CHAPTER 67

MATHEMATICAL MODELING OF SHORELINE EVOLUTION

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ABSTRACT

A mathematical model for long term shoreline evolution is developed. The combined effects of variations of sea level, wave refraction, wave diffraction, loss of sand by density currents during storms, by rip currents and by wind, bluff erosion and berm accretion as well as effects of man-made structures such as long groin or navigational structures and beach nourishment are all taken into account. A computer program is devloped with various subroutines which permit modification as the state-of-the-art progresses. The program is applied to a test case at Holland Harbor, Michigan.

I. INTRODUCTION

The purpose of this paper is to establish a mathematical model for shoreline evolution and to calibrate it with a test case, located at Holland Harbor, Michigan. The present mathematical model includes many of the characteristics already covered in the literature. In addition, it presents an integrated approach covering a large number of phenomena previously neglected. It is extracted from a more general investigation on three dimensional modeling of shoreline evolution.

It is recalled that three time scales of shoreline evolution can be distinguished:

(a) Geological evolution taking place over centuries;
(b) long-term evolution from year-to-year or decade; and
(c) short-term or seasonal evolution and evolution taking place during a major storm.

Associated with these time scales are distances or ranges of influence over which changes occur. The geological time scale deals, for instance, with the entire area of the Great Lakes. The long-term evolution deals with a more limited stretch of shoreline and range of influence; e.g., between two headlands or between two harbor entrances. The short-term evolution deals with the intricacies of the surf zone circulation; e.g., summer profile-winter profile, bar, rhythmic beach patterns, etc.

For the problem under consideration, long-term evolution is of primary importance, the short-term evolution appearing as a superimposed perturbation on the general beach profile. Evolution of the coastline is characterized by low monotone variations or trends on which are superimposed short bursts of rapid development associated with storms.

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The primary cause of long-term evolution is water waves or wave-generated currents. Three phenomena intervene in the action which waves have on shoreline evolution:

(a) Erosion of beach material by short period seas versus accretion by longer period swells;
(b) effect of (lake) level changes on erosion; and
(c) effect of breakwaters, groins, and other structures.

II. MATHEMATICAL FORMULATION

Let us consider a coastal zone limited by boundaries at a small distance from the surf zone (Figure 1). The bottom topography is defined in a three coordinate system, $oxyz$, by a function $z_b = f(x,y,t)$ where the axis, ox, is parallel to the average shoreline direction, the axis, oy, is perpendicular seaward and, oz, is positive upward from a fixed horizontal datum. The angle of the shoreline with the axis, ox, is small. The shoreline is defined by $y = y_s$, $z = z_s = z_b\ (x, y_s, t)$ which also defines the sea level as function of time.

The deepwater limit of the beach is, $y = y_c$. (This limit defines the contour line where the sand is no longer moved by wave action). The water depth at $y = y_c$ is D_c. It will be assumed that D_c remains constant as sea level and beach profiles change. Therefore $\partial z_s/\partial t = \partial z_c/\partial t$.

B is the height of the bluff in case of erosion $\left(\text{i.e., when } \frac{\partial y_s}{\partial t} < 0\right)$, and the height of the berm in case of accretion, $\left(\text{i.e., when } \frac{\partial y_s}{\partial t} > 0\right)$,

The quantity of sand over a stretch of shoreline, Δx, unity and bounded by the datum, $z = o$, $y = o$, and the beach profile z_b at time, t, is: $V(t) = \int_o^{y_1} z_b\ (x,y,t)\ dy$.

Let us assume that, for some reasons, the beach profile changes during an infinitesimal amount of time, dt. Let us further assume that the initial beach profile which is considered at time, $t = t_i$, could be the normal "equilibrium profile".* The departure and modification from this initial beach profile can be characterized by:

(a) A translation in the yz plane defined by an elementary vector of components.

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*The "equilibrium profile" may never exist under varying prototype conditions (similarly two-dimensional wave never exists), but it is a convenient idealized concept which could be approached in two-dimensional wave tank experiments. In the present case, it could be defined as the statistical long term average beach profile which exists under a given wave climate. The model presented herewith is actually independent from this definition.

$\frac{\partial y_s}{\partial t}$, $\frac{dD}{dt}$ $\qquad$ $\frac{dD}{dt}$ is the rate of change of sea level.

Note that this translation is independent from the beach profile and in particular, if the beach profile normally exhibits a number of significant bar formations, under normal conditions, this translation will reproduce this characteristic at the same water depth.

(b) A perturbation characterizing the departure or variation from the initial profile. Since the rate of the vertical component of translation is $\frac{dD}{dt}$, the perturbation can be defined by only a horizontal displacement. This effect is neglected in the present layer and will be presented in a sequel at a later date.

