Chapter 3

WIND TIDE AND SEICHES IN THE GREAT LAKES

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INTRODUCTION

Because of the unusually high lake stages of recent years, the Weather Bureau was called on to forecast the short period variations of lake level, which were believed to be caused by wind stress and atmospheric pressure gradient. It became necessary to investigate the feasibility of such forecasts. In a review of the available literature, many papers were found which described methods of computing the free periods of oscillation for lakes when no external forces were acting. Other papers were found which described methods of computing the steady state relation between a constant atmospheric force and the lake surface when inertial forces are neglected.

But in nature, a steady state different from the equilibrium state is rarely achieved, and neither the external forces nor the inertial forces can be neglected in any attempt to relate the atmospheric disturbances to lake level disturbances.

A theory which relates the build up of the water level disturbance to the concurrent or previous weather conditions is needed, if forecasts are to be prepared with any degree of confidence.

This report gives the preliminary results of an effort to develop such a theory. Since the theoretical study requires some knowledge of the phenomenon to be explained, the first part of the report is devoted to a description of observed water level disturbances. The middle section gives the essential development of the hydrodynamic equations leading to a more generalized theory of the forced oscillations of a lake. This is followed by a short discussion of the consequences of the extended theory.

Most of the mathematical treatment not required for an understanding of the physics of the problem has been relegated to the appendix.

In scientific discussion the term "seiche" is usually restricted to the inertial oscillations which persist after the external force has ceased to act. In popular usage this term is frequently applied to any disturbance which is believed to be produced by some meteorological force and whose period is longer than that of the surface waves.

In the Great Lakes, these disturbances appear to fall naturally into two classes: those which involve all or a large part of a major lake, and those of a more local character. The first class is illustrated by figure 1. The surface water in Lake Erie is driven toward the eastern end of the lake by the wind. This causes an increase in the water level at Buffalo and a decrease in the water level at the western end of the lake. After the wind shifts or decreases in speed, the lake undergoes a series of damped oscillations as it returns to normal.

The second class is illustrated by figure 2. This is a tracing of the actual water level record for Two Harbors, Minn. for a seven-hour period on May 5, 1950. This oscillation appears to be of a more local character than that shown in figure 1.

For the purposes of this report, disturbances of the first class will be referred to as long period and those of the second type as short period. The division appears to occur with a period of about one hour.

THE DATA

In order to test the various hypotheses that have been proposed for the generation of lake level disturbances, all of the continuous lake level records of the U. S. Lake Survey for the year 1950, and selected portions of the records of several other agencies and for other periods have been examined.

The ten or fifteen most prominent disturbances at each of the Lake Survey gages during 1950 were compared with synoptic weather charts and other meteorological data. The location of gages used in this part of the study is shown in figure 3. It was found that short period disturbances tend to occur in zones of disturbed weather, that is, near fronts, squall-lines, thunderstorms, etc., and that long period disturbances seem to conform to the classical picture of the surface water being driven to the leeward end of the lake by high winds.

Both long and short period disturbances occur on all lakes. However, the long period disturbances are most prominent on Lake Erie, and the short period disturbances are most prominent on the other lakes. Accordingly, the detailed study of long period disturbances was largely confined to Lake Erie. Southern Lake Michigan was selected for a detailed study of the short period disturbances because of the relatively large number of lake level gages in the neighborhood of Chicago.

SHORT PERIOD DISTURBANCES

The location of gages used in this study is shown in figure 4. The Wilson Avenue crib gage is located about 3 miles off shore in the open lake. Two gages, Chicago River and Navy Pier, are located in the Chicago Harbor. The Filtration Plant gage is located in a small basin protected by a breakwater open at both ends. The Calumet Harbor gage is located in the Calumet River. The Waukegan gage, also used in this study, is just inside the Waukegan harbor about 35 miles north of Navy Pier.

Figure 5 shows an example of a short period disturbance as recorded by four of these gages. It should be noticed that the agreement between the two records for Chicago Harbor, is much greater than that between these records and those of the Filtration Plant and Calumet Harbor.

Figure 6 gives a comparison between the records of a disturbance as recorded at the Wilson Avenue Crib in the open lake, and the Calumet and Waukegan harbors. It should be observed that the amplitude of the disturbance is much greater at the harbor gages than at the crib gage. This relation also holds for other harbors in this area and appears to be typical of at least the largest disturbances in the records examined.

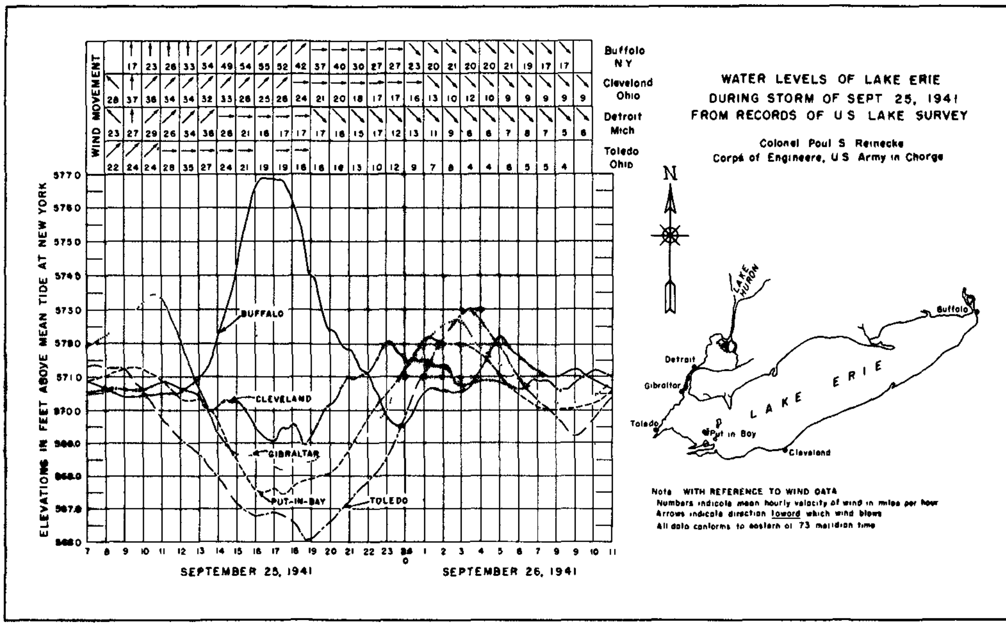

Fig. 1. Lake Erie wind tide, adapted from original drawing by U.S. Lake Survey.

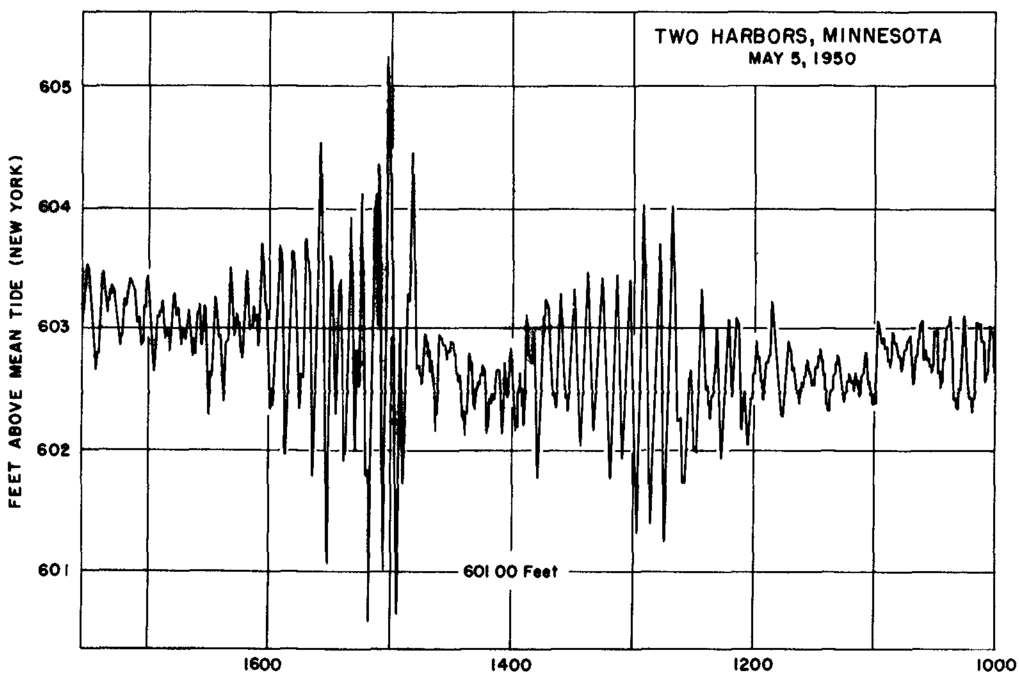

Fig. 2. Example of short period harbor oscillation at Two Harbors, Minn.

