CHAPTER 4

NUMERICAL CALCULATION OF
WIND WAVES IN SHALLOW WATER

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SYNOPSIS
For the purpose of estimating the waves raised by typhoons approaching continental shelf and inland seas, one of the authors (1960) devised graphical method to the forecasting the waves in the fetches travelling over shallow water area in 1960. The method has been widely adopted to evaluate the waves of the bays and inland seas in Japan and the western coast of Taiwan, since it was proved that calculated results considerably agreed with measured records.

On the account of the spread of electronic computers, numerical analysis will be more expedient than graphical operations nowadays. Wilson's numerical integration method (1961)(1962) has been extended to facilitate the calculation of the waves of shallow water area. The procedures of calculation are described and example of hindcasting of waves in typhoon by the machine run are also submitted in this paper.

PROPOSED RELATIONSHIPS
GOVERING SHALLOW WATER GENERATION

Based on the measured data of Bretschneider (1958), the significant wave height $H$ and period $T$ are expressed by the following equations in shallow water.

\[
\frac{gH}{U^2} = \alpha \tanh \left[ k_i \frac{g^2 D}{U^2} \right] \tanh \left[ k_i \frac{g^2 x}{U^2} \right] \quad ... (I)
\]

$U$: wind velocity  
$D$: water depth  
$g$: gravity  
$x$: fetch length

\[
\alpha = 0.26 \quad k_i = 0.01 \quad k = 0.578
\]
\[
\frac{gT}{2\pi U} = \beta \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \frac{1}{2} \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \ldots \ldots (2)
\]

\[\beta = 1.40 \quad k_x = 0.0436 \quad k_x = 0.520\]

Both of the equations are approaching to Wilson's

\[\frac{gH}{U^2} = \alpha \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \ldots \ldots (3)\]

\[\frac{gT}{2\pi U} = \beta \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \ldots \ldots (4)\]

Equations (1) and (2) are illustrated by Fig. 1 and 2.

The ratio between group velocity and wind velocity in shallow water is:

\[
\frac{G}{U} = \frac{1}{2} \left( 1 + \frac{4\pi D/L}{\sinh 4\pi D/L} \right) \frac{gT}{2\pi U} \tanh \frac{2\pi D}{L} \ldots \ldots (5)
\]

Let

\[
S = \frac{gD/U^2}{gT/2\pi U} = \frac{2\pi D}{L_0} = \frac{2\pi D}{L} \tanh \frac{2\pi D}{L} = y \tanh y \ldots (6)
\]

Next, designate

\[
M = \frac{G/U}{\left( gD/U^2 \right)^\frac{1}{2}} \ldots \ldots \ldots \ldots (7)
\]

From eq. (5)

\[
M = S - S^3 - y^3 \frac{2yS^\frac{1}{2}}{2yS^\frac{1}{2}} \ldots \ldots (8)
\]

If S approaches to 0, \( y^3 = S, M = 1 - S/2 \), whereas \( S = [S] \), is the case of deep water wave, the ratio between group velocity and wind velocity is expressed by:

\[
\frac{G}{U} = \frac{1}{2} \frac{gT}{2\pi U} = \frac{\beta}{2} \tanh \left( k_x \left( \frac{gD}{U^2} \right)^\frac{1}{2} \right) \ldots \ldots (9)
\]

In the region of 0 < S < [S], following equation can be approximately established

\[
l - M = a_1 S + a_2 S^2 + \ldots + a_6 S^6 \ldots \ldots (10)
\]

\[
a_1 = 0.4536 \quad a_2 = 0.0931
\]

\[
a_3 = -0.2745 \quad a_4 = 0.17033
\]

\[
a_5 = -0.04760 \quad a_6 = 0.005067
\]
From this equation, in case \( S > \pi, M < 0.288 \), and \( S \) can also be expressed as follows if \( M > 0.288 \)
\[
S = b_1(I-M) + b_2(I-M)^2 + \cdots + b_7(I-M)^7 \quad \cdots \cdots (II)
\]
\[
b_1 = 2.464857 \quad b_2 = -7.35305
\]
\[
b_3 = 52.74583 \quad b_4 = -162.2
\]
\[
b_5 = 275.83 \quad b_6 = -247.2
\]
\[
b_7 = 101.19046
\]

The group velocity and period of shallow water wave can be calculated when \( S \) and \( M \) are worked out.

**CALCULATION PROCEDURES**

The calculation of waves in shallow water is also to be carried out by stepwise method from the lattice of wind field, only evaluating \( S \) instead of calculating period or celerity directly. At the initial point of the fetch, Wilson's method will be adopted all the same. If the wave height \( H_a \), group velocity \( G_a \) at the point "\( a \)" on the space time wind field are known, the problem is to calculate the wave features of the point "\( b \)" in the lattice as shown in Fig. 3.

