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On the Construction of Constitutive Relations in Hyperelasticity

L. Tao[a,*], K.R. Rajagopal[b]

[a] *Department of Aerospace Engineering, Indian Institute of Technology Madras, Chennai 600 036, India.*
[b] *Department of Mechanical Engineering, Texas A&M University, College Station, Texas 77843, USA.*

Abstract

The notion of abstract bodies and their configurations in a three-dimensional Euclidean space as used in rational continuum mechanics is examined with a view towards assessing how well it reflects physical perspective, by considering the traditional formulation of constitutive relations between the stress and the deformation gradient for homogeneous compressible hyperelastic materials. It is found that the notion of deformation gradient tensor that is currently used needs to be modified, the notion of material symmetry should be introduced in a different manner, and the traditional formulation needs to be recast and extended.

Key words: deformation gradient; elasticity; elastoplasticity; hyperelasticity; material symmetry; material frame-indifference; rheology; viscoelasticity

1. Introduction

Recently, Rajagopal and Srinivasa (Rajagopal and Srinivasa (2009)) have developed a completely Eulerian framework to describe the response of elastic bodies that are not hyperelastic, without having recourse to introducing the notion of deformation gradient. Such a formulation is demanded when one tries to describe the response of biological matter that can grow or atrophy, which makes the notion of Lagrangian tracking from a reference configuration other than the current configuration irrelevant. This paper is one in a series of the recent papers by Rajagopal and co-workers (see Rajagopal (2003), Rajagopal (2007), Bustamante and Rajagopal (2010)) that have greatly enlarged the scope of what one means by elastic bodies.

In this note, within a Lagrangian framework we examine the traditional construction of the general constitutive relations for homogeneous compressible hyperelastic solids (Truesdell and Noll (2004)) with two main objectives in mind: (i) We examine the notion of an abstract body or to be more precise, certain quantities whose definitions rest with the abstract body or a reference configuration acting as its surrogate as introduced from a mathematical standpoint, with regard to its aptness from a physical perspective, (see Rajagopal and Tao (2008) where the need for the same is articulated). This is carried out by considering specifically the nature of the deformation gradient tensor and the role it plays in the traditional development of constitutive theory. We will show that the deformation gradient tensor has to be viewed differently and its invariance properties has to be different if it is to be compatible with experimental measurements and tests. (ii) Once a new interpretation for the deformation gradient is given, we discuss the necessity to reformulate and extend the traditional construction, especially the specific stored energy, in order to make the results applicable to elastic solids other than those that are isotropic and to make the results compatible with the principle of Galilean invariance (and also compatible with the assumption of material frame-indifference, if one so desires).

*Email: *luoyitao@iitm.ac.in*

This work is motivated by the observation that one major aim of the traditional treatment is to provide a constitutive relation for the stored energy in terms of the deformation gradient for hyperelastic materials. To determine the specific form of the stored energy for a specific hyperelastic material, we usually need to carry out experiments stretching/shearing samples of the material and measuring the deformations and the corresponding tractions; then with the aid of the experimental data and on the basis of the model that we have in mind, we need to carry out data reduction which usually reduces to finding the constants or material functions that appear in the specific form for the model that we have chosen. As most of the experiments at our disposal are one dimensional, or at best two dimensional in nature, we are reduced to curve-fitting in order to obtain the constants. This procedure requires us to have quantities in the relation, especially the deformation, that are experimentally measurable.

We need to point out that we use the treatment of Truesdell and Noll (2004) as the reference, instead of the updated version of Noll (2006), due to the following considerations. From the viewpoint of testing, calibrating and applying a constitutive model in engineering, it is preferable to deal with a set of variables that are amenable to physical interpretations and measurements. For example, it is not straightforward to see the role of the principle of Galilean invariance in the updated version, and such invariance requirement should be taken as an essential element in the formulation of elasticity within the context of Newtonian mechanics. Though Noll (2006) has reinterpreted the assumption of material frame-indifference, we will show that this assumption does not play a critical role in the case of hyperelasticity. The main defect of the traditional development lies in its adoption of an abstract body and peculiar status that is given to the reference configuration, which seems to lack sound physical basis; as a result, the deformation gradient tensor is not defined in a physically sound way and the material symmetry is not represented in a mathematically appropriate fashion to reflect what it means from the physical standpoint.