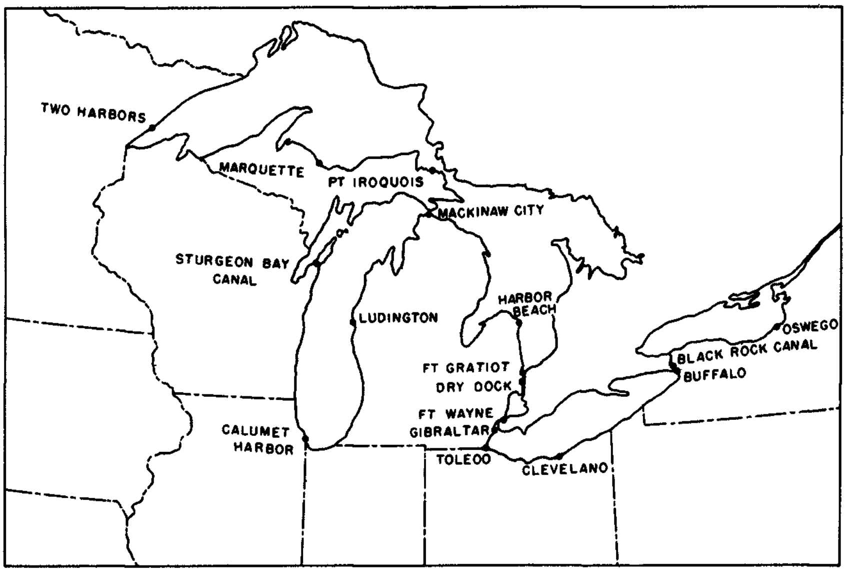

Fig. 3. Locations of U.S. Lake Survey gages used in the study of wind tides and seiches.

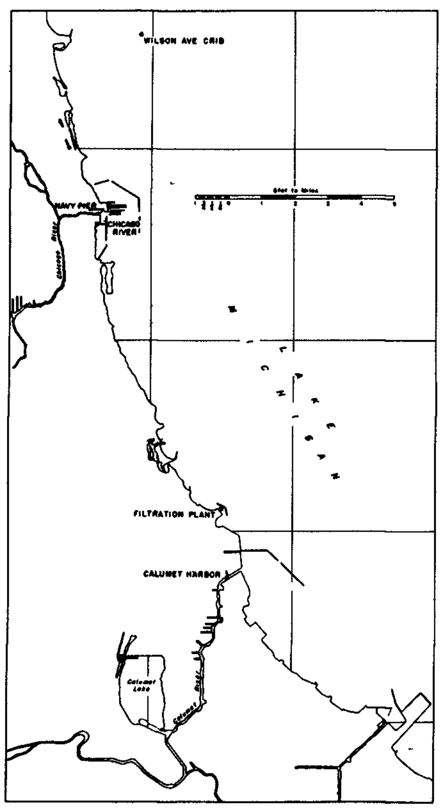

Fig. 4. Location of lake level gages in Chicago.

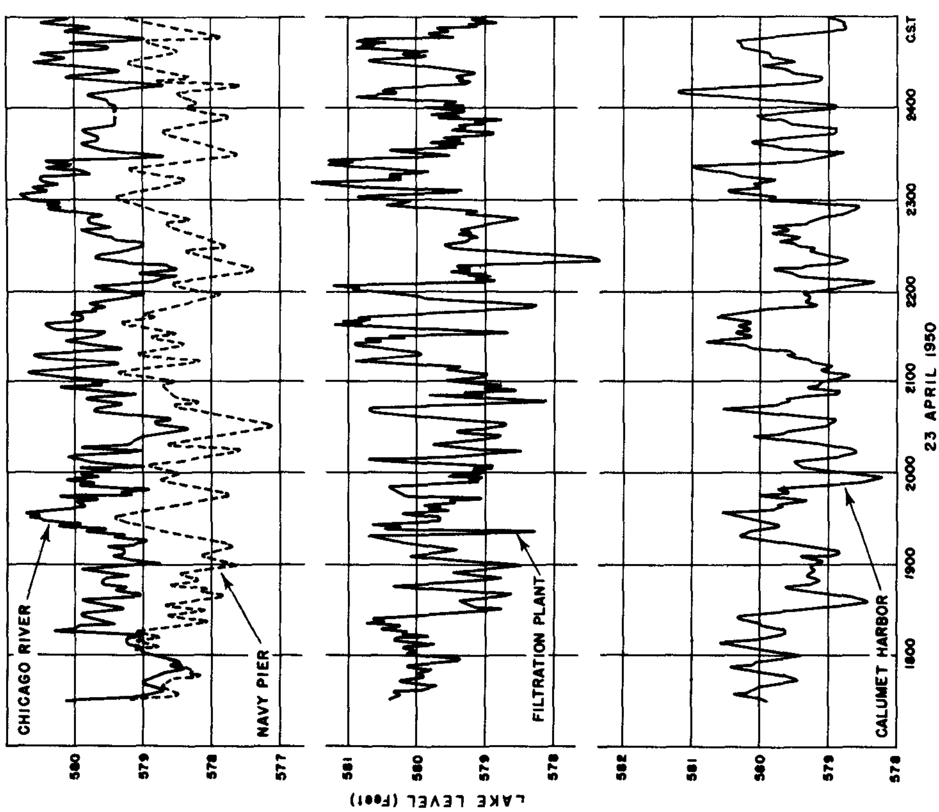

Fig. 5. Comparison of lake level at four Chicago gages.

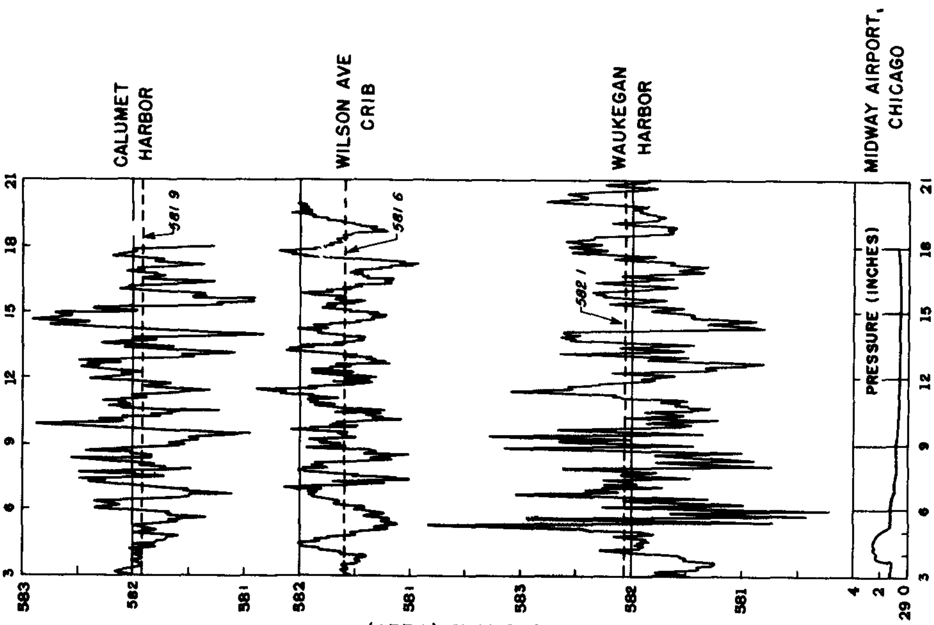

Fig. 6. Comparison of lake level records, as recorded in the open lake and in two harbors, and the barograph record, June 8, 1953.

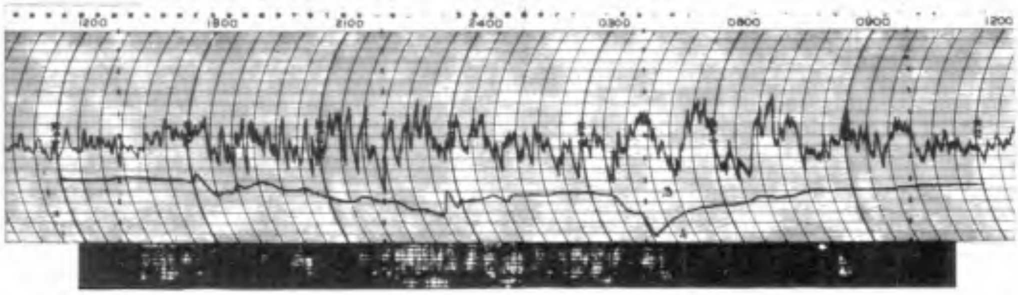

Fig. 7. Comparison of the records of lake level, pressure, and wind speed as recorded at the 79th Street Filtration Plant, Chicago, April 23-24, 1950.

