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On the derivation of jump conditions in continuum mechanics

James Casey*

Department of Mechanical Engineering, University of California, Berkeley, CA 94720

Dedicated to Professor K.R. Rajagopal with Warmth and Admiration

Abstract

For continua which contain surfaces of discontinuity, such as shock waves or acceleration waves, the integral balance laws imply certain thermomechanical jump conditions, which may be derived in various ways. In the present expository article, a form of Reynolds' transport theorem is obtained that is particularly useful for deriving these jump conditions.

Key words: continuum mechanics; singular surfaces; surfaces of discontinuity; jump conditions

1. Introduction

In many branches of continuum mechanics, the idealization of a *singular surface*, across which jump discontinuities occur in some mechanical or thermodynamical variables, or in their spatial or temporal partial derivatives of some order, plays a major role. The singular surface is a mathematical represention of a narrow region across which very large changes occur in some field properties of the medium. Examples of such surfaces of discontinuity include shock waves, vortex sheets, acceleration waves, and interfaces between different materials.[1] The theory of singular surfaces is of relevance to a wide variety of materials, ranging from fluids, to elastic solids, to elastic-plastic and viscoelastic materials. In some applications, the surfaces may be fixed in the deforming body, while in others, they propagate relative to the body as wavefronts. In general, the jump discontinuities are of finite amplitude.

The subject has an impressive history. In the 1760's, Euler, in his acoustical studies, permitted discontinuities in derivatives of velocity. In a paper on acoustics in 1848, Stokes considered the possibility of jumps in velocity and density across a surface of discontinuity. He obtained jump conditions corresponding to the conservation of mass and the balance of momentum.[2] Other important contributions followed, including those of Helmholtz, Riemann, Christoffel, Rankine, Hugoniot, Duhem, Hadamard, Rayleigh, and Taylor.[3] This early body of research laid the mathematical foundations for the spectacular advances that took place during the 20th century in the

* *Email: jcasey@me.berkeley.edu*

[1]The width of a typical shock zone in air is about 10^{-7} meters.

[2]Stokes expressed considerable surprise at the conclusions he was being led to, in particular that discontinuous solutions were dynamically possible. See the original paper, Stokes (1848); the version in Stokes (1883) shows a change of heart, with omissions and a substituted paragraph. A fascinating account of Stokes' exchanges with Lord Kelvin and Lord Rayleigh may be found in Chapter 28 of Truesdell (1984) and in Salas (2007, 2010). This episode in the history of science teaches us an invaluable lesson.

[3]An excellent collection of many of the founding papers (in English) has been compiled by Johnson and Chéret (1998).

area of supersonic flow. An elegant, powerful, and definitive formulation of the theory of singular surfaces was developed by Hadamard (1903). An authoritative presentation of Hadamard's theory, with extensive citation of original sources, is elaborated in Chapter C of the treatise by Truesdell and Toupin (1960). Other notable references include Thomas (1949, 1961, 1969), Hill (1961), Gurtin (1972), Chen (1973), Nunziato *et al.* (1974), Geiringer (1973), and Kosinski (1986).

In addition to the works already cited, there is a vast literature on shock waves (see e.g., Emmons 1945, Courant and Friedrichs 1948, Liepmann and Roshko 1957, von Mises 1958, Landau and Lifschitz 1959, Serrin 1959, Band and Duvall 1961, Zel'dovich and Raizer 1966/67, Knowles 1979, Griffith 1981, Anderson 1990, O'Reilly and Varadi 1999, Chapman 2000, and Salas 2010). Accounts of acceleration waves are contained in Truesdell (1961), Hill (1962), Truesdell and Noll (1965), and Wang and Truesdell (1973). A succinct treatment of singular surfaces and waves is given in Truesdell and Rajagopal (2000).

In continuum thermodynamics, the balance laws for mass, linear momentum, angular momentum, and energy are postulated in integral form for subbodies, i.e., definite portions of matter. Additionally, an integral entropy production inequality, such as the Clausius-Duhem inequality, is postulated. For bodies without singular surfaces, these integral statements lead, through the use of Reynolds' transport theorem, to the standard partial differential field equations plus a dissipation inequality. But, for bodies containing one or more singular surfaces, the integral balance laws additionally produce jump conditions that hold at the surfaces of discontinuity. The jump conditions can be derived in somewhat different ways (see Thomas 1949, 1961, 1969, Truesdell and Toupin 1960, and Green and Naghdi 1965), but always require an extension of the transport theorem. The main purpose of the present article is to obtain a form of the transport theorem that is very convenient for the derivation of jump conditions associated with the balance laws.

After summarizing requisite background material in Sections 2 and 3, Reynolds' transport theorem is discussed in Section 4. For a subbody containing a singular surface, extra terms appear in the transport theorem, which are due to discontinuities in the field variables and the motion of the singular surface. We utilize the device of hypothetical "sampling motions" of the continuum as a means to establish the extended form of the transport theorem (see (71)). Further, a novel form of the transport theorem (75b) is established: it involves three material derivatives and an integral taken over the singular surface. This form of the transport theorem is employed in Section 5 to derive the jump conditions for mass, linear momentum, angular momentum, and energy.[4] Pointwise results are collected in Section 6. Standard direct notation is employed (see e.g., Truesdell and Noll 1965, Gurtin 1981, or Chadwick 1999).

2. Preliminaries

Consider a deformable three-dimensional continuum $\mathcal{B}$ moving in inertial space $\mathcal{E}$ under the influence of applied forces and also subject to heating. Mathematically, the body $\mathcal{B}$ is a differentiable manifold with boundary $\partial\mathcal{B}$, and is endowed with a non-negative mass measure (Noll 1959, Truesdell and Noll 1965). Let X be an arbitrary element (or particle, or material point) of $\mathcal{B}$. A subbody $\mathcal{S}$ is any non-empty three-dimensional submanifold of $\mathcal{B}$; the boundary of $\mathcal{S}$ is denoted $\partial\mathcal{S}$. The space $\mathcal{E}$ is a three-dimensional Euclidean point space, with origin O; associated with $\mathcal{E}$ is an inner-product vector space $\mathcal{V}$. The set of all linear mappings on $\mathcal{V}$ may be formed into a vector space, which is the nine-dimensional space of second-order tensors.

2.1 Kinematics

The manner in which the body $\mathcal{B}$ is embedded in space is described by a set of mappings, the configurations (or placements) of $\mathcal{B}$: a *configuration* of $\mathcal{B}$ is a smooth homeomorphism of $\mathcal{B}$ onto a region of $\mathcal{E}$, a region being regarded as a compact set having a piecewise smooth boundary.[5] All the configurations of $\mathcal{B}$ are homeomorphic (or topologically equivalent) to one another.

[4] The jump inequality for entropy is derived using the form (71) of the transport theorem.

[5] A homeomorphism, or topological mapping, is a continuous mapping that also has a continuous inverse.

