

Developments In Business Simulation & Experiential Exercises, Volume 17, 1990

MODELING COST FUNCTIONS IN COMPUTERIZED BUSINESS SIMULATION: AN APPLICATION OF DUALITY THEORY AND SHEPPARD'S LEMMA

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ABSTRACT

The paper develops an algorithm to model jointly, cost and production functions in computerized business simulations. The algorithm utilizes the concepts of duality theory to derive a generalized cost function, where costs depend both on the level of input prices and the production rate. Sheppard's Lemma is applied to derive the cost minimizing input demand levels based on the characteristics of the cost function. The application of Sheppard's Lemma is shown to help avoid inconsistencies between the cost structure and production technology of the firm. A recommended set of equations is presented and discussed to simulate the cost function of the firm. A numerical example is given to illustrate how the function may be used to demonstrate the stability of the system.

INTRODUCTION

In 1982 Kenneth R. Goosen presented paper at the ABSEL Conference identifying the need to expand research pertaining to the internal design of computerized business simulations. A review of the literature by Goosen (1982) showed that past research on simulation design was simply not extensive enough to assist in a meaningful way in the development of new and better simulations; and he concluded:

"The designing and developing of simulations, appears to be an art form, a creative skill based on intuitive feel rather than acquired knowledge."

Although research in the area of simulation design has increased, a study by Goosen in 1966 raised a concern about the nature of the research relating to mathematical modeling of functional relationships, specifically he stated:

"Very little research has been published concerning the development of functional equations for business games... Satisfactory mathematical equations that have inflection points or maximum and minimum values at the desired points over a desired range of values are difficult to develop. In many cases equations that appear suitable only give desired results over a limited range of values."

The focus of this paper is to address this concern with respect to the modeling of cost functions in computerized business simulations. A review of a number of contemporary business simulations by Gold and Pray (1989) identified a problem in the design of cost functions. Almost all simulations reviewed displayed a Linear relationship between production and costs in both the short-run and long run, implying constant returns to the variable inputs and no economies-of-scale in the cost structure.

Economists have published a great many studies on the cost structure of firms and industries, utilizing wide variety of statistical, engineering, and accounting methods. Although there were some disagreements in the results of these studies, a comprehensive review of the literature by Walters (1963), showed that economies-of-scale were pervasive and existed to some degree in almost all industries. Despite these findings, the cost structure embodied in most computerized business simulations appear to be linear, as indicated by Gold and Pray (1989), who concluded: "Most simulations that permit capital expansion have fixed dollar ratios for plant expansion. This fixed ratio approach imposes constant returns to scale on all firms over any time horizon." (p.26) Gold and Pray (1989) also noted that some designers modeled economies of scale by changing input prices while keeping productivity constant. While this approach may seem adequate it creates an inconsistency between the cost function and the production function. In this case productivity is constant but average costs are declining. Although this result is possible, it is not the general scenario. Duality theory argues that economies of scale are derived, more generally, from increasing returns in the production process and then manifest themselves in the cost structure.

PURPOSE AND PROCEDURE

The paper will proceed to develop an algorithm for the modeling of cost functions in computerized business simulations. The model allows the designer to develop the cost relationship in the simulation first and then derive, jointly, the levels of input usage and the production function implied by the cost structure. The advantages of this approach are threefold. First, it guarantees that the behavior of the production function will be consistent with the cost structure of the firm. Second, the cost information needed to model "real world" firms is more accessible in published sources than production data. Since the approach in this paper uses cost information to develop the cost function first, and then derives the implied production technology, it is easier to simulate. Third, the impact of costs in the simulation on financial performance are more direct. Costs impact profits directly, whereas productivity changes first impact costs.

The paper proceeds in the following manner.