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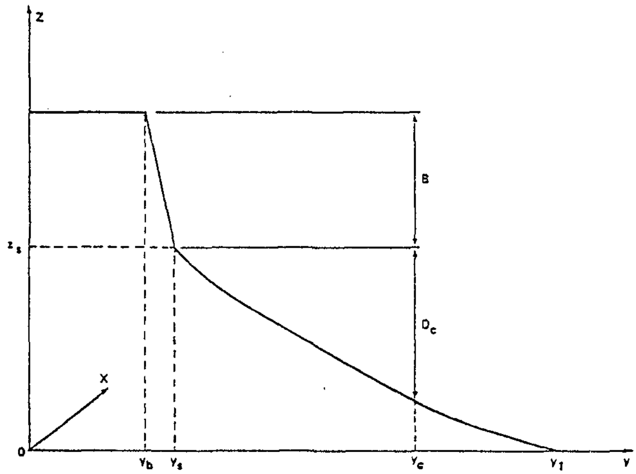

Figure 1 Notation

The variation of sand quantity in the considered domain is:

$$\frac{dV}{dt} = \int_0^{y_1} \left(\frac{\partial z_b}{\partial t} + \frac{\partial z_b}{\partial y}\frac{dy}{dt} + \frac{\partial z_b}{\partial x}\frac{dx}{dt} \right) dy$$

On which if one neglects the variation of z_b with respect to x yields:

$$\frac{dV}{dt} = (B + D_c)\frac{\partial y_s}{\partial t} - (y_c - y_b)\frac{dD}{dt}$$

This variation of volume is due to the variation of littoral drift along the ox axis and the onshore-offshore motion. The following terms are included:

(a) The discharge of sand leaving the beach per unit of width which includes:

1. Q_{yw} due to loss of sand by wind.

2. Q_{ys} (x) due to the quantity of silt contained in the bluff and which tends to move offshore by suspension. This loss occurs only in case of erosion $\frac{\partial y_s}{\partial t} < o$ and is equal to $Q_{ys} = K_s\ B\ \frac{\partial y_s}{\partial t}$ where K_s is the percentage of silt in the bluff.

3. Q_{yf} due to the loss of sand from the beach by density current during storm. Q_{yf} is a function of the size distribution and density of material. A beach of fine material ($< 0.1^{mm}$) will tend to erode more rapidly than beach made of coarse material ($> 1^{mm}$). The coarse material tends to move along shore while the fine sand moves offshore.

The determination of these three quantities are given from sand budget investigations.

(b) A general term M(x,t) expressing the local variation in the sand budget due to

1. loss of sand by rip currents along groins.
2. sudden dumping of sand in case of beach nourishment or flood.

(c) The variation of littoral drift along the axis ox which is

$$Q_s\ (s) - Q_x\ (x + dx) = - \frac{\partial Q_s}{\partial x}\ dx$$

$Q_s = 7.5 \times 10^3\ P_\ell$ where Q_s is in yd^{3}/ year. P_ℓ is in ft - lbs/sec/ft of shoreline and is expressed by the relationship:

$P_\ell = \frac{\rho g^2}{64\pi}\ H_o^2\ T\ K_R^2\ \sin 2\alpha_b$ where K_R is the refraction coefficient from deep water to the line of breaking inception: T is the wave period, H_o is the deepwater wave height, α_o is the angle of the deep water wave with the shoreline, α_b is the angle of breaking with the shoreline.

This formula will be assumed to hold in case of gentle beach curvature. The refraction coefficient K_R and angle α_b can be determined as functions of the deep water wave characteristics H_o, T, α_o (or α) and the angle of the shoreline at breaking, $\simeq \frac{\partial y_s}{\partial x}$ At $x \to -\infty$, the deep-water wave angle α_o with bottom contours is equal to α since the shoreline has the same direction as the axis ox. In the general case, i.e., for any value of x

$$\alpha_o = \alpha - \tan^{-1} \frac{\partial y_s}{\partial x}$$

The breaking waves characteristic: wave height, H_b, water depth d_b, and the angle breaking α_b, can be obtained from the deepwater wave characteristics, H_o, T, and α_o. α_o is given by the previous equation in terms of α and $\frac{\partial y_s}{\partial x}$ which takes into account the curvature of the shoreline. The following equation is valid provided the bottom contours are parallel along a wave ray, (Le Mehaute and Koh, 1967) (Figure 2):

$$\alpha_b \cong \alpha_o \left[0.25 + 5.5\ H_o/L_o\right] \text{ where } L_o = \frac{gT^2}{2\pi}$$

Therefore, the refraction coefficient,

$$K_R = \left[\frac{\cos\left(\alpha - \tan^{-1}\frac{\partial y_s}{\partial x}\right)}{\cos\left[\left(\alpha - \tan^{-1}\frac{\partial y_s}{\partial x}\right)\left(0.25 + 5.5\frac{2\pi H_o^2}{gT^2}\right)\right]} \right]^{1/2}$$

Now it is possible to formulate the variation of littoral drift:

$$\frac{\partial Q_s}{\partial x} = A\, H_o^2\, K_R^2\, 2\cos 2\alpha_b\, \frac{\partial \alpha_b}{\partial x} + A\, H_o^2\, 2K_R\, \frac{\partial K_R}{\partial x}\, \sin 2\alpha_b$$

where $A = 7.5.10^3\ \frac{\rho g^2}{64\pi}\,T$. On the other hand one also has

$$\frac{\partial \alpha_b}{\partial x} = \frac{0.25 + 5.5\frac{H_o}{L_o}}{1 + \left(\frac{\partial y_s}{\partial x}\right)^2}\ \frac{\partial^2 y_s}{\partial x^2}$$