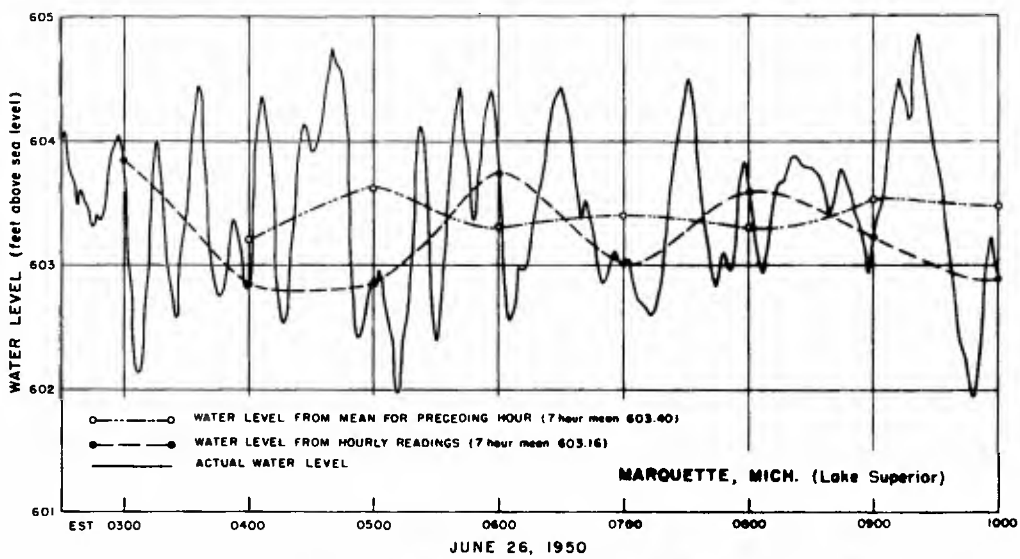

Fig. 8. Water level variations at Marquette, Mich., June 26, 1950.

It should also be noticed that the mean water level at the harbor gages is about .3 to .5 feet higher than that recorded at the crib gage. This tendency for the mean water level to be higher in a harbor than in open water has been reported by McNown (1952) and will be mentioned again.

The possibility that these short period disturbances may be produced by changes in atmospheric pressure has often been mentioned. A well organized atmospheric pressure wave crossed southern Lake Michigan on the morning of June 8, 1953. A copy of the barograph record for Midway Airport, Chicago, is shown on the bottom of figure 6. The more intense pressure change wave passed the western shore of Lake Michigan about an hour before the sudden increase in the amplitude of the lake level oscillations. Supplementary data show that this wave of rising pressure had passed completely across the lake by the time that the large disturbance was recorded on the Waukegan record.

In general all of the gages tend to become excited at about the same time, but the frequency and amplitude, as well as the phase of the most prominent components of the disturbance, vary from one harbor to the next in an apparently random manner.

Figure 7 shows a comparison between the pressure, wind, and water level as recorded at the Filtration Plant on April 23, 1950. The resemblence between the wind and lake records is much greater than that between the pressure and lake records. In neither case, however, is the comparison between either wind or pressure changes and water level changes close enough to imply a one-to-one relationship.

There were other well defined pressure waves unaccompanied by any significant change in the character of the lake records. There were also periods of large amplitude lake oscillations accompanied by a comparatively smooth barograph record. If a time discrepancy of about two hours is allowed, approximately half of the pressure disturbances are accompanied by lake oscillations, and vice versa. Comparisons between the lake records and wind records lead to similar conclusions.

The tendency for these short period oscillations to occur at the approximate time of an atmospheric disturbance indicates that there is some connection between the meteorological disturbances and the lake level fluctuations. However, these data show that the relation cannot be very simple or direct.

So far as could be determined, no wave recorders were in operation in Chicago during 1950, however, the Beach Erosion Board has prepared a "hindcast" of waves in southern Michigan based on the 6-hourly synoptic weather maps prepared by the WBAN Analysis Center in Washington. (Saville; this volume.) Little correlation was found between the amplitude of the observed lake oscillations and the expected amplitude of the surface waves.

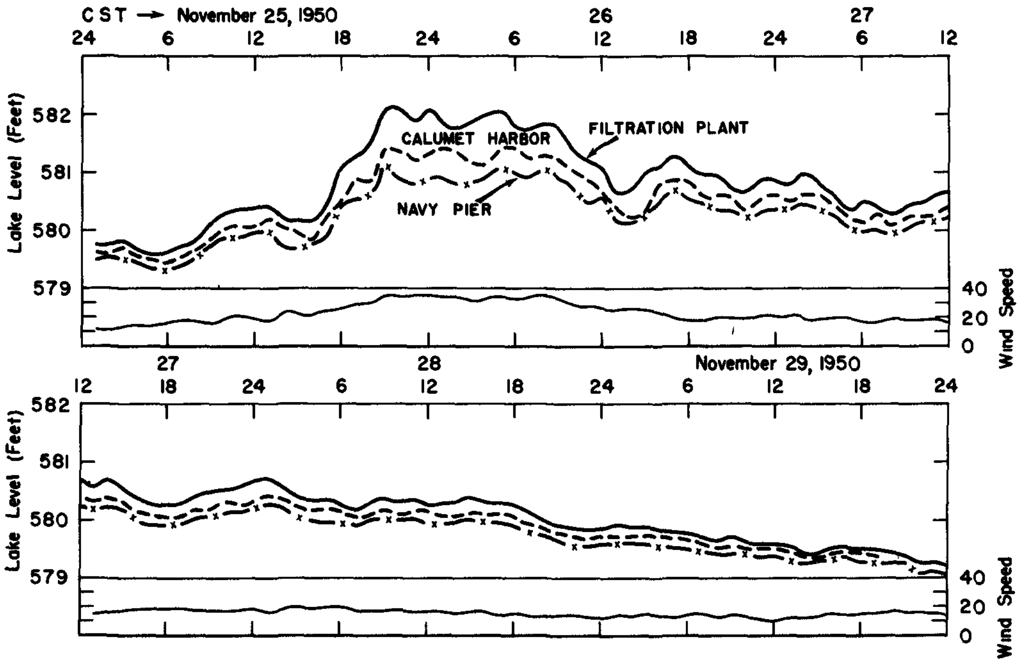

Fig. 9. Hourly mean lake levels, as recorded at 3 Chicago gages, November 25-29, 1950. Northwesterly winds November 25-28, becoming westerly on November 29.

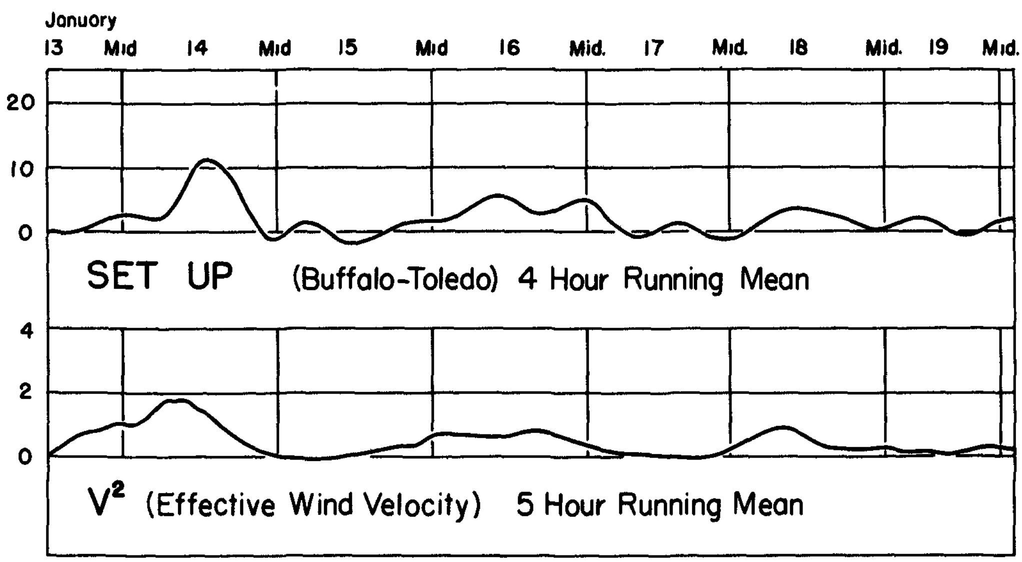

Fig. 10. Set-up on Lake Erie, and V^2(effective wind velocity) for the period, January 13-19, 1953, as defined by Keulegan.