- (1) Summarizing the theoretical properties of cost functions that are most important in the design of computerized business simulations; especially the properties of duality between cost and production implied by Sheppard's Lemma.
- 2) Developing a stable and flexible system of cost equations that encompass the key theoretical properties implied by duality theory and empirical research. The recommended system of equations will permit the

designer to specify, simultaneously, the degree of economies of scale in the cost structure and returns to scale in production.

(3) Presenting a procedure to derive the parameters of the cost system based on the appropriate specifications of the designer. A numerical example is given to illustrate the procedure.

(4) Deriving the input (or factor) demand equations from the parameters of the cost function by applying Sheppard's Lemma.

(5) Simulating the cost system and the derived production function given the parameters derived in the numerical example to demonstrate and discuss how the system functions.

DUALITY AND COST THEORY

Duality theory states that the cost function and the production function are associated and must behave in a consistent manner. There is a direct relationship between costs and production, such that an increase in production efficiency will decrease costs or, conversely, a decrease in production efficiency will increase costs.

The dual relationship between cost and production may be illustrated, succinctly, by first assuming there are only two inputs, Labor (L) and material (M); however, these results may be generalized easily to "n" inputs. The total variable costs (TVC) are then:

$$\begin{aligned} \text{TVC} &= (PL) + (Pm)M \\ \text{Where: } PL &= \text{price of labor } (\$/\text{hr}) \\ Pm &= \text{price of Mat'l } (\$/\text{lb}) \end{aligned}$$

In equation 1 the decision variables are the hours of labor and pounds of material to be used by the firm. The exogenous parameters facing the firm are the price of labor and materials.

It follows that the production function is also dependent on the same two inputs: Dividing both sides of equation 1 by the quantity produced Q, we get the average variable cost

$$Q = f(L, M) \quad [2]$$

where: Q = quantity produced

equation (AVC):

Equation 3 shows average variable costs (AVC) are inversely related to the average products of

$$AVC = PL/APL + Pm/APm \quad [3]$$

where: AVC = average variable costs
APL = average product of labor = Q/L
APm = average product of mat'l = Q/M

labor and materials. As the average product of Labor (APL) or material (APm) increase the AVC will decrease. The marginal cost equation may be derived by taking the derivative of equation 1 with respect to Q:

$$MC = PL/MPL + Pm/MPm \quad [4]$$

where: MC = marginal cost of production
MPL = marginal product of labor = dQ/dL
MPm = marginal product of material = dQ/dM

Equation 4 shows marginal costs (MC) are inversely related to the marginal products of the variable inputs. As the marginal product of labor or materials increase, the MC will decline.

DERIVING THE GENERALIZED COST FUNCTION

Applying the approach presented by Sheppard (1970) it may be shown that costs can be expressed as a function of the Level of input prices and production. The first step is to formulate the lagrangian equation for minimizing total variable costs, equation 1, subject to the production constraint, equation 2:

The cost minimizing input usage may then be obtained by setting the partial derivatives of the

$$Z = (PL)L + (Pm)M + g(Q - f(L, M)) \quad [5]$$

where: g = lagrangian multiplier of constraint Q.

lagrangian equation Z with respect to the inputs equal to zero, giving us the following first order conditions

$$\begin{aligned} dZ/dL &= PL - g \cdot dQ/dL = 0 & [6] \\ dZ/dM &= Pm - g \cdot dQ/dM = 0 & [7] \end{aligned}$$

Solving the equation set simultaneously, the cost minimizing levels of labor and capital are a function of PL, Pm and Q.

Substituting into equation 1, the generalized cost function may be written as:

$$TVC = f(PL, Pm, Q) \quad [8]$$

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Equation is significant for the purpose of modeling cost functions in business simulations. Equation 6 shows that total variable Costs may be expressed as ~ function of input prices and production levels, without directly specifying the level of input use. The level of input use may be derived by applying Sheppard's Lemma.