Consider a time-parametrized family of configurations $\breve{\boldsymbol{\chi}}_t$, continuous in the variable t, where t belongs to some specified interval I of time. We may write

$$\boldsymbol{x} = \breve{\boldsymbol{\chi}}_t(X), \tag{1}$$

where $\boldsymbol{x}$ is the position vector of the point that X is mapped into at the instant t. For each value of t, $\breve{\boldsymbol{\chi}}_t$ is a homeomorphism. We denote its inverse by $\breve{\boldsymbol{\chi}}_t^{-1}$. Intimately connected with the family of functions $\breve{\boldsymbol{\chi}}_t$ is the single function $\breve{\boldsymbol{\chi}}$ of two independent variables, which is defined by

$$\breve{\boldsymbol{\chi}}(X,t) = \breve{\boldsymbol{\chi}}_t(X) \tag{2}$$

for all $X \in \mathcal{B}$ and $t \in I$. The function $\breve{\boldsymbol{\chi}}$ is called a *motion* of $\mathcal{B}$.[6] Also, we call $\boldsymbol{\kappa} = \breve{\boldsymbol{\chi}}(\mathcal{B},t)$ the current configuration of $\mathcal{B}$. For convenience, we may choose as a fixed reference configuration, $\boldsymbol{\kappa}_0$, one of the configurations that is actually occupied by $\mathcal{B}$ at some particular instant during its motion, or could be reached from such a configuration by some other motion. Let $\boldsymbol{X} = \boldsymbol{\kappa}_0(X)$. Clearly, one may employ any coordinate system $(\zeta^1,\zeta^2,\zeta^3)$ on the reference configuration and map it into the body manifold to form a global coordinate system for $\mathcal{B}$. Each particle X then has a fixed label $(\zeta^1,\zeta^2,\zeta^3)$. Such a system of coordinates is called "convected."

The position vector $\boldsymbol{x}$ may be expressed as

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X},t) = \boldsymbol{\chi}_t(\boldsymbol{X}), \tag{3}$$

i.e., either as a single function of $\boldsymbol{X}$ and t, or alternatively, as a family of homeomorphisms. Clearly, for any given motion $\breve{\boldsymbol{\chi}}$ of $\mathcal{B}$, there are infinitely many different functions $\boldsymbol{\chi}$ that correspond to it, each depending on the choice of reference configuration. We shall suppose that, except at singular surfaces, the function $\boldsymbol{\chi}$ is continuously differentiable jointly in $\boldsymbol{X}$ and t, as many times as may be desired.[7] As in (3), we may express $\boldsymbol{X}$ as a function of two variables: $\boldsymbol{X} = \boldsymbol{\chi}_t^{-1}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x},t)$. This function is assumed to have the same smoothness properties as $\boldsymbol{\chi}$. Let $\mathcal{R}_0$ and $\mathcal{R}$ denote the regions that $\mathcal{B}$ occupies in its reference and current configurations, respectively. The region $\mathcal{R}$ may contain one or more singular surfaces (Section 3).

Let θ (> 0) be the absolute temperature at X in the current configuration of $\mathcal{B}$. The temperature history of $\mathcal{B}$ may be described by

$$\theta = \breve{\theta}(X,t) = \Theta(\boldsymbol{X},t). \tag{4}$$

The function Θ may be assumed to be as smooth as desired, except possibly at singular surfaces. The pair of functions $\{\boldsymbol{\chi},\Theta\}$ may be said to define a thermomechanical process for the body $\mathcal{B}$. Other thermomechanical fields will be introduced shortly.

At each value of t, the homeomorphism $\boldsymbol{\chi}_t^{-1}$ may be employed to "pull-back" points, lines, surfaces, and volumes from the region $\mathcal{R}$ to the region $\mathcal{R}_0$; likewise, the homeomorphism $\boldsymbol{\chi}_t$ may be used to "push-forward" sets of points from $\mathcal{R}_0$ to $\mathcal{R}$.[8] Similar remarks hold for the homeomorphisms $\breve{\boldsymbol{\chi}}_t$ and $\breve{\boldsymbol{\chi}}_t^{-1}$. Any subset of the body $\mathcal{B}$ is called a material set. For any given material set, a motion $\breve{\boldsymbol{\chi}}$ generates a one-parameter family of sets, one in each configuration of $\mathcal{B}$. Such a time-parametrized family of sets is said to be material with respect to

[6] Note that at each instant t, the homeomorphism $\breve{\boldsymbol{\chi}}_t$ is, by definition, invertible. Since it is possible for a body in motion to occupy the same region of space at two different times, the function $\breve{\boldsymbol{\chi}}$, depending on two variables, is, in general, not invertible.

[7] For most developments in continuum mechanics, it is sufficient to have continuous partial derivatives of order three. Smoothness assumptions on $\breve{\boldsymbol{\chi}}$ and other functions defined on $\mathcal{B} \times I$ may be phrased in terms of the convected coordinates $(\zeta^1,\zeta^2,\zeta^3)$ and t.

[8] Pull-back and push-forward operations may also be performed on other mathematical objects besides sets (e.g., on vectors or tensors), and then involve the deformation gradient in various ways (see Casey and Papadopoulos 2002).

the motion $\check{\boldsymbol{\chi}}$.[9]

Consider any distributed scalar property ϕ of the body, assumed to be describable by a function of the form

$$\phi = \check{\phi}(X, t). \tag{5}$$

Using the mapping $\check{\boldsymbol{\chi}}_t^{-1}$, we may construct another mathematical representation, $\widetilde{\phi}$, of the property ϕ by

$$\phi = \check{\phi}(\check{\boldsymbol{\chi}}_t^{-1}(\boldsymbol{x}), t) = \widetilde{\phi}(\boldsymbol{x}, t). \tag{6}$$

The mapping $\widetilde{\phi}$, used together with the mapping $\boldsymbol{\chi}_t$, yields yet another representation, $\widehat{\phi}$, of ϕ:

$$\phi = \widetilde{\phi}(\boldsymbol{\chi}_t(\boldsymbol{X}), t) = \widehat{\phi}(\boldsymbol{X}, t). \tag{7}$$

We call the functions $\check{\phi}$, $\widehat{\phi}$, and $\widetilde{\phi}$ the material, referential (or "Lagrangian"), and spatial (or "Eulerian") descriptions of the property ϕ [10]. If any one of these representations of ϕ is given and if the mapping $\boldsymbol{\chi}$ is also given, then the other representations can be found by means of the pull-back and push-forward operators.

The displacement of the particle $X \in \mathcal{B}$ at the instant $t \in I$ is defined by

$$\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{X}, \tag{8}$$

and may be expressed in material, referential, or spatial form.

Whenever the functions $\check{\boldsymbol{\chi}}$ (and hence also $\boldsymbol{\chi}$) are differentiable with respect to time, the velocity $\boldsymbol{v}$ of the particle $X \in \mathcal{B}$ is given by

$$\boldsymbol{v} = \dot{\boldsymbol{x}} = \frac{\partial \check{\boldsymbol{\chi}}}{\partial t} = \frac{\partial \boldsymbol{\chi}}{\partial t} = \dot{\boldsymbol{d}}. \tag{9}$$

The material, referential, and spatial descriptions of $\boldsymbol{v}$ are denoted by $\check{\boldsymbol{v}}$, $\widehat{\boldsymbol{v}}$, and $\widetilde{\boldsymbol{v}}$, respectively. The velocity may be discontinuous at singular surfaces. Also, whenever the function $\boldsymbol{\chi}$ is differentiable with respect to $\boldsymbol{X}$, the deformation gradient is defined by

$$\boldsymbol{F} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}}, \tag{10}$$

and is a second-order tensor. Since the reference configuration $\boldsymbol{\kappa}_0$, as chosen above, is always reachable from the current configuration $\boldsymbol{\kappa}$ via some motion, it follows that its determinant J is always positive. The deformation gradient may be discontinuous at singular surfaces.