Unlike Rajagopal and Srinivasa (2009), we adopt the notion of a special reference configuration which is stress-free in this work. There are several reasons for this practice. A stress-free reference configuration offers a convenient and straightforward way to characterize material symmetry possessed by a hyperelastic body, especially in the case that the body is anisotropic.[1] When an elastic body deforms, its material symmetry changes; If the material symmetry of the body in its stress-free configuration is known (say, characterized by certain unit vectors $\{\mathbf{N}^{(i)}\}$ of symmetry together with the transformations $\{\mathbf{S}\}$ among the vectors), the material symmetry of the elastic body in its stressed configuration is supposed to be completely determined by $\{\mathbf{N}^{(i)}\}$ and the modified deformation gradient. Therefore, the adoption of the special reference configuration and the explicit presence of $\{\mathbf{N}^{(i)}\}$ make it unnecessary to track the changes to the material symmetry. Furthermore, the explicit presence of $\{\mathbf{N}^{(i)}\}$ is essential for us to have a general constitutive relation for homogeneous compressible hyperelastic solids, isotropic or not, when we use our definition of the modified deformation gradient and enforce the principle of Galilean invariance, as will be shown below.

In the next section we will first summarize the procedure and the main results of the traditional treatment and then examine the aptness and efficacy of the treatment from an experimental perspective. We conclude with some remarks regarding our results in Section 3.

2. Basic analysis

Let $\mathcal{B}$ denote the abstract body, and let κ be a one-parameter (time) family of mappings of the abstract body into a three-dimensional Euclidean space that we will call placers, and let $\kappa(\mathcal{B})$ denote the configurations occupied by the abstract body (see Fig. 1). The one-parameter family of placers essentially defines the motion of the body, however we find it more convenient to introduce a one-to-one mapping $\boldsymbol{\chi}$ that maps a reference configuration of the body in time. That is, if $\mathbf{X} = \kappa_0(\mathcal{P})$, $\mathcal{P}$ belonging to $\mathcal{B}$, and $\mathbf{x} = \kappa_t(\mathcal{P})$ then we can define

[1] We have given a definition of material symmetry for hyperelastic solids in Rajagopal and Tao (2008). Also see Eq. (23) below for the same.

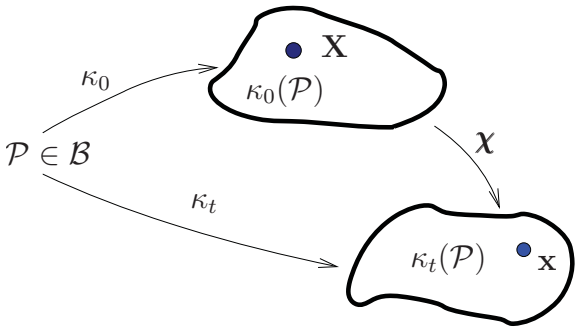

Figure 1: Abstract body, placer and configurations

a function:

$$\mathbf{x} = \boldsymbol{\chi}\big(\mathbf{X}, t\big). \tag{1}$$

We need to formulate the constitutive relation for the Cauchy stress tensor for elastic materials in order to determine the motion $\boldsymbol{\chi}$ of $\mathcal{B}$ under adequately given initial and boundary conditions.

2.1. Traditional development

To understand clearly the problems with the current formulation of constitutive equations for elastic bodies, we summarize the results from the traditional development of compressible hyperelastic solids below Truesdell and Noll (2004).

a. From the second law of thermodynamics, one can obtain

$$\sigma_{ij} = \rho F_{iI} \frac{\partial \psi}{\partial F_{jI}}. \tag{2}$$

Here $\boldsymbol{\sigma}$ is the Cauchy stress tensor and ρ the mass density of $\mathcal{B}$ in the present configuration; $\mathbf{F}$ is the deformation gradient tensor defined through

$$F_{iI} := \frac{\partial \chi_i(\mathbf{X}, t)}{\partial X_I}, \tag{3}$$

and

$$\psi = \tilde{\psi}(\mathbf{F}, T) \tag{4}$$

is the specific stored energy and T is the absolute temperature. The dependence on T will be suppressed from now on for the sake of brevity. One may also replace T with the specific entropy s.