REPRESENTATIVENESS OF OBSERVATIONS

The official records of the U. S. Lake Survey consist, for the most part, of the hourly readings of instantaneous lake level taken from the continuous recorder records. The primary purpose of this record, is the computation of the daily, weekly, and monthly mean lake level. Since each of these means represents the average of a great many individual readings, the method of sampling appears to be adequate for this purpose.

However, these hourly reports are sometimes used as the basis for a study of lake level oscillations. The inadequacy of this procedure is clearly indicated by figure 8. It can be seen that the use of hourly values only gives the appearance of spurious periodicities, and does not represent the true behavior of either the actual water level or the long period trend at the gage. Although these cases represent disturbances of unusual amplitude, the general lack of agreement between the true and apparent periodicities, and the tendency to obscure extreme values are typical of most of the gages in the Great Lakes network.

In an effort to eliminate the effect of harbor oscillations from the lakewide disturbances, hourly means were computed from a number of gages on Lake Michigan for periods when systematic lakewide disturbances were suspected. The hourly mean lake levels for three of the Chicago gages, for the period November 25-29, 1950 are shown in figure 9. It will be noticed that the spread between the record of the three gages increases with the wind speed. Similar studies of other periods show that the sign of this difference depends on the wind direction.

This report is concerned only with cases of unusual lake behavior, and the figures are selected to show extreme cases. It is pointed out that the hourly readings, and even the means computed from a harbor gage, may fail to represent the true behavior of the lake surface during a disturbed period. However, disturbances of the magnitude shown here are of short duration and infrequent occurrence, and these disturbances will not often lead to any significant error in the monthly mean.

LONG PERIOD DISTURBANCES

Keulegan (1951,1952) investigated the wind stress coefficient by studying the relation between the effective wind over Lake Erie, and the set-up. The set-up is defined as the difference in water level at the opposite ends of the lake, as measured by the gages in Buffalo and Toledo. He gave a formula for the effective wind velocity as a function of the observed wind at four stations on the southern shore.

In deriving his formula, Keulegan assumed the existence of a steady state relation between the effective wind velocity and set-up. He sought to offset the error involved in this assumption by time averages of the wind stress and set-up.

Figure 10 is a plot of the set-up and effective wind velocity as defined by Keulegan for the period January 13-19, 1950. It can be seen that a steady state did not exist at any time during this period. The effective wind velocity rises and falls three times during this period.

Keulegan recognized that the steady state could not exist for any protracted period, but he assumed that it would hold at the time midway between the maximum displacement at Toledo and Buffalo. Roughly this corresponds to the peaks in the set-up curve as shown in figure 10.

Using the above assumptions, Keulegan derived the data shown in figure 11 for the relation between the set-up and the effective wind velocity. The linear regression line (dashed line) has been added. The correlation coefficient between this line and the plotted data is .94. The straight line was used because of its greater simplicity. It does not appear that the correlation could be greatly improved by considering the curved line.

It is evident from figure 10, that Keulegan's results can not give all the information needed for forecasting. However, the high correlation between set-up and wind velocity, even though obtained under somewhat special conditions, leads one to believe that a useful forecasting procedure could be obtained from a similar analysis in which changes with respect to time are considered.

HYDRODYNAMIC THEORY

In order to derive an expression relating the atmospheric pressure and wind variations to the behavior of a lake, it is necessary to consider the hydrodynamic equations of motion. Since attention is focussed on the displacement of a free surface it will be convenient to integrate the equations from the bottom to the top of the lake and to consider only the mean motion, averaged through the vertical. The appropriate linearized equations of motion become

$$\partial u/\partial t = -g\partial h/\partial x + F_x + fv \qquad (1)$$

$$\partial v/\partial t = -g\partial h/\partial y + F_y - fu \qquad (2)$$

$$\partial h/\partial t + \partial(Du)/\partial x + \partial(Dv)/\partial y = 0 \qquad (3)$$

where:

h = displacement of water surface

D = mean depth of the water

F_x, F_y = components of atmospheric force

f = Coriolis parameter - effect of earth's rotation.

and the other symbols have the usual meaning.

A slightly different form of these equations is given by Lamb (1932 Chapter VIII). The method of derivation is given in detail by Haurwitz (1951).

SEICHE EQUATIONS -- TWO DIMENSIONAL THEORY

It will be convenient to consider the inertial motion, that is the true seiches, before taking up the forced motions. In order to accomplish this, the applied forces, F_x and F_y, are neglected, and it is assumed that u, v, and h are periodic in time. That is, it is assumed that

$$u = f_1(x,y)e^{i\nu t}$$
$$v = f_2(x,y)e^{i\nu t} \qquad (4)$$
$$h = \Phi(x,y)e^{i\nu t}$$

where f_1, f_2 and Φ are as yet undetermined functions of x, and y and ν is the frequency of the disturbance.

If equations (4) are differentiated and substituted into equations (1-3), u and v can be eliminated from equation (3) to obtain

$$\frac{\partial}{\partial x}\left(D\frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial \Phi}{\partial y}\right) + \lambda\Phi = i\frac{f}{\nu}\left(\frac{\partial D}{\partial x}\frac{\partial \Phi}{\partial y} - \frac{\partial D}{\partial y}\frac{\partial \Phi}{\partial x}\right) \qquad (5)$$

where

$$\lambda = (\nu^2 - f^2)/g \qquad (6)$$

The terms on the right in equation (5) are at least one or two orders of magnitude smaller than those on the left over most of the lake, and if they are neglected, we obtain

$$\partial(D\partial\Phi/\partial x)/\partial x + \partial(D\partial\Phi/\partial y)/\partial y + \lambda\Phi = 0 \qquad (7)$$

The boundary condition is determined by the requirement that no fluid passes through the sides of the lake. This may generally be expressed in the form

$$\partial\Phi/\partial n = 0 \quad \text{on the boundary}$$

where n is a unit normal to the boundary.

Equation (7) can be solved only for certain discrete values of λ, known as eigenvalues. Corresponding to each eigenvalue, there will be one natural frequency of oscillation and one or more eigenfunctions. Each eigenfunction will describe a different mode of oscillation, and each mode of oscillation will be completely specified by the number and location of the nodes, that is the lines of no vertical displacement.

The effect of the earth's rotation is often neglected in the derivation of the equations for seiches. This is equivalent to assuming that $f = 0$, or that

$$\lambda = \nu^2/g \qquad (9)$$

The permissible values of λ are obtained by solving the differential equation subject to the appropriate boundary conditions, and the frequencies of free oscillation can then be obtained by means of equations (6) or (9). Since the Coriolis parameter, f, is not involved in the computation of λ, it appears that the principal effect of the rotation of the earth is to increase the frequency of the free oscillations above that which would be experienced on a nonrotating earth. The amount of this increase can be obtained by eliminating λ between equations (6) and (9), or

$$\nu^2(\text{rotating earth}) = \nu^2(\text{nonrotating earth}) + f^2 \qquad (10)$$

Since f is approximately 10^{-4} in middle latitudes, this correction will amount to about 1% for a period of 2.5 hours and to about 10% for a free period of 7.8 hours.

It appears that satisfactory results can be obtained by neglecting the Coriolis term initially and correcting the computed frequency as indicated in equation (10).

The actual solution of equation (7) for an arbitrary lake is quite difficult, but some insight into the character of such a solution may be obtained from an examination of the known solutions for lakes of regular outline. We consider first a circular lake. The nodal lines are given by concentric circles and equally spaced diameters. (See fig. 12a.) If the circle is flattened slightly so as to form an ellipse, the nodal circles become ellipses, and all but two of the nodal diameters break away from the center to form nodal hyperbolas as shown in figure 12b. As the ellipse is flattened still further the hyperbolas tend to become more nearly straight lines perpendicular to the lake axis. This is illustrated in figure 12c which has the approximate proportions of Lake Erie. For rectangular lakes the nodal lines are parallel to the shores as shown in 12d.

SEICHE EQUATIONS -- ONE DIMENSIONAL THEORY

Since the more important nodal lines for long narrow lakes appear to be approximately straight lines at right angles to the lake axis, it appears that further simplification may be gained by reducing the problem to one dimension. This can be accomplished by considering the average value of u and h for each cross section of the lake. The resulting differential equation is

$$\partial(A\partial\Phi/\partial x)/\partial x + \lambda bg^{-1}\Phi = 0 + \text{higher ordered terms} \qquad (11)^*$$

$A = A(x)$ = Area of cross section
$b = b(x)$ = Width of the lake

The boundary condition is determined by assuming that no fluid flows through the ends of the lake, or

$$\partial\Phi/\partial x = 0 \text{ at } x = 0 \text{ and } x = L. \qquad (12)$$

This is equivalent to the equations used by Chrystal (1905) and Defant (1925), and several methods have been given for obtaining a solution.