In case of wave diffraction, the wave height varies significantly along a wave crest. Then the previous refraction coefficient K_R has to be replaced by a combined coefficient, say $K_D\,K_R$. Also, in a diffraction zone, α_b is due to the sum of variation of shoreline direction $\tan^{-1}\frac{\partial y_s}{\partial x}$ and because of diffraction, the rotation of the wave crest around the end of the groin: Θ (Figure 3). Θ is the angle which has the end of the groin as apex and extends from the limit of the "shaded" area to the considered location defined, therefore,

$$\alpha_b \cong \tan^{-1}\frac{\partial y_s}{\partial x} + \alpha_o - \tan^{-1}\frac{x}{\ell},$$

$$-\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta'}{\partial x} = \frac{1}{\ell}\ \frac{1}{1 + \left(\frac{x}{\ell}\right)^2}$$

An empirical formulation for determining the combined effect of diffraction and refraction is more suitable to quantitative analysis of a real sea spectrum than more exact theories of wave diffraction which are valid for periodic waves over a horizontal bottom and are represented by Fresnel integral.

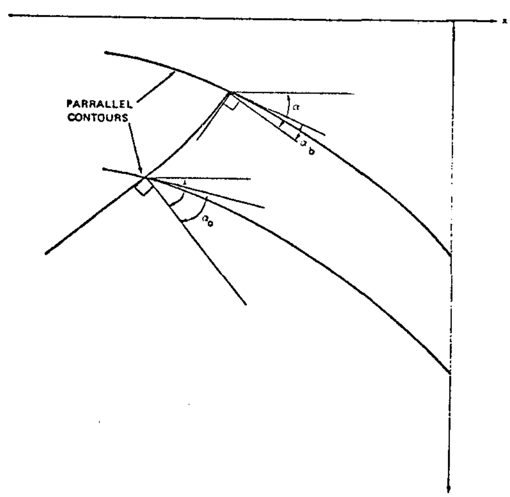

FIGURE 2 EFFECTS OF WAVE REFRACTIONS ON A CURVE BEACH

For this, it will be assumed that the energy travels laterally along a wave crest as well as along a wave ray. This lateral transmission of energy results into a decrease of wave energy from the exposed area to the shaded area,

$$\int_A^C H^2\,dx = \int_0^D H_o^2\,ds$$

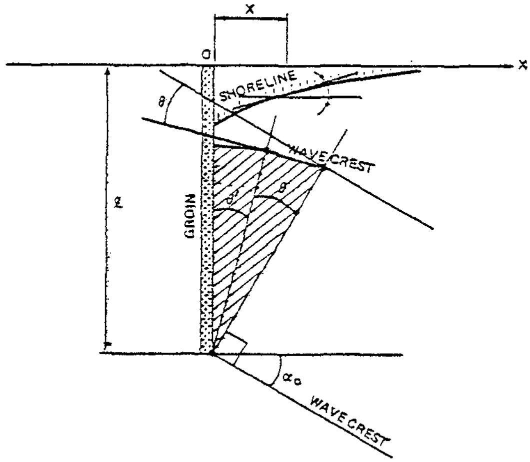

Figure 3 Diffraction Zone Notation

In the case where the long groin is in the previously defined wave diffraction zone as in Figure 3, it is assumed that the wave energy which reaches the groin is absorbed by friction. It is assumed that the combined effects of diffraction refraction of a wave spectrum can simply be represented by a sinusoidal variation of wave height along the breaking line. (Mobarek and Weigel,1966). One has finally

$$K_D(x) = \left(\frac{\sqrt{2}}{2}\,\frac{\cos(2\alpha_o)}{\sin\alpha_o}\right)^{1/2} \sin\left[\frac{\pi\cos(2\alpha_o)}{4\ell}\,(x + \ell\tan(45^o - \alpha_o))\right]$$

Inserting this value in the littoral drift formula previously described permits us to complete the mathematical model.

III. TRANSFORMATION OF THE PHENOMENOLOGICAL EQUATION AND NUMERICAL SCHEME

Now that all the phenomenological equations have been established we will find it more convenient to express them in dimensionless form. The general equation expressing the sand budget balance can still be written. (The loss terms have been dropped for sake of simplicity and can easily be included whenever necessary).

$$(B(x,t) + D_c)\frac{\partial y_s}{\partial t} + (y_c - y_b)\frac{dD}{dt} = \frac{\partial Q}{\partial x}$$

For purposes of analysis we consider this equation in dimensionless form.

$$\text{let } (\text{length})^{\wedge} = \frac{\text{Length}}{B_o + D_c} \quad B_o \text{ to be chosen later}$$

$$\hat{t} = \frac{At}{(B_o + D_c)^3}$$

$$Q = \frac{A}{2} K_R^2 K_D^2 \sin 2\alpha_b$$

The general equation is thus transformed to:

$$\frac{B + D_c}{B_o + D_c} \frac{\partial \hat{y}_s}{\partial \hat{t}} + (\hat{y}_c - \hat{y}_b) \frac{d\hat{D}}{d\hat{t}} = \frac{\partial}{\partial \hat{x}} \frac{K_D^2 K_R^2 \sin 2\alpha_b}{2}$$