b. One imposes the principle of material frame-indifference (FI), i.e., invariance under the mapping,

$$\mathbf{x}^* = \mathbf{x}^*\big(\mathbf{X}, t^*\big) = \mathbf{c}(t) + \mathbf{Q}(t)\,\boldsymbol{\chi}\big(\mathbf{X}, t\big), \quad t^* = t - a, \tag{5}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time t, $\mathbf{Q}(t)$ an arbitrary orthonormal tensor which is a function of t, and a an arbitrary constant. It has been assumed that the reference configuration is unaffected by the change of frame, that is, there is no $\mathbf{X}^*$ corresponding to $\mathbf{X}$, the reference configuration acts as a surrogate for the abstract body and is not viewed as a configuration occupied by the body at some time t, in which case the points in it would be viewed differently by the observers in the two different frames. Under (5),

the deformation gradient tensor transforms according to

$$\mathbf{F}^* = \mathbf{Q}(t)\,\mathbf{F}, \tag{6}$$

and the specific stored energy obeys

$$\tilde{\psi}(\mathbf{F}) = \tilde{\psi}(\mathbf{F}^*) = \tilde{\psi}(\mathbf{Q}(t)\,\mathbf{F})\,. \tag{7}$$

With the help of the polar decomposition theorem, $\mathbf{F} = \mathbf{RU}$, and $\mathbf{Q}(t) = \mathbf{R}^T$, one obtains from (7) that

$$\psi = \tilde{\psi}(\mathbf{U}) = \bar{\psi}(\mathbf{C})\,, \quad \mathbf{C} := \mathbf{UU} = \mathbf{F}^T\mathbf{F}. \tag{8}$$

One implication of this result is that $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$.

c. Finally, if $\mathcal{B}$ possesses a certain material symmetry, that is, if $\mathbf{S}$ belongs to a specific subset of the set of unimodular tensors, the form of the specific stored energy is further constrained by

$$\tilde{\psi}(\mathbf{F}) = \tilde{\psi}(\mathbf{FS})\,, \quad \bar{\psi}(\mathbf{C}) = \bar{\psi}\big(\mathbf{S}^T\mathbf{CS}\big)\,. \tag{9}$$

In the following, we will examine the above construction from the perspective of physical measurements. Such an approach of examination is justified as an important aim of the construction above is to obtain a model for the elastic behavior of real materials. To make the discussion easier, we consider the restrictions due to the principle of Galilean invariance before discussing material symmetry possessed by the body.

2.2. Examination of the traditional development

From a physical perspective, it is not clear whether the transformation for the deformation gradient, Eq. (6), can be justified, as a reference configuration has meaning only if the configuration of the body is in a three-dimensional (physical) space; and if the abstract body is placed in a three-dimensional space by a "placer" at some time, then each particle in the configuration will be accessible to observers associated with the frames. Otherwise, the reference configuration is not really a configuration in the sense that it is a part of the three dimensional space which forms a part of four dimensional space-time, and as such is just acting as a stand-in or substitute for the abstract body. That all the configurations are observable is of paramount importance as one cannot discuss correlation with experiments other than with regard to observations made with reference to some frame of a body in some configuration that is observable.

Let $\kappa_0(\mathcal{B})$ denote a reference configuration of the abstract body. Truesdell makes the observation that a reference configuration need not be associated with any specific configuration of the body. In this viewpoint the reference configuration is treated as a surrogate for the abstract body and is not necessarily observable, though one can use it to make mathematical manipulation. From both a philosophical and physical standpoint one is hard pressed to make sense of this statement for obviously such a reference configuration has to be one that could possibly be taken by the body; otherwise calling it a reference configuration will make no sense. More importantly, one could equally make the choice of a configuration that was actually taken by the body as the reference configuration, and one should then have comparison between how the deformation gradient observed using such a configuration as the reference transforms and how the deformation gradient defined as it is done in the traditional treatment transforms. In the usual parlance of rational continuum mechanics, if the reference configuration is a configuration actually taken by the body, it is a relative deformation gradient, and this linear transformation transforms in a manner different from that given by (6). It is this reference configuration that is meaningful and such a measure is more than adequate to meaningfully define an elastic material.