SHEPPARD'S LEMMA AND THE DERIVED DEMAND FOR INPUTS

Sheppard <1970> proved that the demand for inputs may be obtained by differentiating the cost function with respect to the variable input prices. Given the generalized Cost function, equation ~, and applying Sheppard's Lemma we get:

Equation 7 specifies that the quantity of labor used by the firm may be determined through the cost

$$L = d(TVC)/dP_L \quad [7]$$

$$M = d(TVC)/dP_M \quad [8]$$

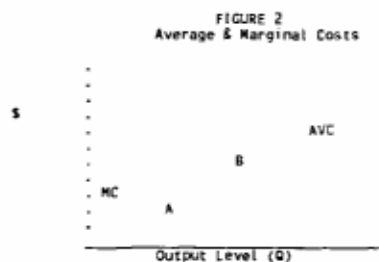
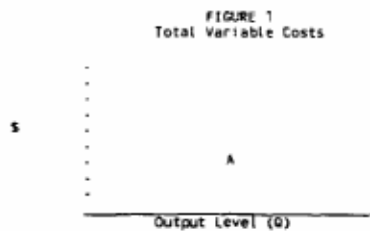
function by taking the derivative of total variable costs with respect to the price of 'labor. Similarly, equation B specifies the quantity of material used by the firm is the derivative of total variable costs with respect to the price of materials.

Sheppard's Lemma is a powerful theoretical tool for the design of cost and production functions. Once the cost function is specified, the demand for inputs (labor and materials) may be ascertained in a manner consistent with duality theory. In this case, increases in average variable costs or marginal costs would imply decreases in average products or marginal products of the variable inputs (as described by equations 3 & 4).

SHAPE OF THE COST FUNCTION

A cost function is an expression relating the costs of doing business to the level of production. The general shapes of the total, average, and marginal variable costs are illustrated in Figures 1 & 2.

Figure 1 shows total variable Costs are generally "S" shaped. Initially, variable Costs rise at a



decreasing rate with respect to output. After point 'A' variable costs rise at an increasing rate. Point "A" is referred to as the point of diminishing returns, and indicates that the productivity of the variable input starts to decline after this point. In the short-run this is due to fixed factors of production and capacity constraints. In the long-run, this is due to decreasing returns to scale in production. Decreasing returns to scale implies diseconomies of scale ~r rising average costs (given fixed factor prices).

Figure 2 shows the marginal and average cost Curves are "U" shaped. Initially the marginal cost declines up to point 'A' and then begins to increase. Average variable costs continue to decline with increases in output until point 'B' and then begin to increase. At point "B" MC is equal to AVC. After point "B" MC exceeds AVC. Duality between production and cost flies marginal products of the variable inputs (MP) would rise until point "A" and then decline; whereas the average products of the variable inputs (AP) would continue to decline until "B" and there rise. At point "B", the MP should also equate the AP. After point "B" MP is Less than AP'.

Economies of scale exist in the long-run if AVC exceeds MC. Economies of scale implies increasing returns to scale in production given fixed factor prices. Constant economies of scale occurs when MC exceeds AVC Diseconomies of scale implies decreasing returns to scale in production given fixed factor prices. Constant economies of scale occur when AVC = MC. The degree of economies of scale (E) is measured by the ratio of AVC to MC, such that

$$E = AVC/MC$$

If $E > 1$ then economies of scale exist; if $E < 1$ then diseconomies of scale exist; and if $E = 1$ then there are constant economies. Generally, it is expected that at low levels of output the firm would be able to achieve economies of scale, and after some point would only be able to obtain constant economies; and eventually diseconomies of scale would occur.

A RECOMMENDED COST SYSTEM

A recommended system of cost equations for modeling business simulations is presented that is consistent with the theory of cost and duality. The cost function is multiplicative in nature and is flexible enough to model Increasing and decreasing returns to the variable input; as well as economies and diseconomies of scale. For clarity of exposition, the two input cases will be illustrated, but the function is easily generalized to any number of arguments.

