The variations of longitudinal velocity and amplitude across the channel are also in terms of ψ. In particular, the variation of amplitude across the channel is

$$-\frac{\partial^2 \zeta}{\partial x^2}\,\psi(y,0),$$

where $z = 0$ is the undisturbed water level. As may be expected ψ also appears in the equations of motion, which are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \zeta}{\partial x} = 0, \tag{3}$$

$$\frac{\partial}{\partial t}\left(\zeta + \tfrac{1}{2}b\zeta^2\right) + \frac{\partial}{\partial x}[h_0 + \zeta)u] + \frac{\Psi_B - \Psi_A}{g}\frac{\partial^3 u}{\partial x \partial t^2} = 0, \tag{4}$$

where $b = B'(0)/B_0$, if $B(z)$ is the width of the channel at height z, and

$$\Psi_B - \Psi_A = \frac{1}{B_0}\int_{B_0} \psi(y,0)\,dv - \frac{1}{A_0}\iint_{A_0} \psi(y,z)\,dydz.$$

These equations are no more difficult to deal with than the corresponding two-dimensional equations, and in many cases of interest it is simple to transform these equations into the two-dimensional ones. There is, for example, a solitary wave solution, and it is interesting to note that its velocity c is independent of $\psi(y,z)$ and depends only on the geometry of the channel:

$$c^2 = g[h_0 + (1 - \tfrac{1}{3}bh_0)a]$$

where a is the amplitude of the wave.

This theory is of most interest where it departs from the two-dimensional theory and from the first approximation, in particular in the variation of surface level across the channel. One general result can be found fairly easily by considering the boundary conditions on ψ at the shore line. If the shore line is at an angle α to the vertical, measured in the y direction, then the slope of the water surface at the shore,

$$\frac{\partial\zeta}{\partial y} = -\frac{\partial^2\zeta}{\partial x^2}h_0 \tan \alpha.$$

This result shows that the crest of a wave $(\frac{\partial^2\zeta}{\partial x^2} < 0)$ will slope up towards the shore if $\alpha > 0$ and thus reach a higher level than at points away from the shore. Conversely if the banks are over-hanging, i.e. $\alpha < 0$, then the level of the crest will be depressed near the shore. Similarly the depth of a wave trough is increased or decreased near the shore if α is greater than or less than zero respectively.

It is possible to solve equation (2) analytically in a number of simple cases; but for most channels a numerical solution is needed. One particularly simple solution is for any triangular channel where

$$\psi(y,z) = \frac{1}{4}(y^2 + z^2)$$

when the origin of the coordinates is taken at the bottom of the channel. In this case the transverse surface profile is always part of a parabola.

Attempts to solve equation (2) analytically for trapezoidal channels led to an interesting result concerning wide channels. That is, as such a channel gets wider so the difference in crest height between the centre and the shore increases like the width of the channel. Thus, an essential part of the derivation of the solution, that such variations be small, no longer holds. It seems likely that long waves travelling along wide channels will usually break at the shore line.

Figure 1 shows how the function $\psi(v,0)$ varies for a sequence of three channels of different widths. For wider channels the value of ψ at the shore line increases linearly with L.

