Chapter 19

HARMONICS OF A WAVE

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INTRODUCTION

In a hydraulics laboratory, waves are produced generally by a flap generator ; such a wave propagates in an open canal with a horizontal bottom and with a rectangular section. Because of the importance of its applications in practice, the detailed study of such a wave has a major interest. Considerable efforts are devoted, at present, to the production of a wave as pure as possible.

Experience has shown, that in spite of all precautions taken, the wave is propagated always with some of its harmonics. It is useful to investigate analytioally, on the basis of Stokes' linear theory, the conditions in which these harmonics originate. The conclusions of the theory presented here are not yet tested experimentally ; it forms part of a programme of research of the Laboratoire de Mécanique des Fluides of the Institut Polytechnique of Grenoble.

Most of the results presented here have been already published ; a bibliography is given at the end of this article. It was thought desirable, however, to give a succinct review of the question, in the hope that it might be useful to the engineer.

POSITION OF THE PROBLEM

We shall take as starting point, the theory of the production of a linear wave by means of a generator, as initiated by Havelock and developed recently by Biesel. We shall recall here only the boundary condition which expresses analytically, the action of the wave-machine paddle on the liquid.

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We shall suppose in conformity with the usual assumptions of Stokes' theory that the paddle, considered rectilinear for simplicity, oscillates around a mean position, which can be regarded vertical. Whatever may be the kinematical construction of the machine, we may suppose, at least as a first approximation, that the immersed portion of the paddle occupies the entire depth of water in the wave-canal. The motion of the liquid will be considered as two-dimensional, which means neglecting the wall-effects and the effects of leakage. Finally, and it is the fundamental hypothesis, we suppose that the horizontal component of velocity of a point on the oscillating paddle is given by the formula :

$$u = f(y) \cos 2\pi \frac{t}{T} \tag{1}$$

correct to the second order ; T is the period of the generator, y is the vertical distance of a point on the paddle in its mean position measured from the bed of the canal, and f(y) is a function determined by the mechanical characteristics of the generator and of the mode of its articulation.

In the case of the piston-type generator performing a simple sinusoidal translational motion parallel to the bed of the canal, $f(y) = e$; while for a generator consisting of a paddle which oscillates about an axis situated in the bed of the canal $f(y) = (y/k)e$. In both cases, the constant e is the maximum elongation of the paddle ; obviously e is small, so also is $|f(y)|$.

If the wave-machine satisfies these kinematic properties, then the formulae of Havelock and Biesel give, for every function f(y), the amplitudes of a simple, monoperiodic wave (period T) generated by the movement of the paddle. It is necessary, however, to note that the theory does not take account of the production of harmonics when the movement of the generator is described by (1).

It should be remembered that the theoretical formulae giving the amplitude a of the wave in terms of kinematic parameters of the generator, are in accordance with experiments. One must not, however, expect a perfect accord between calculated values and measured values of a, since the wave-machines are very imperfectly water-tight ; this is not the only reason for the difference between theory and experiment. But it is not our aim here to discuss the practical importance of Havelock's theory ; it is rather the generation of harmonics.

For the practical engineer, the question takes the following form : Does a given wave-generator satisfy a priori the assumptions of Havelock's theory ? Or, in particular, is the movement of the generator governed by equation (1) ?