α_b = function of α_o, say f (α_o), (the function f depends only on H_o/L_o as previously shown). The hats and subscripts will be dropped from all variables from this point on.

$$\text{Therefore } \frac{1}{2} K_D^2 K_R^2 \sin 2\alpha_b = K_D^2 \cos \alpha_o \sin \alpha_b$$

$$\text{Note } \frac{\partial}{\partial x} \cos \alpha_o \sin \alpha_b = \left[\cos \alpha_o \cos \alpha_b \frac{\partial f}{\partial \alpha_o} - \sin \alpha_o \sin \alpha_b\right] \frac{\partial \alpha_o}{\partial x} = F(\alpha_o) \frac{\partial \alpha_o}{\partial x}$$

The general equation then becomes (after some rearrangements)

$$\frac{\partial y}{\partial t} = \frac{B_o + D_c}{B + D_c} F(\alpha_o) \frac{1}{1 + \left(\frac{\partial y}{\partial x}\right)^2} \frac{\partial^2 y}{\partial x^2} + R(x,y,t)$$

$$\text{where } R(x,y,t) = \frac{B_o + D_c}{B + D_c} F(\alpha_o) \frac{\partial \alpha}{\partial x} - (y_c - y_b) \frac{dD}{dt} + 2K_D \frac{\partial K_D}{\partial x} \cos \alpha_o \sin \alpha_b$$

$$\alpha = \alpha\ (x) \text{ in the diffraction zone}$$

The above equation is the general dimensionless form which gives us the time dependent sand budget. This general equation is nonlinear and appears to be impossible to solve analytically. Some numerical results are presented

The uniform depth theory of Penny and Price is used as an approximation (not substantiated) for diffraction about the end of the breakwater. The shoreline is calculated for various multiples of a fixed Δt (Figure 4a). Of interest is to note that the undulatory patterns of the shoreline seen in Figure 4a disappear in Figure 4b. Hence, diffraction induced undulations in natural shoreline probably rarely appear since offshore wave climates are usually multi-directional.

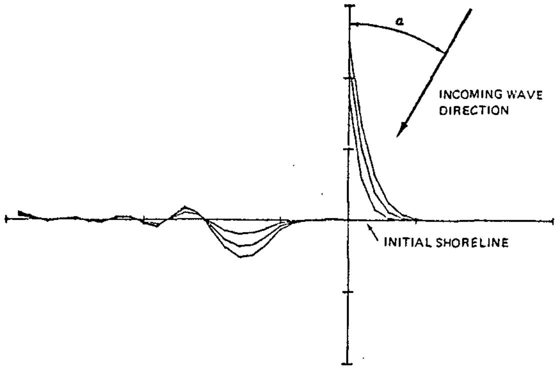

FIGURE 4a SHORELINES AT SUCCESSIVE TIMES 5Δt, 10Δt, 15Δt

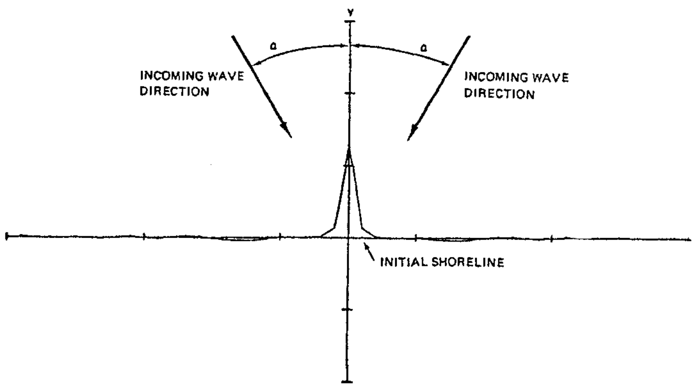

FIGURE 4b SHORELINES FOR TIME 2Δt

The numerical scheme generally used to solve this problem is based on the use of implicit finite differences. Such schemes, whether implicit or explicit, or both, are commonly used to efficiently solve parabolic problems. However, even in the case where only refraction is considered, the boundary condition

$\frac{\partial y}{\partial t} = -\tan\alpha$ at $x = 0$ numerically gives a solution which initially may not conserve mass, i.e., the integrated transport equation

$\frac{\partial}{\partial t}\int_0^L y\,dx = Q(L)$ may not be satisfied. Unfortunately this feature is unavoidable for most such schemes (the exceptions will be discussed below) as the following will demonstrate. Shown on Figure 5 is an initially straight shoreline. In any finite difference scheme, after 1 time increment Δt the shoreline is bounded below by the solid shoreline of Figure 5.