Let $\kappa_0(\mathcal{B})$ represent the reference configuration of a physical body of our interest[2]. For the sake of simplicity, we will assume that the body occupies $\kappa_0(\mathcal{B})$ at $t = 0$ and that $\kappa_0(\mathcal{B})$ is stress-free, the latter makes the description of material symmetry easier to handle. Let the present configuration of the body be denoted by $\kappa_t(\mathcal{B})$. We can

[2]One may view $\mathcal{B}$ as representing the physical body below, if one desires so.

measure how much deformation the body undergoes from $\kappa_0(\mathcal{B})$ to $\kappa_t(\mathcal{B})$. We can also use (1) to describe the motion of the body,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{x}^0, t), \quad \boldsymbol{\chi}(\mathbf{x}^0, 0) = \mathbf{x}^0. \tag{10}$$

That is, $\mathbf{X}$ of (1) is identified as an element belonging to $\kappa_0(\mathcal{B})$ actually occupied by the physical body at, say, $t = 0$. This identification is essential for the application of the constitutive relation (2) to a specific elastic material, especially in determining, through experiments and data reduction, the constants contained in the relation. On the basis of this observation, it is physically difficult to justify the assumption underlying (6) that the reference configuration is unaffected by the change of frame or observer and difficult to understand the relevance of the role of the notion of an abstract body other than through its association with the reference configuration when it comes to the constitutive modeling of hyperelastic materials; Also, it is physically difficult to understand why a deformation undergone by a body will make the present configuration observer-dependent from the supposedly observer-independent reference configuration, even when the deformation is very small. The position of a typical particle of the body is observer-dependent, both in $\kappa_0(\mathcal{B})$ and in $\kappa_t(\mathcal{B})$, which is, under Galilean transformation, related by

$$\mathbf{x}^{0\prime} = \mathbf{b} - \hat{\mathbf{c}}\, t_0 + \hat{\mathbf{Q}}\, \mathbf{x}^0, \quad \mathbf{x}' = \mathbf{b} + \hat{\mathbf{c}}\, (t - t_0) + \hat{\mathbf{Q}}\, \mathbf{x}, \quad t' = t - b, \tag{11}$$

where $\mathbf{b}$ and $\hat{\mathbf{c}}$ are real constant vectors, t_0 and b real constants, and $\hat{\mathbf{Q}}$ a constant proper orthonormal tensor.

We shall now introduce the Relative Deformation Gradient (Relative to the initial configuration)[3]. On the basis of (10), we introduce the relative deformation gradient tensor

$$\hat{F}_{iI} := \frac{\partial \chi_i(\mathbf{x}^0, t)}{\partial x^0_I}. \tag{12}$$

It then follows from (11) and (12) that

$$\hat{\mathbf{F}}' = \hat{\mathbf{Q}}\, \hat{\mathbf{F}}\, \hat{\mathbf{Q}}^T, \tag{13}$$

which means that the relative deformation gradient tensor transforms as a second order tensor under Galilean transformation, in contrast to (6) of the traditional treatment. That the relative deformation gradient transforms according to (13) is well known.

Like Eq. (2) in the traditional development, we can also obtain, from the second law of thermodynamics that

$$\sigma_{ij} = \rho \hat{F}_{iI} \frac{\partial \phi}{\partial \hat{F}_{jI}}, \quad \phi = \tilde{\phi}\big(\hat{\mathbf{F}}, \dots\big). \tag{14}$$

Besides the absolute temperature T or the specific entropy s, the unspecified quantities in $\tilde{\phi}$ are supposedly independent of time. To determine specifically what quantities they should be, we apply the principle of Galilean invariance to get

$$\tilde{\phi}\big(\hat{\mathbf{F}}, \dots\big) = \tilde{\phi}\big(\hat{\mathbf{F}}', \dots\big) = \tilde{\phi}\big(\hat{\mathbf{Q}}\hat{\mathbf{F}}\hat{\mathbf{Q}}^T, \dots\big). \tag{15}$$

There are two possible choices for the quantities as follows.