Whenever the partial derivatives with respect to time of $\check{\phi}$ (and $\widehat{\phi}$) exist, we write

$$\dot{\phi} = \frac{\partial \check{\phi}}{\partial t} = \frac{\partial \widehat{\phi}}{\partial t} \tag{11}$$

for the material derivative of ϕ, corresponding to a motion $\check{\boldsymbol{\chi}}$ of the continuum. We also often use Stokes' notation $D\phi/Dt$ for $\dot{\phi}$. Whenever the conditions of the chain rule of calculus are satisfied by the functions $\widetilde{\phi}$ and

[9] It is evident that not every continuous time-parametrized family of sets is material: Take, for example, a material family of planes in a deforming elastic solid; a family of planes that are perpendicular to these, will not, in general, be a material family. Also, a family of sets may be material with respect to one motion and not for another. Thus, take a family of unit spheres, one in each configuration of a continuum, and centered at the same particle. For any rigid motion of the body, this family is material, but for general motions, it is not.

[10] Despite popular usage, the terminology "Lagrangian" and "Eulerian" is not historically accurate. The spatial description of fluid motion was formulated in 1749 by d'Alembert for some special cases and was generalized shortly thereafter by Euler. Euler himself discovered the referential description. See Section 14 of Truesdell (1954), where the informative footnotes trace the correct attributions to the original 18th century literature on hydrodynamics.

$\boldsymbol{\chi}$, the material derivative of ϕ may be expressed as

$$\dot{\phi} = \frac{\partial \widetilde{\phi}}{\partial t} + \frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}} \cdot \boldsymbol{v}. \tag{12}$$

Corresponding results hold for distributed vector and tensor properties.

Remark 2.1. Suppose that the spatial description $\widetilde{\phi}$ of some property of the continuum is known. As the body travels through the field, each of its particles X experiences a time-rate of change of the field given by the material derivative (11). If the body were to travel through the same field with a different motion, $\check{\boldsymbol{\chi}}'$ (say), then a different referential description, $\widehat{\phi}'$ would be obtained:

$$\phi = \widetilde{\phi}(\boldsymbol{x}, t) = \widetilde{\phi}(\boldsymbol{\chi}'(\boldsymbol{X}, t), t) = \widehat{\phi}'(\boldsymbol{X}, t), \tag{13}$$

where $\boldsymbol{\chi}'(\boldsymbol{X}, t) = \check{\boldsymbol{\chi}}'(X, t)$. In the new motion, the particle X would experience a new rate of change of the field, given by

$$\left(\frac{D\phi}{Dt}\right)' = \frac{\partial \widehat{\phi}'}{\partial t}(\boldsymbol{X}, t) = \frac{\partial \widetilde{\phi}}{\partial t} + \frac{\partial \widetilde{\phi}}{\partial \boldsymbol{x}} \cdot \boldsymbol{v}', \tag{14}$$

where

$$\boldsymbol{v}' = \frac{\partial \boldsymbol{\chi}'}{\partial t} \tag{15}$$

is the new velocity of X at time t. The following observation should be kept in mind: Even though the two motions, $\check{\boldsymbol{\chi}}$ and $\check{\boldsymbol{\chi}}'$, may be very different from one another, if <u>at the instant</u> t, X occupies the same position $\boldsymbol{x}$ in both motions and X has the same velocity in both motions, then

$$\left(\frac{D\phi}{Dt}\right)' = \left(\frac{D\phi}{Dt}\right), \tag{16}$$

i.e., at the instant t, X experiences exactly the same rate of change of the field in both of these motions.

For any subbody $\mathcal{S} \subseteq \mathcal{B}$, let $\mathcal{P}_0$ and $\mathcal{P}$ be the regions occupied by $\mathcal{S}$ in the configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$, respectively, and let $\partial\mathcal{S}$, $\partial\mathcal{P}_0$ and $\partial\mathcal{P}$ be the corresponding boundaries. Then,

$$\mathcal{P} = \check{\boldsymbol{\chi}}_t(\mathcal{S}) = \boldsymbol{\chi}_t(\mathcal{P}_0), \tag{17}$$

with similar relations holding for the boundaries.

Using the mappings $\check{\boldsymbol{\chi}}_t$, etc., in a similar manner as we previously did for ϕ, the amount of the property ϕ that $\mathcal{S}$ possesses at any time t may be expressed in the equivalent forms

$$\Phi = \check{\Phi}(\mathcal{S}, t) = \widehat{\Phi}(\mathcal{P}_0, t) = \widetilde{\Phi}(\mathcal{P}, t), \tag{18}$$

which are the material, referential, and spatial descriptions of Φ. The functions $\check{\Phi}$, $\widehat{\Phi}$, and $\widetilde{\Phi}$ depend on sets and time, rather than points (or vectors) and time. Whenever the partial derivative of $\check{\Phi}$ with respect to t exists, we write

$$\dot{\Phi} = \frac{D\Phi}{Dt} = \frac{\partial \check{\Phi}}{\partial t} \tag{19}$$

for this material derivative. For bodies that do not contain any singular surfaces, the existence of $\dot{\Phi}$, together with a formula for calculating it, is provided by Reynolds' transport theorem. For bodies containing a singular surface, the transport theorem has to be carefully extended to take account of discontinuities (Section 4).

There is one other useful representation of Φ: Let $\overline{\mathcal{P}}$ be the <u>fixed</u> region of space with which the time-dependent region $\mathcal{P}$ coincides at the instant t. Then, we may also express Φ in the form

$$\Phi = \overline{\Phi}(\overline{\mathcal{P}}, t). \tag{20}$$

It should be kept in mind that although the subbody $\mathcal{S}$ instantaneously occupies the fixed region $\overline{\mathcal{P}}$ at the instant t, it will not, in general, do so a short interval before t, and a short interval after t. In other words, different material sets move in and out of $\overline{\mathcal{P}}$. Let the boundary of $\overline{\mathcal{P}}$ be denoted by $\partial\overline{\mathcal{P}}$.

2.2 Balance laws

Let $\breve{m}$ denote the mass measure specified for the body $\mathcal{B}$. It is assumed to be an absolutely continuous function of volume, with corresponding non-negative bounded mass density. Let ρ_0 and ρ be the mass densities of $\mathcal{B}$ in its configurations $\boldsymbol{\kappa}_0$ and $\boldsymbol{\kappa}$, respectively. The law of conservation of mass is: For any subbody $\mathcal{S} \subseteq \mathcal{B}$, the mass m of $\mathcal{S}$ is

$$m = \breve{m}(\mathcal{S}) = \int_{\mathcal{S}} \mathrm{d}m = \text{time-independent function}, \tag{21}$$

where $\mathrm{d}m$ is the element of mass of $\mathcal{B}$. Equivalently,

$$m = \tilde{m}(\mathcal{P}, t) = \int_{\mathcal{P}} \rho \, \mathrm{d}v = \text{time-independent function} = \int_{\mathcal{P}_0} \rho_0 \, \mathrm{d}V, \tag{22}$$

where $\mathrm{d}v$ and $\mathrm{d}V$ are the elements of volume of the regions $\mathcal{P}$ and $\mathcal{P}_0$, respectively. Further, (21) is also equivalent to