The effect of the earth's rotation can be taken into account in computing the natural frequency by using the definition of λ obtained in the study of the two dimensional problem in which the Coriolis terms were considered.

It should be emphasized however, that the averaging process eliminates all transverse modes of vibration. It can give only the average value of h as a function of distance along the axis of the lake. The nodal lines predicted by the one dimensional theory are necessarily straight lines crossing the lake. The true nodal lines are, in general, curvilinear, and will not necessarily intersect the shore. The importance of this point is illustrated in figure 13 which shows one mode of oscillation of a circular lake. This mode of oscillation in a laboratory

*Derivation of equation (11) is given in Appendix I

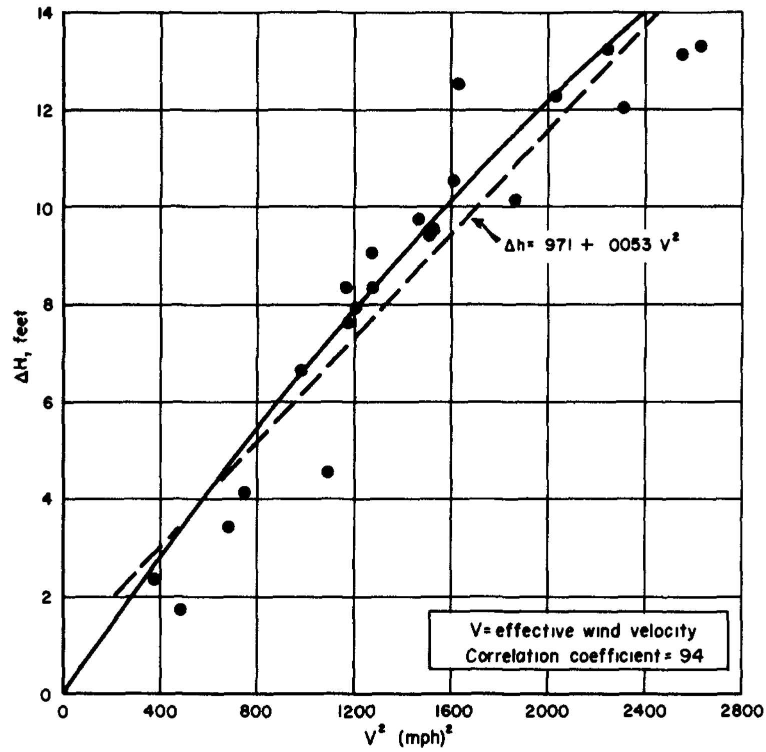

Fig. 11. Relation between set-up and effective wind velocity, from Keulegan.

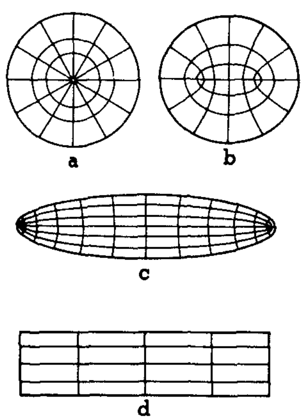

Fig. 12.

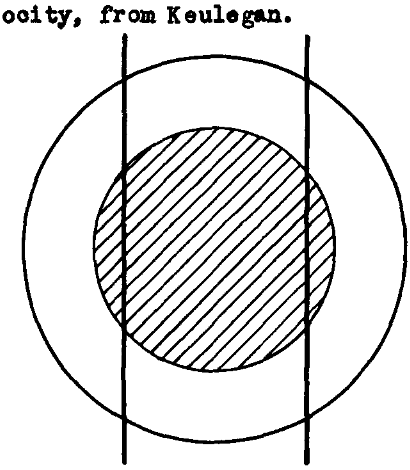

Fig. 13.

Fig. 12. Characteristic nodal lines for lakes of regular outline.

Fig. 13. An example of the true nodal line and the nodal lines as defined by the one dimensional theory for one mode of oscillation of a circular lake.

model has been photographed by McNown (1952). The true nodal line is a circle. The nodal lines in the mean value of h which is to be investigated by the one dimensional theory are given by lines aa' and bb'. It can be shown, however, that all modes of oscillation predicted for a symmetric lake by the one dimensional theory must have a nodal diameter running through the center of the circle. Thus it appears that the one dimensional theory can not give even an approximate description of all modes of oscillation of a lake.

NATURE OF THE EXTERNAL FORCES

Before discussing the equations of forced oscillations of a lake, it may be well to examine the nature of the forces to be considered. The forces considered here are those due to the atmospheric pressure gradient and the frictional stress of the wind on the water surface. We may write

$$F_x = -(\partial p_a/\partial x - \tau x/D)\rho_w \tag{13}$$

where

ρ_w = density of water, assumed constant

p_a = atmospheric pressure

τ_x = component of the wind stress in the x direction.

A similar expression may be given for F_y.

It is generally assumed that τ_x can be expressed in the form

$$\tau_x = \chi \rho_a V^2 \cos\theta \tag{13a}$$

where

χ = wind stress coefficient

V = wind speed

θ = angle between the wind direction and x axis.

We can investigate the approximate ratio between the magnitude of the wind and pressure components of the external force by expressing the pressure gradient in terms of the geostrophic wind by the relation

$$\partial p_a/\partial n = \rho_a f V_g \tag{14}$$

where V_g = geostrophic wind

ρ_a = density of the air,

and writing

$$\frac{\text{Force due to wind stress}}{\text{Force due to pressure gradient}} = \frac{\chi V^2}{fDV_g} \tag{15}$$

Most determinations of the wind stress coefficient show that χ is between 10^{-3} and 3×10^{-3}, and f is approximately 10^{-4} in middle latitudes. Hence if we assume that the true wind is quasigeostrophic we have the approximation

$$\frac{\text{Force due to wind stress}}{\text{Force due to pressure gradient}} = \frac{10V_g}{D}$$

The depth of Lake Erie varies from 10 to 40 meters, and in all lakes the depth within a few miles of the shore is rarely much greater than ten meters. Since important set-ups do not occur until the wind velocity is about 10 m/sec, (20 mph) or more we see that the wind effect will generally be an order of magnitude greater than the pressure effect in the shallow waters near the shores of all lakes, and over all of Lake Erie. In the deeper parts of the other lakes, where the depth may exceed 100 meters, the effects of wind and pressure will be of about equal importance.

In the case of sudden pressure changes such as that shown for Chicago on figures 5 and 6, the actual wind will be much less than the geostrophic value and the pressure gradient term may exceed the wind term.

However, apart from resonance effects, to be discussed later, the effect of the pressure gradient cannot exceed that of the equivalent water barometer, or approximately one foot of water for each inch of pressure. Since even the most violent of these sudden pressure changes rarely exceed .2 or .3 inch it appears unlikely that pressure changes could ever explain as much as one foot of the observed variation in water level.

FORCED MOTION

The mathematics involved in the solution of the equation for forced motion is much simpler in one dimension than in two. However, the physical principles involved are the same. Therefore only the one dimensional problem will be discussed in this paper. The method presented can be extended to cover the two dimensional problem.

The one dimensional differential equation for forced motion is

$$\partial(A\partial h/\partial x)/\partial x - bg^{-1}\partial^2 h/\partial t^2 = g^{-1}\,\partial(AF_x)/\partial x \qquad (17)^*$$

The boundary condition is again determined by the requirement that no fluid can flow through the ends of the lake. This leads to the requirement that the slope of the free surface must be in equilibrium with the applied force at the ends of the lake. This may be expressed in the form

$$\partial h/\partial x = F_x \text{ at } x = 0 \text{ and } x = L. \qquad (18)$$

The solution of this equation can be expressed as a sum of the eigenfunctions obtained in the study of seiches. This solution has the form

$$h = \sum a_n(t)\,\Phi_n(x) \qquad (19)$$

where the coefficients are functions of time. It is shown in Appendix II that the differential equation for $a_n(t)$ has the form

$$d^2a_n/dt^2 + \lambda a_n = R(t). \qquad (20)$$

This is a standard form of the equation for forced oscillations, and its solution is known for many different kinds of functions R(t).