G1. If we adopt the form of

$$\phi = \tilde{\phi}\big(\hat{\mathbf{F}}\big) = \tilde{\phi}\big(\hat{\mathbf{Q}}\hat{\mathbf{F}}\hat{\mathbf{Q}}^T\big), \tag{16}$$

[3] We are aware that the relative deformation gradient is usually defined with the present configuration as the reference configuration (Truesdell and Noll (2004)). To avoid confusion, we have here added 'relative to the initial configuration'.

similar to (4) of the traditional development, we have the stored energy as an isotropic function of $\hat{\mathbf{F}}$. In the case that the Cauchy stress tensor is symmetric, we can show (Rajagopal and Tao (2008))

$$\phi = \bar{\phi}(\hat{\mathbf{C}}), \quad \hat{\mathbf{C}} := \hat{\mathbf{F}}^T\hat{\mathbf{F}}. \tag{17}$$

And consequently, (16) becomes

$$\bar{\phi}(\hat{\mathbf{Q}}\hat{\mathbf{C}}\hat{\mathbf{Q}}^T) = \bar{\phi}(\hat{\mathbf{C}}). \tag{18}$$

That is, ϕ is an isotropic function of $\hat{\mathbf{C}}$, and thus, the resultant relation for the Cauchy stress tensor is restricted to isotropic compressible hyperelastic solids. It implies that we should not take $\phi = \tilde{\phi}(\hat{\mathbf{F}})$ in general.

The above conclusion also holds if a relation similar to (9) is adopted. To demonstrate this point, we combine (9) and (16) to get

$$\phi = \tilde{\phi}(\hat{\mathbf{F}}) = \tilde{\phi}(\hat{\mathbf{F}}\mathbf{S}) = \tilde{\phi}(\hat{\mathbf{Q}}\hat{\mathbf{F}}\mathbf{S}\hat{\mathbf{Q}}^T), \tag{19}$$

where $\mathbf{S}$ transforms supposedly as a second order tensor under Galilean transformation. In the case of a symmetric Cauchy stress tensor, the relation reduces to

$$\phi = \bar{\phi}(\hat{\mathbf{C}}) = \bar{\phi}(\mathbf{S}^T\hat{\mathbf{C}}\mathbf{S}) = \bar{\phi}(\hat{\mathbf{Q}}\mathbf{S}^T\hat{\mathbf{C}}\mathbf{S}\hat{\mathbf{Q}}^T) \tag{20}$$

Or under the transformation of $\hat{\mathbf{Q}} \to (\det \mathbf{S})\hat{\mathbf{Q}}\mathbf{S}$,

$$\phi = \bar{\phi}(\hat{\mathbf{C}}) = \bar{\phi}(\hat{\mathbf{Q}}\hat{\mathbf{C}}\hat{\mathbf{Q}}^T) \tag{21}$$

which has the same form as (18).

G2. We need to provide information concerning the material symmetry of the body while prescribing the stored energy so that we are not forced to consider only isotropic solids, as shown above. This can be done by making the stored energy depend on a set of vectors, together with certain transformations among the vectors that determine the material symmetry of the body; we will not include these transformations explicitly here in the $\tilde{\phi}$ of (14). Suppose that there exist preferred (unit) direction vectors of symmetry $\mathbf{N}^{(k)}$, $k = 1, 2, 3$, (in $\kappa_0(\mathcal{B})$). These direction vectors transform according to $\mathbf{N}^{(k)\prime} = \hat{\mathbf{Q}}\mathbf{N}^{(k)}$, $k = 1, 2, 3$, under Galilean transformation; and together with the appropriate transformations among themselves, they can be employed as a basis to characterize the material symmetry of the body Rajagopal and Tao (2008). Now, we modify $(14)_2$ and use (15) to obtain

$$\phi = \tilde{\phi}(\hat{\mathbf{F}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}) = \tilde{\phi}(\hat{\mathbf{Q}}\hat{\mathbf{F}}\hat{\mathbf{Q}}^T, \hat{\mathbf{Q}}\mathbf{N}^{(1)}, \hat{\mathbf{Q}}\mathbf{N}^{(2)}, \hat{\mathbf{Q}}\mathbf{N}^{(3)}). \tag{22}$$