$$\dot{m} = 0. \tag{23}$$

The linear momentum $\boldsymbol{G}$ possessed by the subbody $\mathcal{S}$ in a given thermomechanical process may be expressed in all of the following forms:

$$\boldsymbol{G} = \breve{\boldsymbol{G}}(\mathcal{S}, t) = \widehat{\boldsymbol{G}}(\mathcal{P}_0, t) = \widetilde{\boldsymbol{G}}(\mathcal{P}, t) = \overline{\boldsymbol{G}}(\overline{\mathcal{P}}, t), \tag{24}$$

where

$$\breve{\boldsymbol{G}}(\mathcal{S}, t) = \int_{\mathcal{S}} \boldsymbol{v} \, \mathrm{d}m, \tag{25a}$$

$$\widehat{\boldsymbol{G}}(\mathcal{P}_0, t) = \int_{\mathcal{P}_0} \rho_0 \, \boldsymbol{v} \, \mathrm{d}V, \tag{25b}$$

$$\widetilde{\boldsymbol{G}}(\mathcal{P}, t) = \int_{\mathcal{P}} \rho \, \boldsymbol{v} \, \mathrm{d}v, \tag{25c}$$

$$\overline{\boldsymbol{G}}(\overline{\mathcal{P}}, t) = \int_{\overline{\mathcal{P}}} \rho \, \boldsymbol{v} \, \mathrm{d}v. \tag{25d}$$

Likewise, the angular momentum of $\mathcal{S}$, taken about the origin O in inertial space, is

$$\boldsymbol{H}_{\mathrm{O}} = \breve{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{S}, t) = \widehat{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{P}_0, t) = \widetilde{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{P}, t) = \overline{\boldsymbol{H}}_{\mathrm{O}}(\overline{\mathcal{P}}, t), \tag{26}$$

where

$$\breve{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{S}, t) = \int_{\mathcal{S}} \boldsymbol{x} \times \boldsymbol{v} \, \mathrm{d}m, \tag{27a}$$

$$\widehat{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{P}_0, t) = \int_{\mathcal{P}_0} \rho_0 \, \boldsymbol{x} \times \boldsymbol{v} \, \mathrm{d}V, \tag{27b}$$

$$\widetilde{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{P}, t) = \int_{\mathcal{P}} \rho \, \boldsymbol{x} \times \boldsymbol{v} \, \mathrm{d}v, \tag{27c}$$

$$\overline{\boldsymbol{H}}_{\mathrm{O}}(\overline{\mathcal{P}}, t) = \int_{\overline{\mathcal{P}}} \rho \, \boldsymbol{x} \times \boldsymbol{v} \, \mathrm{d}v. \tag{27d}$$

The external forces that act on the subbody $\mathcal{S}$ are assumed to consist of body forces and surface tractions. These also may be regarded as measures on the manifold $\mathcal{B}$, with bounded density functions. In the spatial description, body forces are represented by a vector field $\boldsymbol{b}(\boldsymbol{x},t)$ per unit mass, while surface tractions are represented by a vector field $\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$, where $\boldsymbol{n}$ is the outward unit normal to the surface $\partial\mathcal{P}$. Let $\mathrm{d}a$ be the area element of $\partial\mathcal{P}$.

The balance laws of linear and angular momenta may be stated in spatial form as: For any sufficiently smooth thermomechanical process of any subbody $\mathcal{S} \subseteq \mathcal{B}$,

$$\dot{\boldsymbol{G}} = \int_{\mathcal{P}} \rho\, \boldsymbol{b}\, \mathrm{d}v \;+\; \int_{\partial\mathcal{P}} \boldsymbol{t}\, \mathrm{d}a \tag{28a}$$

and

$$\dot{\boldsymbol{H}}_{\mathrm{O}} = \int_{\mathcal{P}} \rho\, \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}v \;+\; \int_{\partial\mathcal{P}} \boldsymbol{x} \times \boldsymbol{t}\, \mathrm{d}a. \tag{28b}$$

The internal energy of the continuum may also be regarded as a measure on the manifold $\mathcal{B}$. Let ε be the corresponding internal energy density function, again bounded. The sum, E, of the internal energy plus the kinetic energy of the subbody $\mathcal{S}$ may be expressed in the alternative forms

$$E = \check{E}(\mathcal{S},t) = \hat{E}(\mathcal{P}_0,t) = \widetilde{E}(\mathcal{P},t) = \overline{E}(\overline{\mathcal{P}},t), \tag{29}$$

where

$$\check{E}(\mathcal{S},t) = \int_{\mathcal{S}} \left\{ \varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} \right\} \mathrm{d}m, \tag{30a}$$

$$\hat{E}(\mathcal{P}_0,t) = \int_{\mathcal{P}_0} \rho_0 \left\{ \varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} \right\} \mathrm{d}V, \tag{30b}$$

$$\widetilde{E}(\mathcal{P},t) = \int_{\mathcal{P}} \rho \left\{ \varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} \right\} \mathrm{d}v, \tag{30c}$$

$$\overline{E}(\overline{\mathcal{P}},t) = \int_{\overline{\mathcal{P}}} \rho \left\{ \varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v} \right\} \mathrm{d}v. \tag{30d}$$

Similarly, using the spatial representation for heating measures, let $r(\boldsymbol{x},t)$ be the heat supply per unit mass to the subbody $\mathcal{S}$, and let $-h(\boldsymbol{x},t,\boldsymbol{n})$ be the flux of heat entering $\mathcal{S}$ across the surface $\partial\mathcal{P}$. The balance law for energy (first law of thermodynamics) may be stated as: For any sufficiently smooth thermomechanical process of any subbody $\mathcal{S} \subseteq \mathcal{B}$,

$$\dot{E} = \int_{\mathcal{P}} \rho \,\{\boldsymbol{b}\cdot\boldsymbol{v} + r\}\, \mathrm{d}v + \int_{\partial\mathcal{P}} \{\boldsymbol{t}\cdot\boldsymbol{v} - h\}\, \mathrm{d}a. \tag{31}$$

As for smooothness assumptions on the fields that have been introduced above, we shall require that: Except at singular surfaces, (a) the fields $\rho(\boldsymbol{x},t)$ and $\varepsilon(\boldsymbol{x},t)$ are continuously differentiable; (b) the fields $\boldsymbol{b}(\boldsymbol{x},t)$ and $r(\boldsymbol{x},t)$ are continuous; and (c) for each fixed value of $\boldsymbol{n}$, the fields $\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})$ and $h(\boldsymbol{x},t,\boldsymbol{n})$ are continuously differentiable.[11]

For bodies containing a singular surface, the material differentiation that appears in the balance statements (28a), (28b) and (31) must be given careful consideration (Section 4).

2.3 Entropy inequality

In the literature on continuum thermodynamics, different viewpoints exist regarding the second law of thermodynamics. A majority of researchers follow the approach introduced by Coleman and Noll (1963), who assume that an entropy function exists for all materials (see also Truesdell and Noll 1965). Rivlin (1973, 1975, 1986) has advocated a more conservative approach, in which one attempts to construct an entropy function for a given class

[11] Slightly lighter or heavier smoothness assumptions appear in the literature (see e.g., Truesdell and Toupin 1960, Gurtin 1972, Gurtin 1981, and Chadwick 1999).

of materials. Rivlin's approach has been successfully adopted for several important classes of materials (Casey 1998, Casey and Krishnaswamy 1998, Krishnaswamy and Batra 1997, Casey 2011). The more complicated the class of materials under consideration is, the more difficult it is to construct an entropy function, and many unresolved issues still remain, particularly for materials with memory.