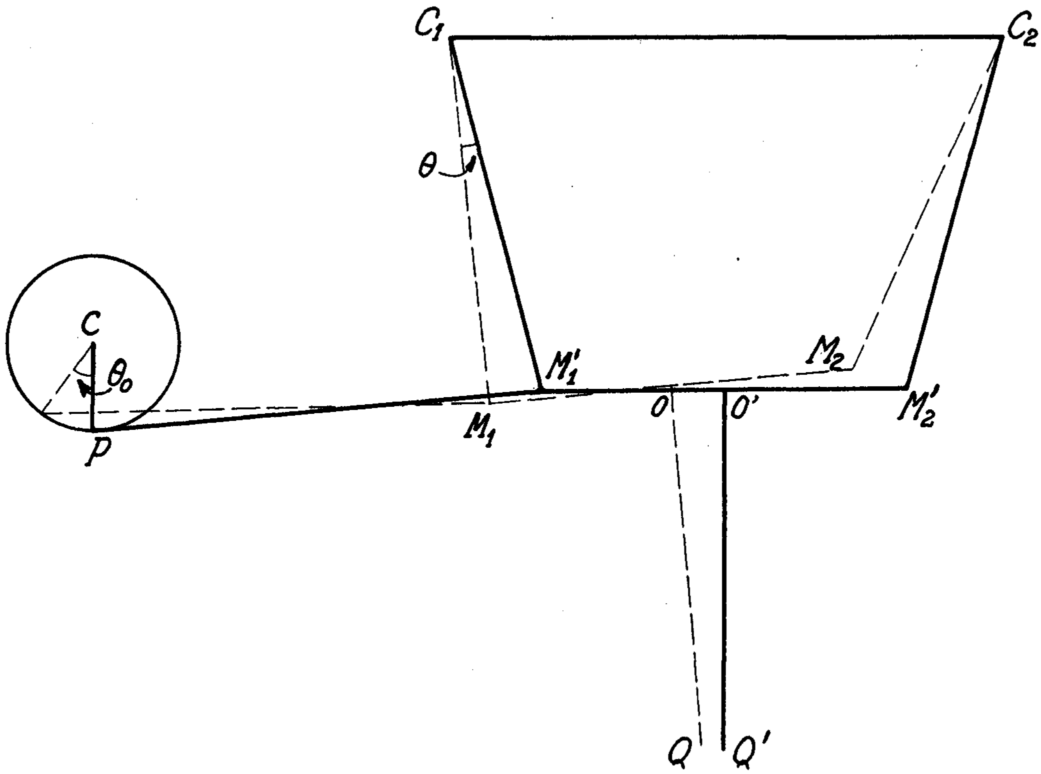

Fig. 1. Wave-generator: "three-bar" type

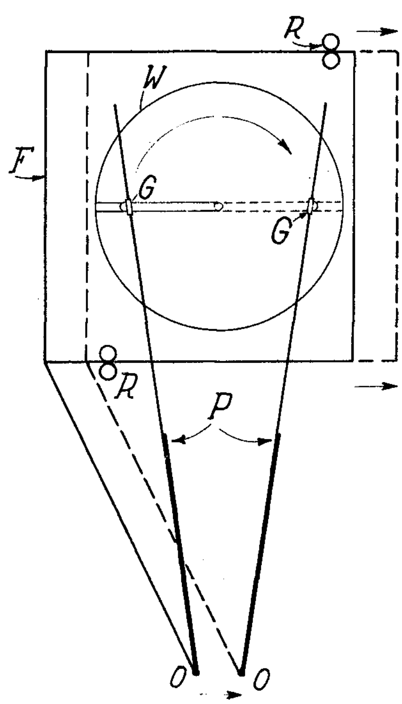

Fig. 2. Wave-generator: Chatou type

Frame F supporting the wave-paddle P; the frame makes horizontal sinusoidal oscillations on the rollers R. Paddle hinged onto frame at O.

Crank wheel W, on which is hinged eccentric guide G; upper part of paddle P slides on G.

We have studied two types of wave-generators from this point of view. The following sections give the conclusions of our theoretical research.

STUDY OF A THREE-BAR WAVE-GENERATOR

A common type of wave-generator is based on the three-bar mechanism. The wave-machine in the Laboratoire de Mécanique des Fluides is symmetrical with respect to the vertical, when the generator is in its mean position. The plane flap situated in the middle of the connecting-rod is thus in the plane of symmetry of the apparatus.

A moving-rod PM_1 is jointed to a common point of the two planes C_1M_1 and M_1M_2 (Fig. 1). The other extremity of the moving rod PM_1 is joined to the excentric P, which turns with a uniform velocity.

The study of this mechanism is made in two stages ; in the first we neglect the law of movement as a function of time ; we only consider the geometric form of the trajectories of points on the flap OQ. Since the amplitude of oscillations of the generator is small, suitable approximations can be made. In the course of the second step we consider the displacements of the generator OQ as a function of time.