We may define the material symmetry set $\mathcal{S}$ for $\mathcal{B}$ through Rajagopal and Tao (2008)

$$\tilde{\phi}(\hat{\mathbf{F}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}) = \tilde{\phi}(\hat{\mathbf{F}}, \mathbf{S}\mathbf{N}^{(1)}, \mathbf{S}\mathbf{N}^{(2)}, \mathbf{S}\mathbf{N}^{(3)}), \ \mathbf{S} \in \mathcal{S}. \tag{23}$$

Such a definition follows from the assumption that physically the body has the same mechanical response when subject to the same relative deformation gradient $\hat{\mathbf{F}}$, while the set of directions $\{\mathbf{N}^{(k)}\}$ is replaced with the set of $\{\mathbf{S}\mathbf{N}^{(k)}\}$. Such an operation may be realized experimentally by (i) deforming the body from $\kappa_0(\mathcal{B})$ to the extent of $\hat{\mathbf{F}}$; (ii) rigidly rotating the stress-free reference configuration of the body such that $\{\mathbf{N}^{(k)}\}$ coincide with $\{\mathbf{S}\mathbf{N}^{(k)}\}$ and then deforming the rotated body to the extent of $\hat{\mathbf{F}}$. We will enlarge $\mathcal{S}$ by including mirror symmetry, inversion symmetry, etc. which the body may possess.

Next, applying Galilean transformation and invariance to (23) and setting $\hat{\mathbf{Q}} = \mathbf{S}^T$ (under $\det\mathbf{S} > 0$), we

have from (22) and (23)

$$\phi = \tilde{\phi}\big(\hat{\mathbf{F}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big) = \tilde{\phi}\big(\mathbf{S}^T\hat{\mathbf{F}}\mathbf{S}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big). \tag{24}$$

This relation may be interpreted experimentally as (i) deforming the body from $\kappa_0(\mathcal{B})$ to $\kappa_t(\mathcal{B})$ to the extent of $\hat{\mathbf{F}}$; (ii) deforming the body from $\kappa_0(\mathcal{B})$ to the extent of $\mathbf{S}^T\hat{\mathbf{F}}\mathbf{S}$. Both deformed have the same amount of the stored energy.

In the case of $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$, we have from (22) and (24),

$$\phi = \bar{\phi}\big(\hat{\mathbf{C}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big) = \bar{\phi}\big(\hat{\mathbf{Q}}\hat{\mathbf{C}}\hat{\mathbf{Q}}^T, \hat{\mathbf{Q}}\mathbf{N}^{(1)}, \hat{\mathbf{Q}}\mathbf{N}^{(2)}, \hat{\mathbf{Q}}\mathbf{N}^{(3)}\big) \tag{25}$$

and

$$\phi = \bar{\phi}\big(\hat{\mathbf{C}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big) = \bar{\phi}\big(\mathbf{S}^T\hat{\mathbf{C}}\mathbf{S}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big). \tag{26}$$

We notice that the relation (26) seems similar to $(9)_2$ of the traditional treatment, with the crucial differences that (i) $\hat{\mathbf{C}}$ and $\mathbf{C}$ are defined on different grounds, one in terms of the relative deformation gradient and the other based on the traditional notion of the deformation gradient, and (ii) the directions of material symmetry are explicitly present in (26). More details concerning the consequences of (25) can be found in Rajagopal and Tao (2008).

We should mention that Rajagopal and Srinivasa Rajagopal and Srinivasa (2009) have discussed the idea of vectors of symmetry in an Eulerian framework. Here, we restrict our consideration of material symmetry to the special case of hyperelasticity in a Lagrangian framework and the adopted set of direction vectors of symmetry $\{\mathbf{N}^{(k)}\}$ corresponds effectively to their tensor $\mathbf{M}$. Our study allows the stored energy to depend on the relative deformation gradient while the study of Rajagopal and Srinivasa Rajagopal and Srinivasa (2009) does not even invoke the concept of deformation gradient or relative deformation gradient.