For present purposes, let us suppose that we are discussing materials for which an entropy density function η exists. For any subbody $\mathcal{S} \subseteq \mathcal{B}$, constituted from such a material, we may represent the entropy $\mathcal{H}$ of $\mathcal{S}$ by

$$\mathcal{H} = \check{\mathcal{H}}(\mathcal{S}, t) = \hat{\mathcal{H}}(\mathcal{P}_0, t) = \tilde{\mathcal{H}}(\mathcal{P}, t) = \overline{\mathcal{H}}(\overline{\mathcal{P}}, t), \tag{32}$$

where

$$\check{\mathcal{H}}(\mathcal{S}, t) = \int_{\mathcal{S}} \eta \, \mathrm{d}m, \tag{33a}$$

$$\hat{\mathcal{H}}(\mathcal{P}_0, t) = \int_{\mathcal{P}_0} \rho_0 \, \eta \, \mathrm{d}V, \tag{33b}$$

$$\tilde{\mathcal{H}}(\mathcal{P}, t) = \int_{\mathcal{P}} \rho \, \eta \, \mathrm{d}v, \tag{33c}$$

$$\overline{\mathcal{H}}(\overline{\mathcal{P}}, t) = \int_{\overline{\mathcal{P}}} \rho \, \eta \, \mathrm{d}v. \tag{33d}$$

Also, if we adopt the Clausius-Duhem inequality, then for any thermomechanical process of $\mathcal{S} \subseteq \mathcal{B}$,

$$\dot{\mathcal{H}} \geq \int_{\mathcal{P}} \rho \, \frac{r}{\theta} \, \mathrm{d}v \; - \int_{\partial\mathcal{P}} \frac{h}{\theta} \, \mathrm{d}a. \tag{34}$$

We shall place the same smoothness conditions on the field $\eta(\boldsymbol{x}, t)$ as we did on the internal energy density at the end of Subsection 2.2.

3. Singular surfaces

We now wish to consider situations in which some thermomechanical variables, or their partial derivatives of some order, experience jump discontinuities across a surface which may be moving across the deforming body. Any such surface is called a *singular* surface. We will be especially interested in discontinuities that may arise in the integrands that appear in the balance equations (23), (28a), (28b), (31), and the entropy inequality (34).

Thus, for the configuration of $\mathcal{S}$ at each instant $t \in I$, suppose that there is a smooth orientable surface $\varSigma(t)$ that divides the region $\mathcal{P}$ into two contiguous parts $\mathcal{P}_1$ and $\mathcal{P}_2$, as indicated in Figure 1. In general, this family of surfaces is not material with respect to the motion $\check{\boldsymbol{\chi}}$ (but in special cases, it may be); it is suggestive to think of these surfaces as being the configurations of a fictitious massless sheet that can sweep across the body, while simultaneously stretching and changing its shape. We may place a pair of Gaussian coordinates q^α, $(\alpha = 1, 2)$, on any particular configuration of the sheet and let them convect with the sheet. The motion of the sheet in the inertial space $\mathcal{E}$ can then be described by an equation of the form

$$\boldsymbol{r} = \hat{\boldsymbol{r}}(q^1, q^2, t). \tag{35}$$

We will assume that the function $\hat{\boldsymbol{r}}$ is differentiable with respect to t. The velocity of the sheet is defined by

$$\boldsymbol{u} = \frac{\partial \hat{\boldsymbol{r}}}{\partial t}. \tag{36}$$

Let $\boldsymbol{n} = \boldsymbol{n}(q^1, q^2, t)$ be a unit normal vector field on the sheet, conventionally chosen to point outwards from $\mathcal{P}_1$ at points where $\varSigma(t)$ forms part of the boundary of $\mathcal{P}_1$ (see Figure 1). The normal component of $\boldsymbol{u}$, i.e.,

$$u_n = \boldsymbol{u} \cdot \boldsymbol{n}, \tag{37}$$

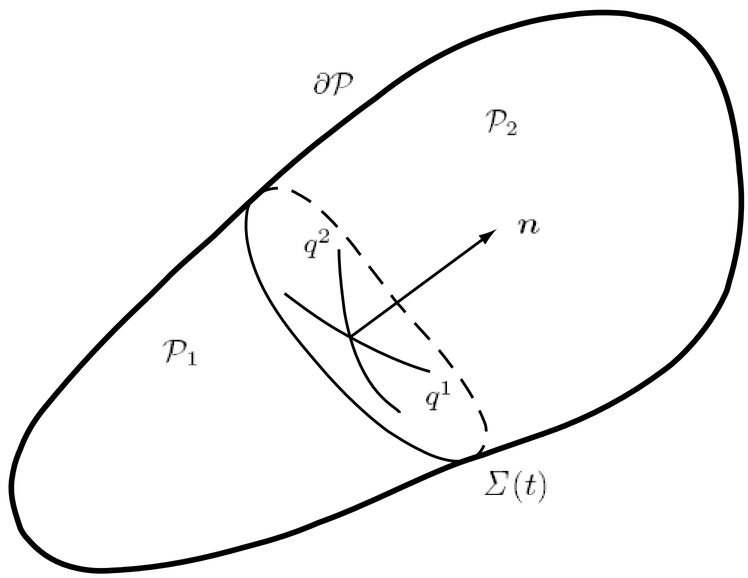

Figure 1: A singular surface $\Sigma(t)$ in the region occupied by a subbody $\mathcal{S}$ at time t.

is called the *speed of displacement* of the moving surface. In the general case, when $\Sigma(t)$ is not a material surface, only the normal component of its velocity is physically relevant.

The homeomorphism $\boldsymbol{\chi}_t^{-1}$, discussed in Section 2, may be employed to pull-back $\Sigma(t)$ and obtain a surface $\Sigma_0(t)$, which, in general, propagates across the fixed region $\mathcal{P}_0$ occupied by $\mathcal{S}$ in its reference configuration, and divides $\mathcal{P}_0$ into two contiguous parts. Similarly, the homeomorphism $\check{\boldsymbol{\chi}}_t^{-1}$ can be used to obtain a corresponding surface that moves across the body manifold. In the special case in which the family $\Sigma(t)$ is material with respect to the motion $\check{\boldsymbol{\chi}}$, the corresponding surfaces in $\mathcal{P}_0$ and $\mathcal{S}$ become fixed.