This shoreline has the least possible area A, where $A = \frac{\Delta x^2}{2} \tan \alpha$

The conservation of mass equation requires $\Delta t\ Q\ (L) = \cos \alpha \sin \alpha_b \geq A$
Thus, Δt must satisfy the inequality

$$\Delta t \geq \frac{1}{2} \frac{\sin \alpha}{\sin \alpha_b} \frac{\Delta x^2}{\cos^2 \alpha}$$

Since the accuracy (and in explicit schemes, stability as well) depends on the ratio $\lambda = \frac{\Delta t}{\Delta x^2}$ the above inequality places a lower bound on the accuracy of the solution which may be unacceptable in practice. The finite difference form of the equation for the conservation of mass may be incorporated directly into the numerical scheme. In this case a solution exists which is similar to the previous case but shows a small erosion throughout the reach. For engineering applications the primary quantity of interest is the amount of sand on a given shoreline. It is then more important to conserve mass than to satisfy the shoreline boundary condition as written in the present form. The general equation will now be used to derive an equivalent equation for the transport Q which, even though subject to similar numerical problems, will satisfy the transport boundary conditions exactly.

Consider, for the moment, the situation in which only refraction is important. The general equation then becomes $\frac{\partial y}{\partial t} = \frac{\partial Q}{\partial x}$
where $Q = \cos \alpha_o \sin \alpha_b$

$$\alpha_b = f\ (\alpha_o)$$

$$\alpha_o = \alpha + \tan^{-1} \frac{\partial y}{\partial x}$$

Differentiating by x gives $\frac{\partial}{\partial t} \frac{\partial y}{\partial x} = \frac{\partial^2 Q}{\partial x^2}$

The transport function Q can be considered as a function of α_o which may be solved for α_o, say $\alpha_o = g(Q)$.

Thus the above transport equation becomes

$$\frac{\partial}{\partial t} \tan\ (\alpha_o - \alpha) = \frac{\partial}{\partial t} \tan\big(g(Q) - \alpha\big) = \frac{\partial^2 Q}{\partial x^2}$$

$$\therefore \frac{\partial g(Q)}{\partial t} = \cos^2\ (g(Q) - \alpha) \frac{\partial^2 Q}{\partial x^2}$$

$$\text{but } \frac{\partial g(Q)}{\partial t} = \frac{dg(Q)}{dQ} \frac{\partial Q}{\partial t}$$

$$\therefore \frac{\partial Q}{\partial t} = \frac{\cos^2(g(Q) - \alpha)}{dg(Q)/dQ} \frac{\partial^2 Q}{\partial x^2}$$

Assuming a solution for this equation is known, the shoreline y can be calculated from the equation

$$y(t,x) = y(o,x) + \int_o^t \frac{\partial Q}{\partial x} (t,x)\ dt$$

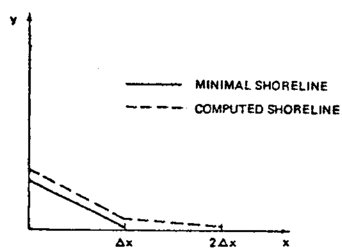

FIGURE 5 COMPUTED AND MINIMAL SHORELINES FOR FINITE DIFFERENCE SCHEME OF TIME = 1Δt

In practice the equation for Q is not solved in the above form. Implicit in the above formulation is the assumption that the function g exists. However, as is illustrated in Figure 6, g is not single valued if the maximum range of the angle α_o is greater than approximately 41 degrees. This difficulty may be removed by considering the equation for Q and y as a system subject to the boundary conditions for Q.

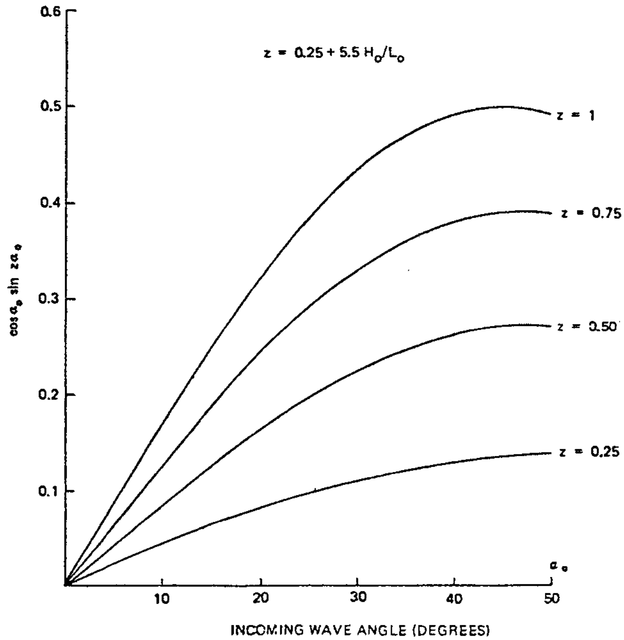

Figure 6: Transport function $Q(\alpha_o) = \cos\alpha_o \sin z\alpha_o$ for selected values of z.

Note that $\dfrac{\cos^2(g(Q)-\alpha)}{dg(Q)/dQ} = \dfrac{dQ}{d\alpha_o} \Big/ \left[1 + \left(\dfrac{\partial y}{\partial x}\right)^2\right]$.

Hence, the equation for Q becomes $\dfrac{\partial Q}{\partial t} = \dfrac{dQ}{d\alpha_o} \dfrac{1}{1+\left(\dfrac{\partial y}{\partial x}\right)^2} \dfrac{\partial^2 Q}{\partial x^2}$

This together with the equation $\dfrac{\partial y}{\partial t} = \dfrac{\partial Q}{\partial x}$ is solved in a cyclic scheme.