While replacing the traditional deformation gradient tensor with the relative deformation gradient tensor that can be given physical interpretation and which can be experimentally measured and by employing Galilean invariance, we have demonstrated that a set of vectors should be explicitly present in the representation of the specific stored energy in order to have a general relation for $\boldsymbol{\sigma}$ beyond isotropic hyperelastic materials. Also, the constraint of $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$ should be imposed additionally.

2.3. Assumption of frame-indifference

In the current practice of rational continuum mechanics, the assumption of frame-indifference (FI) is viewed as a principle. In view of this we shall consider briefly the implications of FI which is germane to our discussion.

Disregarding the controversy about the status of the assumption of material frame-indifference, we will examine how to apply the assumption to the construction of compressible hyperelastic models, while incorporating an experimentally measurable relative deformation gradient tensor like that defined in (12).

It follows from (5) and (10) that

$$\mathbf{x}^{0*} = \mathbf{c}(0) + \mathbf{Q}(0)\mathbf{x}^0, \quad \mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)\,\boldsymbol{\chi}\big(\mathbf{x}_k^0, t\big), \quad t^* = t - a, \tag{27}$$

where $\mathbf{x}_i^{0*}$ is the position of a typical particle of the body observed in the moving frame at $t^* = -a$, corresponding to $\mathbf{x}_i^0$ observed in the inertial frame. The directions of symmetry $\mathbf{N}^{(k)}$, $k = 1, 2, 3$, that appear in the stored energy may be transformed according to

$$\mathbf{N}^{(k)*} = \mathbf{Q}(0)\,\mathbf{N}^{(k)}, \quad k = 1, 2, 3. \tag{28}$$

If we extend (12) to

$$\breve{F}_{iI}^* := \frac{\partial x_i^*}{\partial x_I^{0*}}, \tag{29}$$

we have from (27) and (29) that

$$\breve{\mathbf{F}}^* = \mathbf{Q}(t)\,\breve{\mathbf{F}}\big(\mathbf{Q}(0)\big)^T. \tag{30}$$

Adopting

$$\sigma^*_{ij} = \rho\breve{F}^*_{iI}\frac{\partial\Psi}{\partial\breve{F}^*_{jI}}, \quad \Psi = \tilde{\Psi}\big(\breve{\mathbf{F}},\dots\big) = \tilde{\Psi}\big(\breve{\mathbf{F}}^*,\dots\big) \tag{31}$$

and the restrictions due to FI, we explore below the two cases on the forms of the specific stored energy, like that discussed in Subsection 2.2.

F1. Consider

$$\Psi = \tilde{\Psi}\big(\breve{\mathbf{F}}\big) = \tilde{\Psi}\big(\breve{\mathbf{F}}^*\big) = \tilde{\Psi}\Big(\mathbf{Q}(t)\breve{\mathbf{F}}\big(\mathbf{Q}(0)\big)^T\Big), \tag{32}$$

similar to (7) and (16). With the help of the polar decomposition theorem, $\breve{\mathbf{F}} = \breve{\mathbf{R}}\breve{\mathbf{U}}$, $\mathbf{Q}(0) = \mathbf{1}$ and $\mathbf{Q}(t) = \breve{\mathbf{R}}^T$, the above relation reduces to

$$\Psi = \tilde{\Psi}\big(\breve{\mathbf{U}}\big), \tag{33}$$

which leads to

$$\Psi = \bar{\Psi}\big(\breve{\mathbf{C}}\big), \quad \breve{\mathbf{C}} := \breve{\mathbf{F}}^T\breve{\mathbf{F}}, \tag{34}$$

a form similar to (8). Eq. (34) has the same defect as that of (18) in virtue of

$$\bar{\Psi}\Big(\mathbf{Q}(0)\breve{\mathbf{C}}\big(\mathbf{Q}(0)\big)^T\Big) = \bar{\Psi}\big(\breve{\mathbf{C}}\big)$$

that follows from (32). There is, however, one crucial difference between (18) and (34) in that we have imposed $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$ explicitly in the derivation of the former but $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$ is a consequence of FI in the latter.