At first, compute the velocity \( U_a \) of the wind blowing over point "\( a \)" from the lattice or by a formula \( U = U(x, t) \), which can be derived from the pressure distribution of typhoon or hurricane as well as its moving direction and velocity, also the water depth of this point \( D_a \) is to be determined from the lattice or some approximate function \( D = D(x, t) \).

Secondary, calculate
\[
Ma = \frac{Ga}{Ua} \left( \frac{gD_a/U_a}{\sqrt{g}} \right)^2
\]
if \( M < 0.288 \), the waves at this point are still to be deep water wave, and Wilson's method should be used, whereas \( M < 0.288 \) following procedures are to be adopted.

**CALCULATION IN CASE OF WAVE DEVELOPMENT**

By differentiating Eq. (I)
\[
\frac{dH}{dx} = \frac{k^2}{\alpha} \times \frac{\frac{\alpha \tanh k(xD/U)^{1/2} + gH/U}{\tanh k(xD/U)^{1/2} - gH/U} - \ln(\frac{\alpha \tanh k(xD/U)^{1/2} + gH/U}{\tanh k(xD/U)^{1/2} - gH/U})}{\tanh k(xD/U)^{1/2} \ln(\frac{\alpha \tanh k(xD/U)^{1/2} + gH/U}{\tanh k(xD/U)^{1/2} - gH/U})}
\]
\[
\cdots \cdots \cdots (I2)
\]
at point "\( a \)", if the waves are developing, it must be that \( \alpha \tanh k(xD/U)^{1/2} > gH/U \), and the wave height \( H_b \) can be calculated by:
\[
H_b = H_a + \left( \frac{dH}{dx} \right)_a \Delta x
\]
the choosing of \( \Delta x \) is the same as Wilson's method.
Prior to the evaluation of group velocity, calculate $S_a$ from $M_a$ by eq. (11).

Since

$$ S = \frac{g D}{U^2} \left[ \frac{g T}{2 \pi U} \right]^2 $$

from eq. (2)

$$ S = \frac{g D}{U^2} \left[ \beta \tanh k_x \left( \frac{g D}{U^2} \right) \right] \tanh \left[ \frac{k_x (g x / U)}{\tanh k_x (g D / U)} \right] $$

therefore

$$ \frac{dS}{dx} = \frac{8k_x g}{3 \pi U^2} \left[ \frac{1}{(1 / S x g D / U)^k} \right] \tanh \frac{k_x (g D / U)}{\tanh k_x (g D / U)} \times $$

$$ \left[ \beta \tanh \left[ k_x (g D / U)^k + (1 / S x g D / U)^k \right] \tanh \left[ k_x (g D / U)^k -(1 / S x g D / U)^k \right] \right] $$

For developing waves, naturally $\beta \tanh \left[ k_x (g D / U)^k \right] > (1 / S x g D / U)^k$, and $S_b$ will be determined by following equation,

$$ S_b = S_a + \left( \frac{dS}{dx} \right)_a $$

(15)

$M_b$ can be calculated by eq. (10) and

$$ G_b = M_b \frac{U_a}{U_a^2} \left( \frac{g D_a}{U_a^2} \right)^{1/2} = M_b \left( \frac{g D_a}{U_a} \right) $$

(16)

also

$$ T_b = \left( \frac{4 \pi D_a}{G_b} \right)^{1/2} $$

(17)

CALCULATION IN CASE OF WAVE DECAY

If $\beta \tanh \left[ k_x (g D / U)^k \right] < \beta H / U^2$ and/or $\beta \tanh \left[ k_x (g D / U)^k \right] < (1 / S x g D / U)^k$ are recognized at point "a", it means that the wave series whose height is $H_a$ reaching this point with group velocity $G_a$ can not grown any more under the circumstance $U_a$ and $D_a$. In other words, the waves have already larger than the wind $U_a$ can generate in shallow water area of depth $D_a$. As shown in Fig. 4, $H_a$ is located above the curve $H(U_a, D_a)$. In such a case, following consideration are being made.