One possible method is the centered Crank-Nicolson type implicit-explicit scheme discussed in the following. Suppose y is given for all x at a given time t, and that from time t the wave climate is specified by the (constant) triple (α, H_o, T)

$$\text{Let } L(t,x) = \frac{dQ}{d\alpha_o} \frac{1}{1+\left(\frac{\partial y}{\partial x}\right)^2}$$

$$\lambda = \frac{\Delta t}{\Delta x^2}$$

$$L(t,x) = \text{an approximation to } L(t,x)$$

Integrating the Q equation gives

$$Q(t+\Delta t,x) = Q(t,x) + \frac{\Delta t}{2}\left[L(t,x)\left.\frac{\partial^2 Q}{\partial x^2}\right|_t + \tilde{L}(t+\Delta t,x)\left.\frac{\partial^2 Q}{\partial x^2}\right|_{t+\Delta t}\right]$$

where $\frac{\partial^2 Q}{\partial x^2} = \frac{Q(x+\Delta x) - 2Q(x) + Q(x-\Delta x)}{\Delta x^2}$

Integrating the y gives

$$y(t+\Delta t,x) = y(t,x) + \frac{\Delta t}{2}\left[\left.\frac{\partial Q}{\partial x}\right| + \left.\frac{\partial Q}{\partial x}\right|_{t+\Delta t}\right]$$

where $\frac{\partial Q}{\partial x} = \frac{Q(x+\Delta x) - Q(x-\Delta x)}{2\Delta x}$

The equations are solved numerically, subject to the appropriate boundary conditions, by the cyclic algorithm:

1) Let $L(t+\Delta t,x) = L(t,x)\ \forall x$
2) Calculate $Q(t+\Delta t,x)\ \forall x$ subject to the appropriate boundary conditions
3) Calculate $y(t+\Delta t,x)\ \forall\ x$
   Calculate $L(t+\Delta t,x)$, set this equal to $\tilde{L}(t+\Delta t,x)$
   Calculate new Q
4) If new Q compares with old Q stop, if not go to step 3

Tests with this scheme have shown that it converges to its limit after one application of step 3.

This method can easily be modified to solve the equation where both diffraction and variations in lake level are allowed, i.e.,(s is the beach slope)

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x} K_D^2 \cos\alpha_o \sin\alpha_b - \frac{1}{s}\frac{dD}{dt}$$

For convenience let $Q = \cos\alpha_o \sin\alpha_b$

$$\tilde{Q} = K_D^2\ Q$$

As before

$$\frac{\partial}{\partial t}\frac{\partial y}{\partial x} = \frac{\partial}{\partial t}\tan(g(Q)-\alpha)$$

$$= \frac{1}{\cos^2(g(Q)-\alpha)}\frac{dg(Q)}{dQ}\frac{\partial Q}{\partial t} - \frac{1}{\cos^2(g(Q)-\alpha)}\frac{\partial\alpha}{\partial t}$$

Also $\frac{\partial\tilde{Q}}{\partial t} = K_D^2\frac{\partial Q}{\partial t} + 2K_D\frac{\partial K_D}{\partial t}Q$

The second term in each of the above two equations are negligible in physical situations of usual interest, where the distance between the shoreline and the tip of the breakwater is large compared to the distance the shoreline changes during a time Δt.

Therefore, the transport equation becomes

$$\frac{\partial \tilde{Q}}{\partial t} = \frac{K_D^2}{dg/dQ} \cos^2(g(Q)-\alpha\ ) \frac{\partial^2 \tilde{Q}}{\partial x^2} \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{\partial Q}{\partial x} - \frac{1}{s}\frac{dD}{dt}$$

This system is solved using the same type of algorithm as previously employed.

In the present situation where only refraction is important several approximations are possible which produce problems having analytic solutions. The most direct approximation, and essentially the assumption of Pelnard-Considere, is to approximate

$$\frac{\partial y}{\partial t} = \left[z \cos \alpha_o \cos \alpha_b - \sin \alpha_o \sin \alpha_b \right] \frac{1}{1+\left(\frac{\partial y}{\partial x}\right)^2} \frac{\partial^2 y}{\partial x^2}$$

(subject to the boundary conditions

$$\left.\frac{\partial y}{\partial x}\right|_{x=o} = -\tan \alpha$$

$$y(x,t) = 0 \quad \text{for } x = \infty\ )$$

by $\frac{\partial y}{\partial t} = a \frac{\partial^2 y}{\partial x^2}$ where a is a constant. For the standard breakwater problem, the most logical choice for this constant is given by

$$a = \frac{z}{1 + \tan^2 \alpha} = \left[\qquad \right]_{x=o}$$

since the shoreline in this case is principally governed by its behavior at the breakwater. This problem has solution

$$y(x,t) = 2 \tan \alpha \frac{\sqrt{at}}{\sqrt{\pi}} e^{-x^2/4at} - \tan \alpha\ x \operatorname{erfc}\left(\frac{x}{\sqrt{4at}}\right)$$

which is exactly the same as that of Pelnard-Considere except that the constant a has been changed. This problem however doesn't conserve mass since

$$\frac{\partial}{\partial t} \int_o^\infty y(x,t)dx = z \sin \alpha \cos \alpha \neq \sin z\alpha \cos \alpha$$