The conclusions arrived at are the following : From the geometrical point of view the trajectories of points situated on the generator OQ have been known since a long time to be small arcs of curves of the sixth order, called trimochloides. A complete study of the set of these curves is quite complicated ; but, fortunately, each of the relevant parts of the curves can be approximated by an arc of a unicursal cubic. We have obtained parametric equations of this arc of the cubic in terms of the angle θ , contained between any position of the plane C_1M_1 and its mean position.

From the kinematical point of view, we have expressed θ in terms of the angle θ_0 between the crank CP of the motor and a fixed direction ; this latter angle θ_0 is a linear function of time, when the movement is uniform. We may write $\theta_0 = \omega t$. It remains now to substitute this value of θ in the parametric equations of the trajectories, giving thus the coordinates of points on the generator OQ as a function of time. Consider the following system of axes of reference O'xy, with O' at the bed of the canal and O'y vertically upwards. In this system of axes, the abscissa x of a point of the generator will be given by a formula of type :

$$x = E_0 + \sum_1^\infty E_n \sin\left(n\omega t + \alpha_n\right) + y' \sum_1^\infty F_n \sin\left(n\omega t + \beta_n\right) \qquad (2)$$

where E_o, E_n, F_n are constants depending on the dimensions of the mechanism and y' is the ordinate of the mean position of the point. From this the horizontal movement of the generator results from the superposition of elementary translations :

$$x_n = E_n \sin (n \omega t + \alpha_n), \qquad \omega = \frac{2\pi}{T} \tag{3}$$

and of elementary rotations :

$$x'_n = y' F_n \sin (n \omega t + \beta_n) \tag{4}$$

around the origin O'. Thus the total movement results from the superposition of an infinite number of generators of the piston and oscillating paddle type. Evidently the amplitudes E_n and F_n are very small and decrease rapidly with 1/n. In my article referred to below, explicit formulae are given for n = 1, 2, 3 ; from n = 4 onwards the amplitudes lie outside the limits of sensibility of the current apparatus. From (3) and (4) we have the horizontal velocity components :

$$u_n = \frac{dx_n}{dt} = n \omega E_n \cos (n \omega t + \alpha_n), \tag{5}$$

$$u'_n = \frac{dx'_n}{dt} = n \omega F_n \cos (n \omega t + \beta_n). \tag{6}$$

we see thus that the characteristic function f(y) is equal to $n \omega E_n$ and $n \omega y' F_n$ respectively for the piston-type and for the oscillating-type generator.

After having established this we may go back to the boundary-value problem of Havelock. It imposes homogenous boundary conditions on the unknown potential function Φ (x,y) everywhere except on the paddle of the wave-generator. Along this we may write for the derivative $\partial\Phi/\partial x$:

$$u = \frac{dx}{dt} = \frac{\partial \Phi}{\partial x} ,$$

where u is given by (2). It follows that the solution Φ is arrived at by a summation of ϕ_n and ϕ'_n corresponding to boundary values u_n and u'_n given by (5) and (6). Here the particular solution ϕ_n corresponds to a wave of period T/n produced by a piston-type generator and ϕ'_n corresponds to a wave generated by an oscillating paddle of the same period.

The presence of harmonics can be thus theoretically explained. That there are no other causes of the generation of harmonics can be asserted only if the calculated amplitudes coincide with the observed amplitudes.

WAVE-GENERATOR DESIGNED BY M. GRIDEL

We have referred above to a second type of wave-generator. It is a wave-generator designed by M. Gridel at the Laboratoire National d'Hydraulique at Chatou (Paris) and is based on entirely different principles. It is not our purpose here to describe the apparatus in detail ; we give its essential characteristics.

The paddle of the generator is plane (Fig. 2). The upper part is moved by an eccentric, sliding on the flap and having a uniform circular movement of period T. Its lower extremity describes a simple rectilinear oscillating motion on the bed of the wave-canal with the same period T. But it does not result, unfortunately, that for every point of the generator an equation of the form (1) is satisfied. It appears, however, that the idea of M. Gridel reduces the importance of parasite harmonic waves.