1) If the wind of velocity $U_a$ blew over deep water area, the wave height would increase $\Delta H_a$, while the fetch was being prolonged $\Delta x$, $\Delta H_a$ can be calculated by

$$ \Delta H_a = \left( \frac{dH}{dx} \right)_a \Delta x $$

(18)

$$ \Delta H_a = \frac{k^2 (\alpha + g H_a / U_a) (\alpha - g H_a / U_a)}{\alpha \ln (\alpha + g H_a / U_a) - \ln (\alpha - g H_a / U_a)} \Delta x $$

2) Actually the waves should decrease their height for being suffered by bottom friction at shallow water of depth $D_a$. It is necessary to evaluate the wind velocity $U_a$ which makes the fully arisen wave height just equals $H_a$ in depth $D_a$ as shown by the curve OM in Fig. 4, from eq. (1), let

$$ \frac{g H_a}{U_a} = \alpha \tanh \left[ k_x \left( \frac{g D_a}{U_a}^k \right)^k \right] $$

(19)
Uₚ can be found out by Newton’s method then, ΔH₂ can be calculated as follows

\[ ΔH₂ = \frac{k²}{\alpha} (a + gHₚ/Uₚ^2)(a - gHₚ/Uₚ^2) \alpha \ln(a + gHₚ/Uₚ^2) - \ln(a - gHₚ/Uₚ^2) \Delta x \]  (20)

While the wind of velocity Uₚ is acting on a wave series with a height Hₚ over Δx length in deep water, the height should be increased ΔH₂, however, in shallow water of depth Dₚ, the wave height remain constant, the energy obtained from the wind and lost due to bottom friction are in equilibrium, namely, the lost height of wave series of height Hₚ travelling over Δx is ΔH₂. So that Hₚ is:

\[ Hₚ = Hₚ + ΔH₂ - ΔHₚ \]  (21)

The same consideration can be applied for evaluating group velocity, calculate ΔG, from following equation first.

\[ ΔG₁ = \frac{8 k² g}{3 \beta Uₚ^2} \left[ \ln(\beta/2 + gHₚ/Uₚ^2) - \ln(\beta/2 - gHₚ/Uₚ^2) \right] \Delta x \]  (22)

The relationship of G/U and gD/U when x→∞ can be approximately expressed as bellow:

\[ gD/U > 0.06 \quad \frac{G}{U} = \frac{\beta}{2} \tanh\left( kₚ \frac{gD}{Uₚ^2} \right) \]  (23)

\[ gD/U ≤ 0.06 \quad \frac{G}{U} = \frac{\beta}{3} \left( \frac{gD}{Uₚ^2} \right)^½ \]  (24)

The wind velocity Uₚ which makes the fully arisen group velocity at depth Dₚ just equals Gₚ can be worked out by solving eq. (23) or (24) and ΔG is

\[ ΔG₂ = \frac{8 k² g}{3 \beta U} \left[ \ln(\beta/2 + gHₚ/Uₚ^2) - \ln(\beta/2 + gHₚ/Uₚ^2) \right] \]  (25)

Gₚ will be calculated by

\[ Gₚ = Gₚ + ΔG₁ - ΔG₂ \]  (26)

The period can be calculated as follows.

\[ Mₚ = \frac{Gₚ}{Uₚ} \left( \frac{gDₚ}{Uₚ^2} \right)^½ \]  (27)

\[ Tₚ = \frac{4πDₚ}{gSₚ} \]  (28)

\[ S = b (1-M) + b(1-M)^2 + ... + b(1-M)^n \]  (29)

The position of point 'b' on the lattice are determined by following equations.

if \[ Gₚ > \lambda/\gamma \quad \Delta x = \lambda \quad \Delta t = \gamma/Gₚ \]  (30)

if \[ Gₚ < \lambda/\gamma \quad \Delta t = \gamma \quad \Delta x = Gₚ \gamma \]  (31)

\[ xₚ = xₚ + \Delta x \quad tₚ = tₚ + \Delta t \]  (32)

Same Procedures will be applied to calculate waves of Point 'c' from point 'b'. The flow chart is to be used for Programming as Fig.5.

DISCUSSION ON THE EFFECT OF REFRACTION

The refraction effect of waves in shallow water must not be neglected. In following examples, calculations on refraction have been made by amending the contour line to...
be parallel. The difference is only a few percent both in wave height, group velocity and wave propagation line in comparison with the result of calculation without considering refraction. It is not unnatural because the refraction of waves is caused by decreasing of celerity as the waves advancing to shallow water, however, in this calculation, the decrement in wave celerity owing to shoaling is almost balanced by the increase from wind effect. The wave celerity remains nearly constant, therefore the refraction effect seems not to be appeared. This is very noticeable phenomenon in wind waves of shallow water, further investigations are needed.

**Calculation Example**

The waves along the northern coast of Seto Inland Sea raised by typhoons "Suô" (Aug. 1946) 'Ruth' (Oct. 1951) 'Doya' (Sept. 1954) as well as the waves attacked western coast of Taiwan caused by Typhoon 'Parmela' (Sept. 1961) have been hindcasted by this method. The result of the calculation of typhoon "Suô is submitted here.