When this approximation is used in the transport equation for Q, the problem becomes

$$\frac{\partial Q}{\partial t} = a \frac{\partial^2 Q}{\partial x^2} \qquad \text{subject to the boundary conditions}$$

$$Q(x=o) = 0$$

$$Q(x=\infty) = \cos \alpha \sin \alpha_b$$

which has solution

$$Q(x,t) = \cos\alpha \sin\alpha_b \operatorname{erf}\left(\frac{x}{\sqrt{4at}}\right)$$

Integrating the equation

$$\frac{\partial y}{\partial t} = \frac{\partial Q}{\partial x}$$

gives $y\,(x,t) = \cos\alpha \sin\alpha_b \left(\frac{\sqrt{2}}{\sqrt{\pi a}}\; t^{1/2} e^{-x^2/4at} - \frac{x}{a} \operatorname{erfc}\left(\frac{x}{\sqrt{4at}}\right)\right)$

which is of the same form as the previous solution.

IV. APPLICATION

The evolution of the shoreline at Holland was studied using the present model. The relevant physical data as well as the estimates of offshore sediment losses were used in the analysis (Figure 7). The historical shorelines were interpolated to give the shoreline every 100 feet along the baseline. The results of these computations are not given. The height of the berm is assumed to be 10 feet. The depth to no sediment motion was estimated at 30 feet, based on visual consideration of the offshore bathemetry as well as the use of the method of Weggel (private communication). An offshore line loss of 3.2 yd^3/yr/ft of beach is also included.

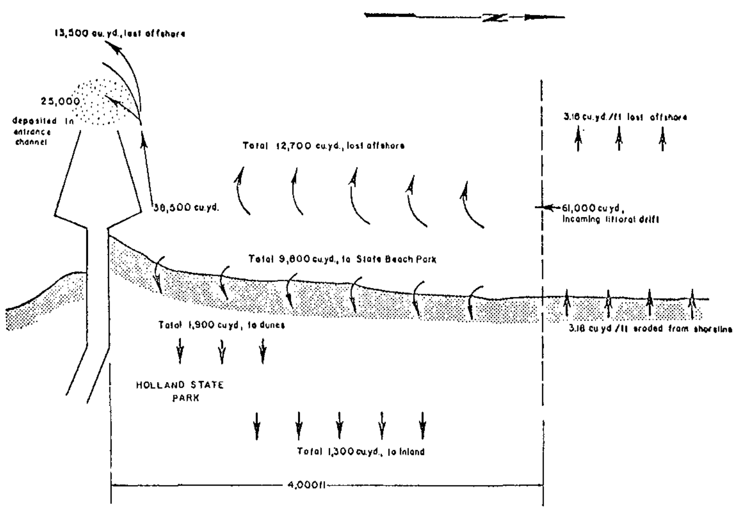

Figure 7.: Summary of sand budget north of Holland, Michigan.

Choice of wave climate is the remaining input parameter to be determined, and is the most controversial. The wave climate most desirable for the study of shoreline evoluation is a time series giving wave height H, period T, and direction D. Unfortunately, this is almost never available and hence statistical summaries must be used. The monthly statistical summaries given by the SSMO for Lake Michigan South previously described are chosen for use. One possible employment of these summaries is to construct monthly times t for each possible (H,T,D) triple, i.e., t (H,T,D), and then calculate the evolution of the shoreline as the (H,T,D) triples are chosen in some deterministic order or at random. This method would be computationally very expensive and is not used. The most simple approach is to assume that these are but 2 (H,D,T) triples representing the gross transport north and south, each occurring for some length of time per month. The entire shoreline is alternately calculated for an incremental time assuming the direction of the incoming wave is positive, then negative. The period T used is taken to be the average T, i.e.

$\bar{T} = \frac{\Sigma p(H,T)T}{\Sigma p(H,T)}$ where the p(H,T) are the (H,T) probabilities given in the SSMO ( ). The choice of (H,D) for north and south, denoted (H_N,D_N) and $(H_S D_S)$ respectively, must now be made. This choice is subject to the condition that the actual northerly transport, as calculated using the statistics and given a straight shoreline for the reach of interest, be preserved, i.e., that

$$t_N H^2_N \bar{T} \cos \bar{\alpha}_o \sin \bar{\alpha}_b = t_h \sum_{\substack{H,T \\ D \text{ giving north} \\ \text{transport}}} p(T|H)p(H,D)TH^2 \cos \alpha_o \sin \alpha_b$$

holds where

t_h = number of hours in a given month
$p(T|H)$ = conditional probability T occurs given H
$p(H,D)$ = probability of (H,D) pair, using SSMO Table 18 as a data base
α_o = D_N - shoreline orientation
α_b = $f(\alpha_o)$
t_N = number of hours the "average" wave condition exists
H_N = "average" wave height
$\bar{\alpha}_o$ = D_N - shoreline orientation
D_N = "average" direction

And similarly for the directions giving southerly transport.