Consider again any distributed property ϕ of the body $\mathcal{B}$. Various representations of ϕ were considered in Section 2. Referring to Figure 1, suppose now that, at each instant t, the function $\widetilde{\phi}(\boldsymbol{x}, t)$ is continuous in the variable $\boldsymbol{x}$ at all points in the open region bounded by $\mathcal{P}$ but excluding the surface $\Sigma(t)$. On the surface $\Sigma(t)$, the values of $\widetilde{\phi}(\boldsymbol{x}, t)$ need not even be defined. However, we suppose that, at each instant t, (finite) one-sided limits of $\widetilde{\phi}(\boldsymbol{x}, t)$ exist as $\Sigma(t)$ is approached alternatively from the interior of $\mathcal{P}_1$ and the interior of $\mathcal{P}_2$. These limits, denoted by

$$\phi_1 = \phi_1(q^1, q^2, t), \qquad \phi_2 = \phi_2(q^1, q^2, t), \tag{38}$$

respectively, furnish two continuous surface fields on $\Sigma(t)$ at each instant t.[12] We will suppose that ϕ_1 and ϕ_2 are also continuous in t. The jump, $[\![\phi]\!]$, of ϕ across $\Sigma(t)$ is defined by

$$[\![\phi]\!] = [\![\phi(q^1, q^2, t)]\!] = \phi_2 - \phi_1. \tag{39}$$

If $[\![\phi]\!]$ is not identically zero on $\Sigma(t)$, then $\Sigma(t)$ is said to be *singular* with respect to ϕ at time t. In view of the continuity of $[\![\phi]\!]$ in t, if $\Sigma(t)$ is singular with respect to ϕ at time t, it will persist in being singular for some open interval containing t. We will suppose that, at each $t \in I$, there exists at least one thermomechanical variable with respect to which $\Sigma(t)$ is singular. (Otherwise, we would not call it a singular surface.)

Certain thermomechanical fields require additional smoothness properties. Included among these are ρ, $\boldsymbol{v}$, ε, and η, which appear inside integrals whose material derivative is needed. At each value of t, such fields will be

[12] A one-sided limit is also assumed to exist on the outer boundary of the subbody.

assumed to be continuously differentiable in the interior of the regions $\mathcal{P}_1$ and $\mathcal{P}_2$. Further, it will be assumed that their partial derivatives with respect to $\boldsymbol{x}$ and t approach (finite) limits as $\Sigma(t)$ is approached from either side; these limits will be continuous on $\Sigma(t)$, and we will suppose that they are also continuous in t.

In accordance with the continuity assumptions that were made on the function $\boldsymbol{\chi}$ in (3), there is no jump in the displacement field at $\Sigma(t)$:[13]

$$[\![\boldsymbol{d}]\!] = \mathbf{0}. \tag{40}$$

Allowing for the possibility of a jump in the velocity field, let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the limits of $\widetilde{\boldsymbol{v}}(\boldsymbol{x}, t)$ as $\Sigma(t)$ is approached from either side. Also, for later convenience, we introduce the relative velocities[14]

$$\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}, \qquad w_n = \boldsymbol{w} \cdot \boldsymbol{n}, \qquad w_{1n} = \boldsymbol{v}_1 \cdot \boldsymbol{n} - u_n, \qquad w_{2n} = \boldsymbol{v}_2 \cdot \boldsymbol{n} - u_n. \tag{41}$$

Then,

$$w_{n1} = w_{1n}, \qquad w_{n2} = w_{2n}, \tag{42a}$$

and

$$[\![w_n]\!] = [\![\boldsymbol{w} \cdot \boldsymbol{n}]\!] = [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]. \tag{42b}$$

A shock surface is, by definition, a singular surface across which there is a jump in $\boldsymbol{v} \cdot \boldsymbol{n}$. In general, this jump will be accompanied by jumps in other variables as well. Across a vortex sheet, the normal component of velocity is continuous, but there is a jump in the tangential component.

In the special case in which a singular surface is material with respect to the motion $\check{\boldsymbol{\chi}}$, $\boldsymbol{w} = \mathbf{0}$. If w_{n1} or w_{n2} is nonzero, the singular surface is in motion relative the the material just behind it or just in front of it, and, following Hadamard (1903), the singular surface is then called a *wave*.[15]

If the velocity field is everywhere and always continuous, but there is a propagating discontinuity in some component of the acceleration, then one has an acceleration wave. Acceleration waves are identified physically with sound waves.

Certain geometrical and kinematical conditions of compatibility must be satisfied on a moving singular surface (see Truesdell and Toupin 1960), but these will not be needed in the present article.

4. Reynolds' transport theorem for bodies with singular surfaces

The balance laws and entropy inequality, stated in Section 2, each involves the material derivative of an integral. In the absence of singular surfaces, the standard form of Reynolds' transport theorem establishes the existence of these material derivatives, and furnishes expressions for them.[16] For bodies containing one or more singular surfaces, the transport theorem must be extended to take into account the effects of the moving discontinuities.

[13]Thus, discontinuities such as dislocations, fractures, and slip surfaces are ruled out of the present discussion. For a treatment of Volterra dislocations in finitely deforming bodies, see Casey (2004). An excellent account of Volterra's theory may be found in Section 156A of Love (1927).

[14]See Green and Naghdi (1965). Alternatively, the negatives of w_{1n} and w_{2n} may be employed; these are called the *local speeds of propagation* of the surface.

[15]This generalizes our ordinary conception of "wave" to the case where the propagating disturbance is a discontinuity. In one of a characteristically thoughtful series of lectures which he delivered at Rice University in September, 1936, Levi-Civita (1938) begins: "It is not easy to give a general definition of waves which would be precise and would at the same time include all cases presenting a character which our intuition attributes to waves." He goes on to note, however, that Leonardo da Vinci, writing in the 15th century, had already grasped an essential feature of waves: "... it often happens that the wave flees the place of its creation, while the water does not; like the waves made in a field of grain by the wind, where we see the waves running across the field while the grain remains in its place." (Indeed, Leonardo wrote extensively about the motions of water and air.)

[16]See Section 81 of Truesdell and Toupin (1960). A discussion of the transport theorem, using the language of differential forms, was recently presented by Lidström (2011).

4.1 A family of material volumes containing no surface of discontinuity

For a moving and deforming subbody $\mathcal{S} \subseteq \mathcal{B}$ containing no surface of discontinuity, we wish to calculate the material derivative, at an arbitrary time $t \in I$, of the amount Φ of the property ϕ possessed by $\mathcal{S}$ during a given thermomechanical process. The function $\widetilde{\phi}$ is assumed to be continuously differentiable for all $\boldsymbol{x} \in \check{\boldsymbol{\chi}}(X,t)$ and for all $t \in I$.

Various equivalent representations of Φ are indicated in (18). We also note that if the integral of ϕ over $\mathcal{P}$ is changed into an integral over $\mathcal{P}_0$, we have

$$\Phi = \int_{\mathcal{P}_0} \phi \; J \,\mathrm{d}V. \tag{43}$$

Since $\mathcal{P}_0$ is a fixed region, the material derivative (19) of Φ is explicitly given by

$$\begin{aligned} \dot{\Phi} &= \int_{\mathcal{P}_0} \frac{D}{Dt}(\phi J)\,\mathrm{d}V \\ &= \int_{\mathcal{P}_0} \left\{ \dot{\phi} + \phi \operatorname{div} \boldsymbol{v} \right\} \; J\,\mathrm{d}V, \end{aligned} \tag{44}$$

where the kinematical formula $\dot{J} = J \operatorname{div} \boldsymbol{v}$ has been utilized. Consequently,

$$\dot{\Phi} = \int_{\mathcal{P}} \left\{ \dot{\phi} + \phi \operatorname{div} \boldsymbol{v} \right\} \,\mathrm{d}v. \tag{45}$$

This is the most basic expression of Reynolds' transport theorem.

A second expression of the theorem may be obtained from (45) through use of the divergence theorem and the formula (12). Thus,

$$\dot{\Phi} = \int_{\mathcal{P}} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial \mathcal{P}} \phi\, \boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}a. \tag{46}$$

Here, $\mathcal{P} = \check{\boldsymbol{\chi}}(\mathcal{S},t)$ is a family of material volumes, with material surfaces $\partial\mathcal{P} = \check{\boldsymbol{\chi}}(\partial\mathcal{S},t)$.