F2. Consider

$$\begin{aligned}\Psi &= \tilde{\Psi}\big(\breve{\mathbf{F}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big)\\ &= \tilde{\Psi}\Big(\mathbf{Q}(t)\breve{\mathbf{F}}\big(\mathbf{Q}(0)\big)^T, \mathbf{Q}(0)\mathbf{N}^{(1)}, \mathbf{Q}(0)\mathbf{N}^{(2)}, \mathbf{Q}(0)\mathbf{N}^{(3)}\Big).\end{aligned} \tag{35}$$

The equation results in

$$\begin{aligned}\Psi &= \tilde{\Psi}\big(\breve{\mathbf{U}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big)\\ &= \tilde{\Psi}\Big(\mathbf{Q}(0)\breve{\mathbf{U}}\big(\mathbf{Q}(0)\big)^T, \mathbf{Q}(0)\mathbf{N}^{(1)}, \mathbf{Q}(0)\mathbf{N}^{(2)}, \mathbf{Q}(0)\mathbf{N}^{(3)}\Big).\end{aligned} \tag{36}$$

In the derivation of the second equality, we have used FI, $\breve{\mathbf{F}}^* = \breve{\mathbf{R}}^*\breve{\mathbf{U}}^*$ and $\breve{\mathbf{U}}^* = \mathbf{Q}(0)\breve{\mathbf{U}}\big(\mathbf{Q}(0)\big)^T$ from the polar decomposition theorem. Furthermore, FI implies that

$$\begin{aligned}\Psi &= \bar{\Psi}\big(\breve{\mathbf{C}}, \mathbf{N}^{(1)}, \mathbf{N}^{(2)}, \mathbf{N}^{(3)}\big)\\ &= \bar{\Psi}\Big(\mathbf{Q}(0)\breve{\mathbf{C}}\big(\mathbf{Q}(0)\big)^T, \mathbf{Q}(0)\mathbf{N}^{(1)}, \mathbf{Q}(0)\mathbf{N}^{(2)}, \mathbf{Q}(0)\mathbf{N}^{(3)}\Big).\end{aligned} \tag{37}$$

That is, Ψ is an isotropic function of the tensor $\breve{\mathbf{C}}$ and vectors $\{\mathbf{N}^{(k)}\}$ a result formally identical to that of (25) from Galilean invariance.

The above discussion has shown that the result of FI is restricted to isotropic hyperelastic solids, if the notion of abstract body is replaced with quantities experimentally measurable and if the vectors of material symmetry are not explicitly included in representing the specific stored energy. This result speaks to the inadequacy of the revised formulation of hyperelasticity as proposed in Noll (2006).

3. Concluding remarks

Motivated by the observation that a constitutive model for the thermomechanical response of a material body should be composed of physically measurable quantities (or quantities inferrable from measurable quantities) so that one might corroborate and calibrate the model, we examine the role played by the notion of an abstract body and its configurations in rational continuum mechanics. We re-examine the methodology of the traditional formulation of the constitutive relation for homogeneous compressible hyperelastic materials.

We have shown that

1. The role of the traditional deformation gradient tensor should be replaced by that of the relative deformation gradient so that one can define a meaningful experimental procedure to develop a constitutive relation for the stored energy that can be experimentally corroborated.
2. The traditional treatment should be reformulated and extended to include explicitly the vectors that define material symmetry or the like in order to have a general constitutive relation for homogeneous compressible hyperelastic solids, isotropic or otherwise, when we use the relative deformation gradient and impose the principle of Galilean invariance.
3. The revised version of hyperelasticity in Noll (2006) does not circumvent the problem that arises in the traditional formulation of hyperelasticity.

More details and results concerning the formulation on hyperelasticity can be found in Rajagopal and Tao (2008). It would be interesting to extend the above procedure to the general framework of elasticity and certain formulations of viscoelasticity and elastoplasticity.

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