The average directions of the shoreline at the breakwater are calculated using the historical records. The directions are chosen for the incoming wave angles since the complex geometry of the harbor breakwater shields the nearby shoreline from waves arriving from most directions. At present, the time duration of waves arriving from the north is assumed to be the same as from the south. Hence, $t_N = t_S = 0.5\, t_h$. The conservation of transport equation is then used to calculate the average wave heights H_N and H_S.

A computer program was used to calculate the evolution of the shoreline from September 1967 to May 1968. The historical 1967 and 1968 shorelines, as well as the computed 1968 shoreline, are shown on Figure 8. The calculation assumed that Δx = 100 feet with $\lambda \simeq y$ which gives a value for Δt which varies from 8 to 20 hours depending upon the month and wave characteristics. The principal discrepancy between the predicted and actual 1968 shoreline occurs in the vicinity of the breakwater. While the shapes agree there is an erosion in the calculated shoreline which is probably due to the approximations used in calculation of the diffraction coefficients, and incoming wave angles which are functions of x in the shadow region of the diffraction zone. The unaltered theory of Penny and Price was incorporated into the numerical scheme since most breakwaters can be represented as line barriers, and hence is almost always useful. However, for the case of Holland Harbor a universally valid prediction of the shoreline would require the detailed calculation of the diffraction effects due to the geometry of the breakwaters. Also, the convenient choice of incoming wave direction obscures the fundamental problem of how to properly use the statistical wave summaries.

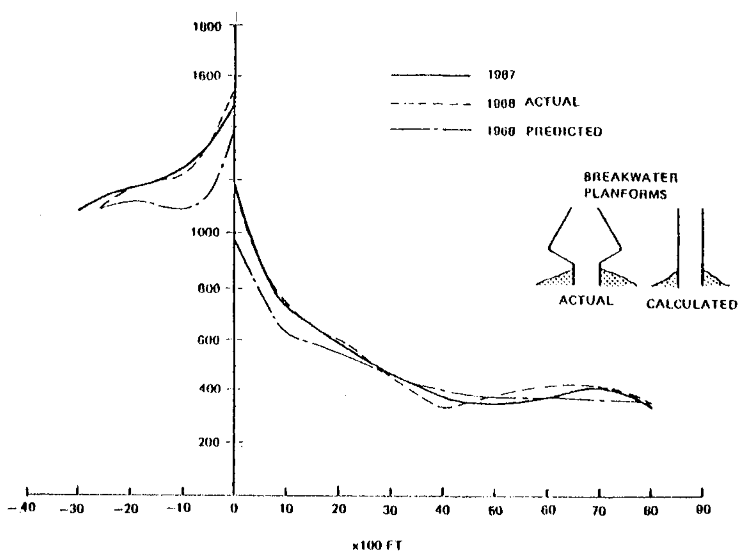

FIGURE 8 COMPUTED vs. ACTUAL DATA

V. CONCLUSION

The basic idea of Pelnard-Considere, i.e., to investigate shoreline evolution by concentrating on conservation of mass as a spacial one dimensional problem, has been generalized to essentially its limits of applicability. These physical processes of refraction and diffraction (where applicable) have been incorporated, as have deterministic variations in lake level, bluff height and beach slope. The inclusion of refraction makes possible the proper use of the known physical relationships between wave energy and littoral drift on a priori basis without necessarily determining these as results from the past recorded shorelines at a given location. Accurate determination of the behavior of the shore in the lee of a breakwater requires inclusion of diffraction in some form. This could be done in a heuristic manner, as presented here either in the global approximation described in a previous section or in the use of the constant depth theory of Penny and Price. It could also be done in a more rigorous manner which would include the effects of a sloping beach. Thus quantitative predictions of the shoreline can, in theory, be attempted in situations where on-offshore transport of sand in either negligible or as known from other sources of information.

The resulting theory is presented in several equivalent forms, one in terms of the behavior of the shoreline $y(x,t)$ alone, the other expressed explicitly in the longshore transport $Q(x,t)$ and implicitly in $y(x,t)$. The former has the advantage that numerical schemes, such as that of Crank-Nicolson described earlier, can qualitatively indicate the behavior of the shoreline in regions of rapid change. However, the conservation of mass is difficult, if not initially impossible to achieve since any approximation of a transport derived term (i.e. a term arising from $\partial Q/\partial x$) will alter the transport balance. On the other hand, the later form allows employment of analytical or numerical approximations in the transport equation which will not disturb the total sand content of the system, but only its local distribution.

The most severe and unavoidable limitations to the engineering application of these methods is the use of the statistical wave summaries. While one possible use of these was attempted, many others are possible. Efficient and accurate employment of the offshore wave statistics is endemic to the problem of large scale shoreline prediction, and must be achieved before any theory, whether one line, multiple lines, or grid can successfully produce accurate results.

Also, the problem of shoreline evolution sensitivity to time step in the input wave climatology would require further research.

Despite this limitation, it is felt that by taking into account effects of wave refraction, wave diffraction and change of lake level, as done in this paper, a mathematical model with multiple bottom contour lines could be formulated which will, if the problem of wave statistics input is solved, permit us to calculate the evolution of the complete bottom topography.

It is important to point out that wave refraction effect on shoreline evolution have been found particularly important. It is particularly necessary in order to determine a planform stability criteria, which can be established from the present formulation.

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