*Derivation in Appendix I

It is shown in Appendix II, that if

$$F_x = \partial F_x/\partial t = h = \partial h/\partial t = 0 \tag{21}$$

at time $t = 0$, then

$$h = \frac{1}{g}\sum\left\{\int_0^L \nu_n \int_0^t F \operatorname{Sin} \nu (t - T)dT\, \Phi_n dx\right\} \Phi_n(x) \tag{22}$$

A study of the observations shows that there are frequent periods when these assumptions are justified.

This equation gives the theoretical relation between the applied meteorological forces and the displacement of the free surface of a lake. It provides the connection between the seiche theories and the wind tide theories which is necessary to the development of any logical forecasting system.

BEHAVIOR OF THE LAKE IN RESPONSE TO CERTAIN SIMPLE FORCES

The significance of equation (22) will be clarified by a few simple examples. If a steady state solution exists, it can be found by integrating the first equation of motion in the proper form (see Appendix I). The resulting expression is

$$h(x) = h_0 + F(x)/g. \tag{23}$$

This is equivalent to the wind tide equation given by Keulegan (1951, 1953). This expression can be expanded in a series of the form

$$h = \sum\left[\int_0^L F(x)\, \Phi(x)\, dx\right] \Phi_n(x) \tag{24}$$

If we assume that a constant force is suddenly imposed on a quiet lake, it is found that

$$h = \sum\left[(1 - \operatorname{Cos} \nu_n t) \int_0^L F(x)\, \Phi_n(x)\, dx\right] \Phi_n(x) \tag{25}$$

The coefficient of each mode of oscillation fluctuates around its steady state value with an amplitude equal to the steady state value, so that the resulting displacement reachs a maximum of approximately twice the steady state value. In the absence of damping it is not obvious that a steady state condition can ever be achieved.

In figure 10, it is noticed that on Lake Erie, the force function rose from a value near zero to a maximum and returned to zero several times. The simplest mathematical expression for a force of this kind is given by

$$\begin{aligned} F &= 0 && \text{for } t \leq 0 \\ &= F^*(x,y)(1 - \cos\omega t) && \phantom{\text{for }} t \geq 0 \end{aligned} \tag{26}$$

In this case we have

$$h = \sum\left\{\left[1 - \frac{\omega^2 \operatorname{Cos} \nu t - \nu^2 \operatorname{Cos} \omega t}{\omega^2 - \nu^2}\right] \int_0^L F^* \Phi(x)\, dx\right\} \Phi(x) \tag{27}$$

The term in square brackets is obviously a function of the ratio ω/ν . If we set $\omega = \theta \nu$, this becomes

$$a_n(t,\theta) = \left[1 - \frac{\theta^2 \cos \nu t - \theta \nu t}{\theta^2 - 1}\right] \qquad (28)$$

A plot of $a_n(t,\theta)$ for low values of t and various values of θ is given in figure 14. For small values of θ, corresponding to forces with a period long in proportion to the natural period, the lake is always in approximate equilibrium with the applied force. For large values of θ, corresponding to forces whose period is short compared to the natural period, the lake behavior approximates that which would result from the sudden imposition of a constant force with the same mean value.

As θ approaches unity, the amplitude of $a_n(t,\theta)$ grows because of the term $(\theta^2 - 1)^{-1}$. The apparent period varies with time, but the average period agrees with the period of the forcing mechanism, or with the natural period, whichever is greater, and the water level disturbance often becomes out of phase with the applied force.

In the above discussion, only one harmonic of the atmospheric force has been considered. The natural atmospheric disturbances may be considered as the sum of many component eddies. These eddies vary in size, speed, and other characteristics. Only those eddies whose size is comparable to that of the lake can be efficiently expanded in terms of the eigenfunctions. The effects of those eddies which are much smaller than the lake are observed as "noise" or interference superimposed on the basic pattern of the water level movements determined by the larger eddies. This noise will always lead to some error in the forecast, and useful forecasts will be possible only in regions in which the amplitude of the noise is small compared to that of the basic disturbance.

This noise appears in the lake level records as fluctuations of the lake level with periods that are very short compared to the fundamental natural period of the lake. It is believed that most of the variations in lake level shown in figures 2, 5, 6, 7 and 8 are due to noise of this type.

There is little evidence of noise of this type in the records for Buffalo, Toledo, and Gibraltar at the ends of Lake Erie. Hence it appears that useful forecasts of the lakewide disturbances on Lake Erie should be possible. However, some additional development work is still needed. Equation (22) should provide the starting point for the development of a practical forecasting procedure.

However, the available records for the other Great Lakes indicate that the amplitude of the noise is usually as great as or greater than that of the lakewide oscillations. In view of this fact, it appears unlikely that useful forecasts of the lakewide disturbances on these lakes can be based on the available records.

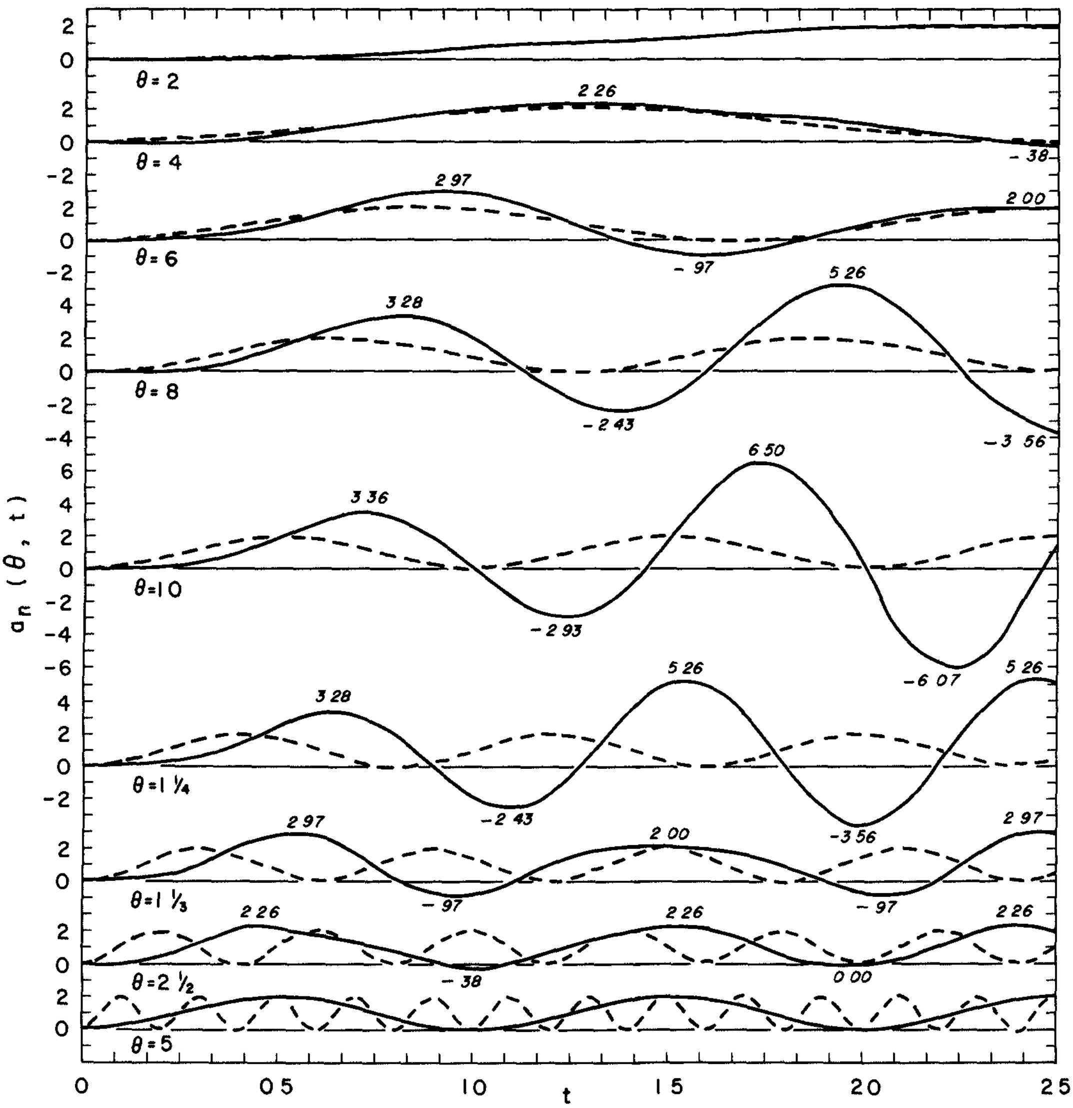

Fig. 14. Plot of $a_n(\theta,t)$ for several values of θ and low values of t. The dashed line gives the value of an applied force. The solid line shows the relative displacement of the lake surface at a point. The unit of time is the free period of oscillation. The slanted numbers inside the chart give the extreme values of the displacement.