**Fundamental Conditions**

The route of typhoon and topographical features of western part of Seto Inland Sea are described in Fig. 6, the fetch length of various direction of every calculated point are also shown in the same figure.

Along this coast, the tidal range is rather large, extraordinary high tide will be recognized as the typhoon center approaching, in this calculation, the deviation of water level by the extrahigh tide has been considered and added into water depth.

Waves diffracted from outer sea have not been considered in this calculation. All waves to be calculated are generated in shallow water area.

**Calculation Conditions**

In general, the pressure distribution in typhoon cycle is as below.

\[ P = P_c + a \exp(-\frac{r}{r_0}) \]  

\( P_c \): pressure at typhoon center (m b)  
\( r \): distance from typhoon center (km)  
\( r_0 \): radius of the largest gradient wind velocity circle (km)  
\( P \): pressure at the circle with radius \( r \)  
\( a \): constant

\( a \) and \( r \) are different in each typhoon.
The gradient wind velocity $V_g$ is

$$V = \sqrt{\frac{a}{r}} \exp \left( -\frac{r}{2\gamma} \right) - \frac{fr}{2} \quad \ldots \quad (34)$$

$\gamma$: air density, $f = 2\omega \sin \phi$ : Coriolis coefficient

Actual wind in typhoon is the resultant of symmetrical wind $U'$ and wind of field $U$, taking typhoon center as the origin, wind velocity of the point $(x, y)$ can be calculated from the following equations.

$$U_x = U_x + U_x', \quad U_y = U_y + U_y' \quad \ldots \quad (35)$$

$$U_x = -0.6(V_g r)(x \sin \alpha + y \cos \alpha) \quad \ldots \quad (36)$$

$$U_y = 0.6(V_g r)(x \cos \alpha + y \sin \alpha) \quad \ldots \quad (37)$$

$$O_x = (0.6 V_{g\max}) V_g V_x \quad \ldots \quad (38)$$

$$O_y = (0.6 V_{g\max}) V_g V_y \quad \ldots \quad (39)$$

$\alpha$ is the angle of symmetrical wind direction and the tangent of isobar. It will be different in latitude, in this calculation $\alpha = 30^\circ$ is adopted. $V_x, V_y$ are the components of progressing velocity of typhoon.

The origin of fixed coordinate is set at 131 E and 33°4 N, EW and NS direction are taken as X-Y- axes respectively. If the linear fetch is at an angle of $\beta$ to the X axis the component of wind velocity can be calculated by following equation

$$U = U_x \cos \beta + U_y \sin \beta \quad \ldots \quad (40)$$

The positions of the center of typhoon "Suō", when she was in the vicinity of Seto Inland Sea at every hour, are listed below

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<th>Date</th>
<th>hour</th>
<th>X(km)</th>
<th>Y(km)</th>
</tr>
</thead>
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<td>14</td>
<td>-101</td>
<td>-207</td>
</tr>
<tr>
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<td>15</td>
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<td></td>
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<tr>
<td></td>
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</tr>
<tr>
<td></td>
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<tr>
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<td>201.5</td>
</tr>
</tbody>
</table>

During calculation, the unit distance on lattice of wind diagram is to be 2km windward and 1km in the region of depth less than 10m leeward, but the time unit $T$ is remaining 30 minutes.

**CALCULATION RESULTS**

Wave which attacked the estuary of Yoshida river and other point from various direction have been calculated by
the electronic computer. Fig. 7,8 illustrated the waves on the SE fetch of Yoshida estuary.

In addition, for the purpose of investigating the distribution of waves over the west part of Seto Inland Sea, a number of parallel linear fetch with a distance of 10km have been set as shown in Fig. 6. Waves on such fetch lines have been calculated and the contour of wave heights and periods are to be delineated for every hour as shown in Fig. 9 and 10.

REFERENCES

2) Ijima, Sato, Aono "Waves raised by Typhoon Ise" Ibid. (In Japanese) (1961)
Fig. 1. Relation of wave height and wind for shallow water.

Fig. 2. Relation of wave period and wind for shallow water.
Fig. 3. Process of calculation.

Fig. 4. Calculation of wave height decrease.

Fig. 5. Flow chart
Fig. 6. West part of inland sea of Seto, and locations of linear fetch.

Fig. 7. A result of numerical calculation Yoshida River Estuary, typhoon Suō Nada SE.

Fig. 8. Time change of wave height and period.
Fig. 9, 10. Distribution of wave height and period by assumed typhoon "Ise Wan" taking the route of typhoon "Suō".

Fig. 9: Distribution of wave height and period by assumed Typhoon "Ise Wan" taking the route of Typhoon "Suō".

SUŌ 17°N Aug. 27

WAVE HEIGHT
WAVE PERIOD