A third expression of the transport theorem involves the <u>fixed</u> region $\overline{\mathcal{P}}$ with which $\mathcal{P}$ is instantaneously coincident at time t. Thus, setting $\mathcal{P} = \overline{\mathcal{P}}$ in (46), and taking $\partial/\partial t$ outside the integral sign, we obtain

$$\begin{aligned} \dot{\Phi} &= \int_{\overline{\mathcal{P}}} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi\, \boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}a \\ &= \frac{\partial \overline{\Phi}}{\partial t} + \int_{\partial\overline{\mathcal{P}}} \phi\, \boldsymbol{v} \cdot \boldsymbol{n}\,\mathrm{d}a, \end{aligned} \tag{47}$$

where the representation $\overline{\Phi}$ is given by (20). In words, the latter expression of the transport theorem may be stated as: The material derivative at time t of the amount Φ of a distributed property ϕ possessed by a deformable subbody $\mathcal{S}$ may be calculated as the sum of two terms, the first of which is the time-rate of change of the integral of ϕ taken over the <u>fixed</u> region $\overline{\mathcal{P}}$, and the second of which is the flux of $\phi\, \boldsymbol{v}$ across the boundary of $\overline{\mathcal{P}}$.

The formulas (44), (46), and (47) hold also when ϕ represents a distributed vector or tensor property of the continuum.

4.2 A family of volumes which are not necessarily material ("sampling volumes") but which do not enclose any surface of discontinuity

The transport theorem may be extended to apply to a continuous time-dependent family of volumes that are not necessarily material with respect to the motion $\check{\boldsymbol{\chi}}$. Let us construct such a family of volumes in the following way.

As before, let $\overline{\mathcal{P}}$ be the fixed region that the subbody $\mathcal{S}$ instantaneously occupies at time t. For each instant τ belonging to an open time interval $\overline{I}$ $(\subseteq I)$ containing t,[17] let $\boldsymbol{g}_\tau$ be a homeomorphism that maps $\overline{\mathcal{P}}$ onto a region of $\mathcal{E}$, having volume $V(\tau)$ and bounding surface $A(\tau)$. Let $\mathrm{d}V(\tau)$ and $\mathrm{d}A(\tau)$ be the volume and area elements of $V(\tau)$ and $A(\tau)$, respectively. We take $\boldsymbol{g}_t$ to be the identity mapping on $\overline{\mathcal{P}}$. For present purposes, we may think of t as representing any fixed value of time and τ as being variable. We have

$$\boldsymbol{g}_\tau(\overline{\mathcal{P}}) = V(\tau), \qquad \boldsymbol{g}_\tau(\partial\overline{\mathcal{P}}) = A(\tau), \qquad\qquad (\tau \in \overline{I}) \tag{48}$$

and, when τ takes on the value t,

$$\boldsymbol{g}_t(\overline{\mathcal{P}}) = \overline{\mathcal{P}}, \qquad \boldsymbol{g}_t(\partial\overline{\mathcal{P}}) = \partial\overline{\mathcal{P}}. \tag{49}$$

Let

$$\boldsymbol{g}(\boldsymbol{x}, \tau) = \boldsymbol{g}_\tau(\boldsymbol{x}), \tag{50}$$

where $\boldsymbol{x}$ is the variable position vector in the fixed region $\overline{\mathcal{P}}$; $\boldsymbol{g}(\boldsymbol{x}, \tau)$ is the position vector, at time τ, of the point that $\boldsymbol{x}$ is sent to by the homeomorphism $\boldsymbol{g}_\tau$. Clearly,

$$\boldsymbol{g}(\boldsymbol{x}, t) = \boldsymbol{x} \tag{51}$$

for all $\boldsymbol{x} \in \overline{\mathcal{P}}$. We call $\boldsymbol{g}$ a "sampling motion."[18] Essentially, we are using $\overline{\mathcal{P}}$ as a fixed reference region and we are defining a one-parameter family of volumes $V(\tau)$, $(\tau \in \overline{I})$. The position vector $\boldsymbol{x}$ is, in fact, the position vector of the particle $X \in \mathcal{S}$ at the fixed time t, but it is important to note that a sampling motion can be defined independently of the actual motion $\check{\boldsymbol{\chi}}$ of the body.[19] It is sometimes suggestive to think of the particles of the subbody $\mathcal{S}$ as being taken on an imaginary tour of the field $\widetilde{\phi}$ to observe the values of the property ϕ being experienced by the continuum, but for present purposes, it is better to think more abstractly in terms of a mapping $\boldsymbol{g}$ that takes the points of $\overline{\mathcal{P}}$ into the space $\mathcal{E}$. In general, the family of volumes $V(\tau)$, $(\tau \in \overline{I})$, is material with respect to the sampling motion $\boldsymbol{g}$, but not with respect to the actual motion $\check{\boldsymbol{\chi}}$.

The velocity of points belonging to the sampling volumes $V(\tau)$ is $\partial\boldsymbol{g}(\boldsymbol{x}, \tau)/\partial\tau$. Let us adopt the notation

$$\boldsymbol{u}^* = \frac{\partial \boldsymbol{g}}{\partial \tau}(\boldsymbol{x}, \tau). \tag{52}$$

In general, even at time t, this velocity field is independent of the velocity field $\boldsymbol{v}$, corresponding to the motion $\check{\boldsymbol{\chi}}$ of the continuum.

We will assume that during the interval $\overline{I}$, each surface $A(\tau)$ encloses a region that is occupied by some subbody of $\mathcal{B}$.[20] As in subsection 4.1, the field $\check{\phi}$ is assumed to be continuously differentiable. We may then use this family of time-dependent volumes to "sample" the field $\widetilde{\phi}$ associated with the motion of the continuum. Thus, let

$$\Phi(\tau) = \int_{V(\tau)} \phi \; \mathrm{d}V(\tau). \tag{53}$$

We may apply the transport theorem in the form (47) to calculate the derivative of $\Phi(\tau)$ with respect to τ, keeping in mind that the family of volumes $V(\tau)$, $(\tau \in \overline{I})$, is a material family with respect to the sampling

[17] It is sufficient in the present discussion that the time interval $\overline{I}$ be a small interval around t: $\overline{I} = (t-h, t+h)$, where h is an arbitrarily small real number.

[18] The function $\boldsymbol{g}$ may be taken to be as smooth as desired.

[19] As an aid to visualization, it may be helpful to think of the shadow of a cloud moving over a rapidly flowing river, or of a wave moving across a wheatfield, as in footnote 15. For later purposes, we also observe that it is always permissible, as a special choice, to employ the actual motion as a particular sampling motion.