MOVING DISTURBANCES

Proudman (1929) has shown that resonance will occur between the atmosphere and a lake of constant depth if the atmospheric disturbance moves with the same speed as the long waves in the water. It can also be shown by the theory developed in this report, that resonance should occur in a rectangular basin of constant depth, if the wave length of the atmospheric disturbance corresponds to the distance between two successive nodes of any mode of oscillation of the lake.

Since the speed of long waves, and the effective wave length of the large scale lake disturbances depend on the depth of the lake, it appears that in a natural lake, these parameters vary from point to point so that a temporary state of resonance between the atmosphere and the lake can occur in some sections of the lake without having much effect on the lake as a whole. These disturbances, once created, are dispersed throughout the lake according to the laws of solitary waves.

HARBOR DISTURBANCES

The differential equation for the disturbances of harbors and other small basins opening into a larger body of water is the sane as that for disturbances in the entire lake. However the boundary conditions are somewhat changed. For seiches in a harbor opening onto a smooth lake, the boundary conditions are

$$\partial h/\partial n = 0 \text{ on the closed boundary} \qquad (29)$$

$$h = 0 \quad \text{at the harbor opening} \qquad (30)$$

There are mouth effects similar to those found in the study of organ pipes in acoustics, and the effective harbor opening will usually be displaced slightly toward open water, from the geometrical opening of the harbor.

In the general case, however, the lake surface is not smooth, and the boundary condition at the opening becomes

$$h_{harbor} = h_{lake} = h(t) \qquad (31)$$

In this case, the oscillations of the lake at the harbor entrance will force an oscillation of the harbor with the same frequency. Since the boundary condition (29) requires that the elevation of the water surface must be maximum or a minimum on the closed boundary, it is evident the amplitude of the disturbance must be greater at the closed boundary of the harbor than at the opening.

By means of a suitable change in the dependent variable, it is possible to transfer the effect of the lake oscillation into the non-homogeneous term of the differential equation of the harbor oscillation. Here it will be added to the effects of the wind stress and the atmospheric pressure gradient to determine a total forcing term. The solution may then be expanded in a series of the eigenfunctions of the harbor defined by the differential equations for seiches and boundary conditions (29), (30), as in the case of an entire lake.

If the periods of any of the individual components of the noise, referred to above, approach any free period of oscillation of any harbor or other restricted basin opening into the lake, that component will be amplified by resonance within the harbor. From the study of a great many records of water level variations in harbors, similar to those shown in the first part of this report, it appears to the writer that most of these disturbances were established in this way. That is to say, atmospheric disturbances give rise to a wide spectrum of disturbances on the open lake. All of these are somewhat amplified by convergence and some are greatly amplified by resonance within the harbor. This hypothesis can explain the following observed characteristics of the short period disturbances.

(1) All harbors in the same area tend to become excited at approximately the same time.

This is to be expected if all harbors are forced by the same agency, either lake or atmosphere.

(2) The characteristic period appears to be different in each harbor.

This is to be expected if the disturbance is due in any way to resonance.

(3) The harbor disturbances may occur as much as an hour or two before or after the apparently associated atmospheric disturbance passes the harbor.

(4) Harbor disturbances may occur when no atmospheric disturbance passes the immediate area of the harbor.

These latter two characteristics imply that the energy of the disturbance must be communicated to the lake at some distance from the harbor and advected to the harbor in the form of a water level disturbance in the lake.

If this theory is correct, it may be impossible to forecast these short period oscillations primarily from meteorological considerations. However, the situation is not altogether hopeless, since it would be possible to minimize the undesirable features of these short period fluctuations of water level in harbors by changes in the harbor design. In order to accomplish this, it would be necessary to observe the spectrum of disturbances occurring in the open lake near the harbor entrance, and to make sure that all changes in harbor design tend to decrease the resonance between the harbor and the more common frequencies of the open lake. The importance of this subject has been discussed more fully by Vanoni and Carr (1951), Carr (1952), McNown (1952), and McNown, Wilson, and Carr (1953).

All disturbances which do not have a node at the harbor entrance, may be regarded as progressive waves which enter the harbor, are reflected by the opposite shore, and after some attenuation in amplitude emerge from the harbor. Since the volume of water carried by the waves is a function of their amplitude, it is evident that the waves carry more water into the harbor than they carry out. While most of this excess water will eventually be carried out of the harbor by a gravity current, the average water level inside the harbor will be greater than at the entrance.

It appears that the water level in all harbors should be greater than that in the open lake, but that the amount of this difference would depend on the amount of water carried into the harbor by wave action. Thus on calm days this difference would be vanishingly small, but on disturbed days it would increase by an amount depending on the meteorological situation and the harbor exposure.

CONCLUSION

This investigation was undertaken to determine the cause of the short period oscillations, and to evaluate the possibility of forecasting oscillations of both long and short periods.

The ultimate cause of the short period oscillations has not been definitely determined. However, a theory which is consistent with the observations and with hydrodynamic principles has been presented. The validity of this theory can be adequately tested only by additional field and laboratory studies planned for this purpose.

It appears that useful forecasts of the larger long period oscillations of Lake Erie should be possible. It appears unlikely that useful forecasts of the short period disturbances or of the long period disturbances on any of the lakes other than Erie will be possible within the next few years.

The discussion of the representativeness of the lake level as measured by gages located in harbors was an unexpected by-product of this study, and no attempt has been made to evaluate the discrepancies between the records of nearby gages in engineering terms. However, it is believed that an effort should be made to locate gages in the open lake whenever this is possible.

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APPENDIX I, DERIVATION OF THE ONE DIMENSIONAL EQUATIONS FOR THE OSCILLATIONS OF A LAKE

By integration from one side of the lake to the other it follows from equations (1) and (3) that

$$\int_{y_1}^{y_2} \frac{\partial u}{\partial t} dy = -g \int_{y_1}^{y_2} \frac{\partial h}{\partial x} dy + \int_{y_1}^{y_2} F_x dy \qquad \text{I-1}$$

$$\int_{y_1}^{y_2} \frac{\partial h}{\partial t} dy + \int_{y_1}^{y_2} \frac{\partial Du}{\partial x} dy + \int_{y_1}^{y_2} \frac{\partial Dv}{\partial y} dy = 0 \qquad \text{I-2}$$

where y_1 and y_2 are single valued functions of x forming the boundary of the lake and the Coriolis term is neglected.

A theorem in advanced calculus states that

$$\frac{\partial}{\partial x} \int_{y_1}^{y_2} G(x,y) dy = \int_{y_1}^{y_2} \frac{\partial G}{\partial x} dy - G(x,y_1)\, \partial y_1/\partial x + G(x,y_2)\, \partial y_2/\partial x \qquad \text{I-3}$$

where G is any function of x and y. The condition that no fluid pass through the boundary requires that

$$v(y_1) = u(y_1)\, \partial y_1/\partial x, \text{ and } v(y_2) = u(y_2)\, \partial y_2/\partial x \qquad \text{I-4}$$

By repeated use of I-3 and I-4, equations I-1 and I-2 may be transformed into

$$\int_{y_1}^{y_2} u\, dy = -g \frac{\partial}{\partial x} \int_{y_1}^{y_2} h\, dy + \int_{y_1}^{y_2} F_x dy + g h(y_1) \partial y_1/\partial x - g h(y_2) \partial y_2/\partial x \qquad \text{I-5}$$

$$\frac{\partial}{\partial t} \int_{y_1}^{y_2} h\, dy + \frac{\partial}{\partial x} \int_{y_1}^{y_2} D u\, dy = 0 \qquad \text{I-6}$$

We introduce the new variables

$$\overline{D} = \frac{1}{b} \int_{y_1}^{y_2} D dy \qquad \overline{F}_x = \frac{1}{b} \int_{y_1}^{y_2} F_x dy$$

$$\overline{u} = \frac{1}{b} \int_{y_1}^{y_2} u dy \qquad \overline{h} = \frac{1}{b} \int_{y_1}^{y_2} h dy$$

where $b = y_2 - y_1$, and set

$$h(y_1) = \bar{h} + \epsilon_1 \qquad h(y_2) = \bar{h} + \epsilon_2$$

With the aid of these substitutions, we may write equations I-5 and I-6 in the form

$$\partial\bar{u}/\partial t + g\partial\bar{h}/\partial x = \bar{F}_x + gb^{-1}(\epsilon_2 \partial y_2/\partial x - \epsilon_1 \partial y_1/\partial x) \qquad \text{I-7}$$

$$b\,\partial\bar{h}/\partial t + \partial(b\overline{Du})/\partial x = 0 \qquad \text{I-8}$$

In most cases the term in parentheses in equation I-7 is small compared to the terms on the left and can be neglected. By neglecting this term and eliminating u between these equations, we obtain

$$\partial(A\,\partial\bar{h}/\partial x)/\partial x - bg^{-1}\,\partial^2 h/\partial t^2 - g^{-1}\,\partial(A\bar{F}_x)/\partial x, \qquad \text{I-9}$$

where A is the cross sectional area. Equation (17) is equation I-9 excepting that the bars indicating average values have been omitted. If the force term on the right is omitted, this corresponds to the equation for wave motion in a canal of variable section given by Lamb (1932, p. 273-274).