Multiplicative Cost Function

$$TVC = c_1(P_L)^{c_2}(P_M)^{c_3}Q^{c_4 + c_5Q} \quad [10]$$

where: TVC = total variable cost
 c_i = parameters; $i = 1$ to 4
 P_L = price of labor input
 P_M = price of mat'l input
 Q = quantity produced

The parameter c_1 is simply a scaling factor to obtain the desired level of cost. Parameters c_2 and c_3 are the input price elasticities and show the proportion of input costs to total variable costs. The exponent term $(c_4 + c_5Q)$ allows for variable cost elasticities with respect to output, 0. Variable cost elasticity is necessary to model Increasing and decreasing returns and economies and diseconomies of Scale. Economies of scale (E) may be derived from equation 10 since $E = AVC/MC = (TVC/Q)/(dTVC/dQ)$.

Economies of Scale

$$E = 1 / (c_4 + c_5 Q(1.0 + \ln Q))$$

Where E = economies of scale

$\ln Q$ = natural log of Q

The resulting input demand equations derived from the cost function using sheppard's lemma are:

INPUT DEMAND EQUATIONS

$$L = c_2 (TVC)/P_L$$

$$M = c_3 (TVC)/P_M$$

A well-behaved cost function also requires the following restrictions:

HOMOGENEITY RESTRICTION

$$c_2 + c_3 = 1.0$$

This restriction guarantees that the cost function is homogenous of degree one. This simply means that if all variable input prices increase by some proportion, say 10%, then total variable costs will increase 10%, given a constant production level. This relationship holds by definition, refer to equation 1.

NUMERICAL EXAMPLE: DESIGNING THE COST FUNCTION AND DETERMINING THE PARAMETERS OF THE SYSTEM

Although the equations appear complex, it is relatively easy to design a cost function and solve for the parameters of the system. To illustrate the ease of application, a numerical example will be given.

Suppose a simulation designer wants to model a cost function that possesses increasing returns or economies of scale at an initial output level of 1000 units, and constant returns at 1400 units of output. Of course any Scenarios consistent with standard cost behavior could be evaluated. To summarize, we have the following:

Economies	Output (Q)
$E = 2.0$	1000 units
$E = 1.0$	1400 units

Assuming a two input Case (labor and materials), the designer needs to specify the proportion of variable costs that are attributed to labor (c_2) and the proportion of variable costs attributed to materials (c_3). The sum of c_2 plus c_3 must equal 1.0. Lets suppose we specify

Inputs	Proportions
L	$c_2 = 0.80$
M	$c_3 = 0.20$
sum	1.00

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finally, the designer needs to specify the total variable costs corresponding to the initial output level of 1000 units; and the input prices:

Given the above data, the first step is to solve for the parameters C4. and C5 by using equation 11

$$\begin{aligned} \text{TVC} &= \$100,000 \text{ at an output level of } 1000 \text{ units.} \\ P_1 &= \$25 \text{ per hour of labor} \\ P_m &= \$10 \text{ per pound of materials} \end{aligned}$$

and substituting in the values for E and 0 given above.

The second step is to solve for the parameter C1. Simply substitute all known parameters and variable's into equation 10 and solve. At this point all parameters and variables are known except for C1:

$$\begin{aligned} 2.0 &= 1 / (c_4 + c_5 1000(1.0 + 1n1000)) ; \text{ and} \\ 1.0 &= 1 / (c_4 + c_5 1400(1.0 + 1n1400)) \end{aligned}$$

Simplifying each equation respectively:

$$0.5 = c_4 + 7907.76 c_5 ; \text{ and}$$

$$1.0 = c_4 + 11541.92 c_5$$

Solving the two equations simultaneously:

$$c_4 = -0.5879$$

$$c_5 = 0.0001376$$

Note that C1 is a scaling factor and does not affect the shape or properties of the cost function. The final total variable cost equation becomes:

$$100000 = c_1(25)^{-0.8}(10)^{-2.0}(0.5879 + 0.0001376 \cdot 1000)$$

Solving the above equation for c1 we get:

$$c_1 = 107782.4$$

Once the parameters of the cost function are known, the input levels for labor and materials may be derived by substituting into equations 12 and 13 the values for TVC given any level of 0, P1, and Pm.