CONCLUSION : EXPERIMENTAL VERIFICATION

To verify the theoretical conclusions stated above, the Laboratoire de Mécanique des Fluides at Grenoble have the following plan of research.

The wave-recorder of MM. Santon and Marcou described in the communication of Prof. Santon enables us to make a harmonic analysis of the periodic phenomena of the wave. It gives not only the amplitudes of the different components, but the phases of each harmonic as well. In view of the sensibility of the apparatus, one can follow the evolution of the progressive waves of period T/3 ; for harmonics of higher orders the experimental results are uncertain.

We may also note the precautions to be taken. The wave canal presents many complex phenomena ; with a large wave amplitude, there appear in certain frequency-bands, parasite phenomena like a transversal clapotis. Harmonic analysis becomes more difficult. Under these conditions we are also not sure of the linearity of the phenomena. We may consequently diminish the wave-amplitude, thereby diminishing the intensity of the harmonics, rendering the phenomena difficult to observe. A precise accord between theory and experiment cannot therefore be expected and we have to content ourselves only with the order of amplitudes calculated by means of the formulae of M. Biesel.

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RESUME

HARMONIQUES DE LA HOULE PROGRESSIVE DANS UN CANAL

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Poursuivant des études théoriques et expérimentales sur la propagation des houles planes dans les canaux à sections rectangulaires, aux Laboratoires de Mécanique des Fluides de l'E.N.S.E.H. de Grenoble, nous avons été amenés à chercher l'origine des harmoniques dont on décèle toujours la présence dans la houle réelle. La question est à la fois d'ordre théorique et expérimental. Notre mémoire insiste surtout sur l'aspect théorique des phénomènes, mais nous donnons aussi des indications sur les recherches expérimentales en cours d'exécution, destinées à contrôler nos prévisions.

Au laboratoire, le générateur de houle est le plus souvent un batteur accouplé par un système de bielles à l'arbre d'un moteur, tournant à une vitesse uniforme. Le batteur que nous avons étudié repose sur le principe d'un "trois-barres". Dans une série de travaux, J. Kravtchenko, L. Santon et moi-même avons étudié le mouvement du volet au point de vue cinématique.

Eu égard à la faible amplitude des oscillations du volet, on constate que les portions utiles des trajectoires des points du corps peuvent être, sans erreur sensible, assimilées aux cubiques unicursales surosculatrices aux trajectoires (qui sont, comme on sait, des courbes du sixième ordre). Cette conclusion est indépendante de toute hypothèse sur le mouvement du moteur. Lorsque celui-ci tourne à une vitesse uniforme, la composante horizontale du vecteur-vitesse d'un point quelconque du volet peut se décomposer en séries de Fourier suivant ωt, où ω est la pulsation du moteur et t le temps. Ce développement se décompose en deux séries de termes. Le nième terme de la première série peut être interprété comme la composante de la vitesse d'un batteur piston fictif animé d'un mouvement sinusoïdal de translation horizontale, de pulsation ω. Le nième terme de la seconde série s'interprète comme la composante horizontale de la vitesse d'un batteur tournant fictif, animé d'un mouvement oscillatoire de rotation autour d'un axe horizontal situé au fond du canal, normal au plan de symétrie de celui-ci et dont la position ne dépend pas de n ; la pulsation du mouvement oscillatoire correspondant est $n\omega$. Or, des théories récentes, dues principalement à Havelock et à Biésel, donnent alors les moyens de calculer l'amplitude et la phase de la houle linéaire, plane de Stokes, engendrée par chacun desvolets fictifs que l'on vient de définir et que nous supposons être plans.

A chacun des mouvements sinusoïdaux simples, ci-dessous, correspond donc une houle de Stokes bien déterminée. La naissance des harmoniques se trouve ainsi être complètement expliquée.

La vérification expérimentale est faite au moyen de l'enregistreur à houle de Santon et Marcou. L'analyse harmonique de la houle réelle que cet appareil permet d'effectuer donne au moins l'ordre de grandeur des phénomènes. C'est là un résultat très encourageant si l'on songe aux approximations consenties pour édifier la théorie de la propagation d'une houle réelle dans un canal.