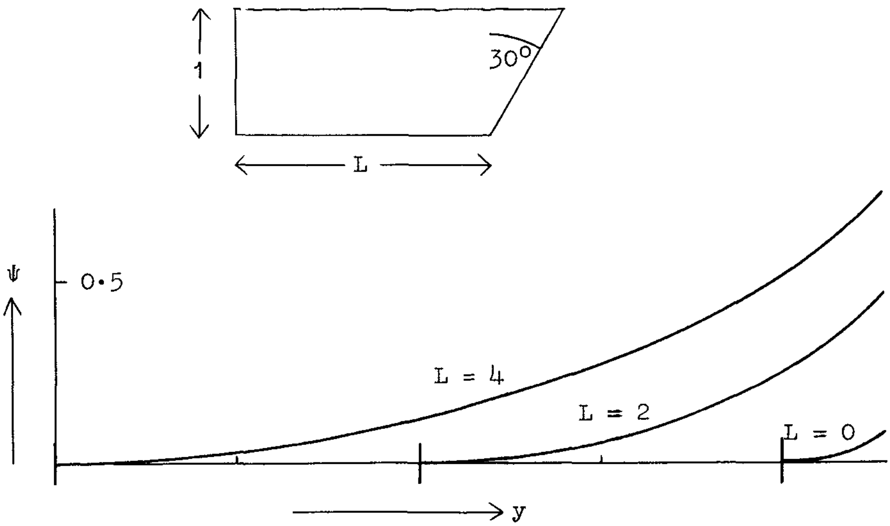

Figure 1 : Values of $\psi(v,0)$ for channels of the cross-sectional form shown at the top of the figure.

Some simple experiments measuring the variation in surface height along the crests of solitary waves have been conducted in three different channels. Two were trapezoidal, with one vertical side and the other side at 30° or 45° to the vertical. The measurements from these channels were in reasonable agreement with the theoretical results. The third channel was triangular, with one side vertical and the other at 60° to the vertical. The results from this latter channel were not very consistent and usually differed substantially from the theoretical values. It is clear from the theory that it is not applicable for shore lines sloping at small angles to the horizontal and it may be that, for waves of the amplitude used, 30° is too small an angle. Further details are given in reference 2.

This theory has been extended to curved channels (Towers, unpublished thesis, University of Bristol 1968). The equations corresponding to equations (1) are

$$\frac{\partial f}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \qquad B_0\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(Q_0 f) = 0, \tag{5}$$

where $f = (1 + \kappa y)u$, $Q_0 = \iint_{A_0} \frac{dydz}{1 + \kappa y}$, and κ is the radius of curvature of the centre-line of the channel. Note that even in the first approximation the velocity is not uniform across the channel, u is the velocity at $y = 0$. As before, x is measured along the channel, y in the horizontal transverse direction and z vertically upwards: however, this now defines curvilinear coordinates which is why $(1 + \kappa y)$ appears in some terms. The surface $y = 0$ is chosen to be midway between the two shore lines. The form of the equations and the values of f, u, κ, Q and x are all slightly different if a different zero is used, but ζ is unchanged, so that it is preferable to look on equations (5) as equations for $\zeta(x,t)$. In this, the first, approximation ζ is constant across the channel.

If $f(x,t)$ is eliminated from equations (5) it is clear that the effect of channel curvature only appears in Q_0. If κ is assumed to be small, which it must be if this long wave approximation is to be consistent, then we may write

$$Q_0 = A_0 - \kappa \iint_{A_0} y \, dy \, dz + \kappa^2 \iint_{A_0} y^2 \, dy \, dz + \ldots \tag{6}$$

If the cross-sectional area is symmetrical about the centre-line the second term in (6) is zero, otherwise it may be non-zero. Hence it appears that the curvature of the channel may have a stronger influence when the channel cross-section is aysymmetrical.

By comparing equations (1) and (5) it may be seen that the effect on wave amplitude of varying curvature along a channel is the same as would occur if the cross-sectional area A_0 varied in a straight channel while B_0 was constant, for example a rectangular channel with varying depth.

The next approximation, which gives the variation of wave amplitude across the channel has been worked out, and solutions obtained for channels of three different cross-sections. The results indicate that these second-order effects are quite small in those circumstances where the theory is most likely to be applicable. For example, in a channel of square cross-section of unit depth with $\kappa = \frac{1}{3}$ and a solitary wave of amplitude of 0·3 the wave crest is only 0·028 higher at the outer edge of the channel than at the inner edge.

References

1. Peregrine, D. H. 1968 Long waves in a uniform channel of arbitrary cross-section. Jour. Fluid Mechanics 32, 353 - 365.

2. Peregrine, D. H. (to be published) Solitary waves in trapezoidal channels. Jour. Fluid Mechanics.