$$\text{TVC} = 107782.4(P_1)^{-0.8}(P_m)^{-2.0}(0.5879 + 0.0001376 Q)$$

The input levels derived in this manner are consistent with the "dual" relationship between cost and production functions.

$$L = 0.80 (\text{TVC})/P_1$$

$$M = 0.20 (\text{TVC})/P_m$$

SIMULATING THE COST FUNCTION

The cost function in the numerical example will be simulated along with the dual production function to illustrate its behavior and the relationship between production and total variable costs. The output level was varied between 900 units and 1600 units, given fixed input prices of \$25 and \$10 for labor and materials respectively. Table I summarizes the results.

Average variable costs are "U" shaped with the minimum level occurring at the production rate of 1400 units. After 1400 units of output, AVC begins to rise. Economies of scale behave as modeled by the designer in the example. Returns to scale start at 2.64 and gradually decline. Constant economies

TABLE 1
Impact of changing Output Levels on Variable Costs

Quantity	TVC	AVC	Scale(E)
900	95,489	106.1	2.64
1000	100,000	100.0	2.00
1100	105,490	95.9	1.60
1200	111,965	93.3	1.34
1300	119,463	91.9	1.14
1400	128,045	91.5	1.00
1500	137,791	91.9	0.89
1600	148,805	93.0	0.80

of 1.00 correspond to the minimum AVC, which is consistent with the theory of cost. After an output rate of 1400 diseconomies of scale are exhibited and the scale coefficient, E, drops below 1.00.

The "dual production function can be determined through the cost function by using the derived input demand equations (estimated in the numerical example). Given the same output levels, Table 2 summarizes the results, focusing on the labor input.

The average product of Labor, AP1, is the dual of the average variable cost. Average product of labor begins at relatively Low Level, 0.295 units per hour, and gradually rises. In tandem, average variable costs fall in response to the increased productivity of labor. The maximum average project of labor

TABLE 2
The Dual Production Function and Labor Productivity

Quantity	Labor	AP1
900	3055.7	0.295
1000	3200.0	0.313
1100	3375.7	0.326
1200	3582.9	0.335
1300	3822.8	0.340
1400	4097.4	0.342
1500	4409.3	0.340
1600	4761.8	0.336

occurs at an output rate of 1400 units, which corresponds to the minimum point on the average variable cost function. The second input, materials, behaves in a consistent fashion but the results are not displayed.

SUMMARY AND CONCLUSIONS

The properties of duality theory need to be addressed when designing cost and production functions. The characteristics of the cost structure embodied in a business simulation imply certain over a wide range of values.

(2) The cost function is restricted to be homogeneous of degree one, a requirement of a well-behaved cost structure.

(3) The Levels of inputs and the production function is derived directly from the cost function, guaranteeing the properties of duality are maintained. As a consequence, only cost information i-characteristics relating to the production technology. If these relationships are not carefully modeled, inconsistencies between production and cost may develop. A review of the Literature has indicated that there are some common problems in the way in which contemporary business simulations have designed there cost and production relationships, especially pertaining to economies of scale.

The system of equations developed in this paper to model the cost structure of the firm possesses a number of desirable properties:

(1) A multiplicative functional form that relates total variable costs directly to input prices and the level of production. The functional form is flexible and allows for variable elasticities, increasing and decreasing returns, and economies and diseconomies of scale. The function also appears to be stable needed to model and simulate both the cost and production functions.

(4) The parameters of the cost system may be easily calculated with only limited data requirements. Only information pertaining to input proportions, total variable costs at the starting point of the simulation and economies of scale coefficients for two discrete points are needed.

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