When the force term is omitted, $\bar{h}$ may be written as the product of two functions of the form

$$\bar{h} = \bar{\Phi}(x)\bar{T}(t) \qquad \text{I-10}$$

where $\bar{T}(t)$ is periodic, and the differential equation for Φ is given in the form

$$\partial(A\,\partial\bar{\Phi}/\partial x)/\partial x + \lambda b g^{-1}\bar{\Phi} = 0. \qquad \text{I-11}$$

When the bars are omitted, this becomes equation (11).

The steady state solution for one dimension is obtained from I-7 by setting $\partial\bar{u}/\partial t = 0$ and neglecting the term in parentheses. This gives

$$\partial h/\partial x = g^{-1}F_x,$$

or

$$h = g^{-1}\int_0^x F_x dx = g^{-1}F \qquad \text{I-12}$$

which gives equation (23).

If we neglect the terms on the right side of equation I-7 and use I-8 to eliminate h from I-7, we obtain the differential equation for u in the form

$$\partial^2 u/\partial t^2 - (gbA)^{-1}\,\partial^2 u/\partial t^2 = 0 \qquad \text{I-13}$$

By introducing a new independent variable defined by the relation

$$v = \int_0^x b dx \qquad \text{I-14}$$

and assuming that u is periodic in time this may be transformed into

$$d^2(Au)dx^2 + \nu^2(gbA)^{-1}(Au) = 0 \qquad \text{I-15}$$

The boundary condition for this equation becomes

$$(Au) = 0 \text{ at } x = 0 \text{ and } x = L \qquad \text{I-16}$$

when (Au) is known, h may be found by the expression

$$h = b^{-1}d(Au)/dx. = d(Au)/dv \qquad \text{I-17}$$

Equations I-15 through I-17 are in the form given by Chrystal. This appears the most convenient form of the equations for the computation of the one dimensional seiche functions. It is not as convenient as equation I-11 for generalization to two dimensions or for the study of forced seiches.

It will be instructive to use equation I-15 for an investigation of the seiches of a circular lake of constant depth and to compare the results with the complete solution as given by Lamb (1932, p. 284 ff) and McNown (1952). If the origin of coordinates is taken at the center of the circle, the width of the circle $2y = 2(a^2-x^2)^{1/2}$, and equation I-15 becomes

$$d^2(Au)/dx^2 + \nu^2\left[Dg(a^2-x^2)\right]^{-1}(Au) = 0 \qquad \text{I-18}$$

with the boundary condition

$$(Au) = 0 \text{ at } x = \pm a.$$

The boundary conditions and the coefficients of the differential equation are all even functions. Hence the solution must be an even function and all solutions will have either maxima or minima at x = 0. The first derivative of the solution therefore must vanish along the y axis. This implies that the vertical displacement at the center of the circle must always be zero. It is well known, however, that there are modes of oscillation in which all nodal lines are circles, and the maximum displacement of the free surface occurs at the center of the circle. This result can be generalized to apply to any basin which is symmetric with respect to some axis. It appears that the one dimensional theory can not even approximate the solution of modes of oscillation, whose nodes do not intersect the shore.

APPENDIX II, SOLUTION OF THE ONE DIMENSIONAL EQUATION FOR THE FORCED OSCILLATION OF A LAKE

The method employed here involves the expansion of the differential equation for forced motion, equation (17), in terms of the eigenfunctions of the lake, obtained as solutions of equation (11). In order to accomplish this we define two new quantities

$$F = \int_0^x F_x d_x \qquad \text{II-1}$$

$$\psi = F - gh \qquad \text{II-2}$$

By means of these substitutions equation (17) is transformed into

$$g\,\partial(A\partial\psi)/\partial x - \partial^2\psi/\partial t^2 = -bg^{-1}\ \partial^2 F/\partial t^2 \qquad \text{II-3}$$

with the boundary conditions

$$\partial \psi / \partial x = 0 \quad \text{at } x = 0, \text{ and } x = L \qquad \text{II-4}$$

If we assume that ψ may be expressed in the form

$$\psi = \Phi(x)T(t) \qquad \text{II-5}$$

where Φ is a solution of equation (11), equation (22) may be expressed in the form

$$\Phi(x)\ (\lambda T + \partial^2 T/\partial t^2) = \partial^2 F/\partial t^2. \qquad \text{II-6}$$

We define two functions of n and t by the relations

$$a_n(n,t) = \int_0^L \psi\ (x,t)\,\Phi_n(x)dx \qquad . \quad \text{II-7}$$

$$R_n(n,t) = \int_0^L \left[\partial^2 F(x.t)/\partial t^2 \right] \Phi_n\,(x)dx \qquad \text{II-8}$$

where $\Phi_n(x)$ is the nth normalized eigenfunction defined by equation (11). Thus we see that $\Phi(n,t)$ and $R(n,t)$ are the generalized Fourier coefficients of $\psi(x,t)$ and $\partial^2 F/\partial t^2$, when these quantities are expressed as a series of the eigenfunctions, Φ_n. These coefficients are functions of time. By introduction of equations II-5, II-7, and II-8 into equation II-6 , we obtain the differential equation for $a_n(t)$ in the form

$$d^2 a_n/dt^2 + \lambda a_n = R_n(t) \qquad \text{II-9}$$

where we can use the symbol for a total derivative since a_n depends only on t. This is a standard form of the equation for forced oscillations. The general solution of equation II-9 may be given as

$$a_n = a_{n,1} + a_{n,2} \qquad \text{II-10}$$

where

$$a_n = a_n\Big]_{t=0} \cos \nu_n t + da_n/dt\Big]_{t=0} \nu_n^{-1} \sin \nu_n t: \qquad \text{II-11}$$

and depends only on the initial conditions.

$$a_n = \frac{1}{\nu_n} \int_0^t R(\tau) \sin(t-\tau)\, d\tau \qquad \text{II-12}$$

and depends only on the applied force.

If the initial conditions are known, $a_{n,1}$ can be evaluated for any future time by standard methods. In the following discussion it will be assumed that $a_{n,1} = 0$, and only $a_{n,2}$ will be discussed. This is equivalent to assuming that at the time $t = 0$, a_n and da_n/dt are both zero. Observations show that there are frequent periods in which this assumption is valid, and we may choose $t = 0$ in any such period.

Equation II-9 is the differential equation for the coefficient of a single mode of vibration. The complete solution is composed of the sum of all modes of vibration. Hence the complete solution for x under the condition assumed here is

$$\psi = \sum \left\{ \int_0^t \int_0^L \left[\partial^2 F(\tau)/\partial t^2 \right] \Phi_n(x) \operatorname{Sin}(t - \tau) dx \, d\tau \right\} \Phi_n(x) \qquad \text{II-13}$$

Since $h = (F - \psi)/g$, and it is permissible to change the order of integration, we have

$$h = \frac{1}{g} \sum \left\{ \int_0^h \left[F - \int_0^t \frac{\partial^2 F(\tau)}{\partial t^2} \operatorname{Sin}(t - \tau) d\tau \right] \Phi_n(x) \, dx \right\} \Phi_n(x) \qquad \text{II-14}$$

After performing the integration this becomes II-15

$$h = \frac{1}{g} \sum \left\{ \int_0^h \left(\frac{\partial F}{\partial t} \right]_{t=0} \operatorname{Sin} \nu_n t + F \Big]_{t=0} \operatorname{Cos} \nu_n t + \nu \int_0^t F \operatorname{Sin} \nu (t - \tau) d\tau \right) \Phi_n dx \right\} \Phi_n(x)$$

If F and $\partial F/\partial t = 0$ at $t = 0$, this becomes

$$h = \frac{1}{g} \sum \left\{ \int_0^L \nu_n \int_0^t F \operatorname{Sin} \nu (t - \tau) d\tau \, \Phi_n \, dx \right\} \Phi_n(x) \qquad \text{II-16}$$

This is equation (22).