[20] Under these circumstances, the region $\overline{\mathcal{P}}$ corresponds to a fixed "control volume" (see Section 12 of Gurtin 1981).

motion $\boldsymbol{g}$ and that they do not enclose any surface of discontinuity. In particular, when $\tau = t$, we obtain

$$\begin{aligned} \frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(\tau)\bigg|_{\tau=t} &= \int_{\overline{\mathcal{P}}} \frac{\partial\widetilde{\phi}}{\partial\tau}(\tau)\bigg|_{\tau=t} \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi\, \boldsymbol{u}^*(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}a \\ &= \int_{\overline{\mathcal{P}}} \frac{\partial\widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi\, \boldsymbol{u}^*(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}a \\ &= \frac{\partial\overline{\Phi}}{\partial t} + \int_{\partial\overline{\mathcal{P}}} \phi\, \boldsymbol{u}^*(\boldsymbol{x},t)\cdot\boldsymbol{n}\,\mathrm{d}a. \end{aligned} \tag{54}$$

Remark 4.1. It is worth observing that the only way in which the sampling motion $\boldsymbol{g}$ comes into play in the formula (54) is through the normal velocity of the bounding surface at time t. In other words, two different sampling motions will deliver the same value for the rate $\mathrm{d}\Phi(\tau)/\mathrm{d}\tau\,\big|_{\tau=t}$ as long as they have the same normal component of surface velocity at the instant t.[21] Consider now the following two special sampling motions: (i) the actual motion $\breve{\boldsymbol{\chi}}$; and (ii) a sampling motion $\boldsymbol{g}$, chosen in any manner such that the surface $A(\tau)$ has the velocity $\boldsymbol{v}\cdot\boldsymbol{n}$ at the instant t, where $\boldsymbol{v}$ is the particle velocity. Then the rate $\mathrm{d}\Phi(\tau)/\mathrm{d}\tau\,\big|_{\tau=t}$ is equal to the material derivative $\dot{\Phi}$ at the instant t. (Compare with Remark 2.1.)

4.3 A family of material volumes containing a singular surface

We now consider the situation sketched in Figure 1, where the surface $\partial\mathcal{P}$ is a member of a family that is material with respect to the motion $\breve{\boldsymbol{\chi}}$, and $\Sigma(t)$ is a surface of discontinuity, whose speed of displacement is given by (37). In general, the two regions, $\partial\mathcal{P}_1$ and $\partial\mathcal{P}_2$, into which $\partial\mathcal{P}$ is divided by $\Sigma(t)$, are not material with respect to $\breve{\boldsymbol{\chi}}$.

Once again, let $\overline{\mathcal{P}}$ be the <u>fixed</u> region with which $\mathcal{P}$ instantaneously coincides at time t. Let $\partial\overline{\mathcal{P}}_i$ be the boundary of $\overline{\mathcal{P}}_i$ $(i = 1, 2)$. Also, let $\partial\overline{\mathcal{P}}' = \partial\overline{\mathcal{P}} \cap \partial\overline{\mathcal{P}}_1$ and $\partial\overline{\mathcal{P}}'' = \partial\overline{\mathcal{P}} \cap \partial\overline{\mathcal{P}}_2$. Then,

$$\begin{aligned} \overline{\mathcal{P}} &= \overline{\mathcal{P}}_1 \cup \overline{\mathcal{P}}_2, \\ \partial\overline{\mathcal{P}}_1 &= \partial\overline{\mathcal{P}}' \cup \Sigma(t), \\ \partial\overline{\mathcal{P}}_2 &= \partial\overline{\mathcal{P}}'' \cup \Sigma(t), \\ \partial\overline{\mathcal{P}} &= \partial\overline{\mathcal{P}}' \cup \partial\overline{\mathcal{P}}''. \end{aligned} \tag{55}$$

Further, let $\mathcal{S}_1 \subset \mathcal{S}$ be the subbody that instantaneously occupies the region $\overline{\mathcal{P}}_1$ at time t, and let $\mathcal{S}_2$ be the subbody that occupies $\overline{\mathcal{P}}_2$; employing the pull-back operator $\breve{\boldsymbol{\chi}}_t^{-1}$, these are given by

$$\mathcal{S}_i = \breve{\boldsymbol{\chi}}_t^{-1}(\overline{\mathcal{P}}_i), \qquad (i = 1, 2). \tag{56}$$

In general, $\Sigma(t)$ moves across the deforming subbody $\mathcal{S}$ and partitions it into a different pair of subbodies at each value of t; but, the union of each pair is always the same:

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2. \tag{57}$$

Utilizing the representation (20), we may write

$$\Phi = \Phi_1 + \Phi_2, \tag{58}$$

[21] Note that both sampling motions are mappings of the same fixed region $\overline{\mathcal{P}}$, and there is no surface of discontinuity.

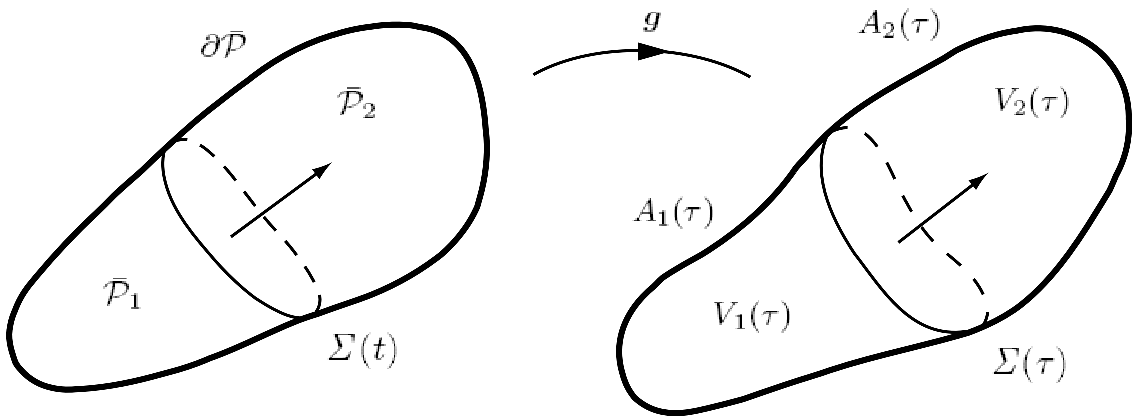

Figure 2: Two families of contiguous volumes, $V_1(\tau)$ and $V_2(\tau)$, are created by a sampling motion $\boldsymbol{g}$ defined on the fixed region $\overline{\mathcal{P}}$ that is instantaneously occupied by a subbody $\mathcal{S}$ at time t. The image of $\Sigma(t)$ under the mapping $\boldsymbol{g}$ is the actual configuration $\Sigma(\tau)$ of the surface of discontinuity.

where

$$\Phi_i = \overline{\Phi}(\overline{\mathcal{P}}_i, t) = \int_{\overline{\mathcal{P}}_i} \phi \, \mathrm{d}v, \qquad (i = 1, 2). \tag{59}$$

We need a form of the transport theorem for the subbody $\mathcal{S} \subseteq \mathcal{B}$ moving through the fixed region $\overline{\mathcal{P}}$, across which the surface of discontinuity $\Sigma(t)$ also sweeps (Figure 2). We may proceed as follows.

We will define a special sampling motion which takes into account the presence of the singular surface. To this end, consider the region $\mathcal{P}(\tau)$ that is occupied by the subbody $\mathcal{S}$ at time τ in its actual motion $\check{\boldsymbol{\chi}}$. We write

$$\boldsymbol{x}(\tau) = \boldsymbol{\chi}(\boldsymbol{X}, \tau) \tag{60}$$

for the position vector of $X \in \mathcal{B}$ in the region $\mathcal{P}(\tau)$. At time τ, the surface of discontinuity $\Sigma(\tau)$ partitions $\mathcal{P}(\tau)$ into two contiguous parts, with volumes $V_1(\tau)$ and $V_2(\tau)$. Let $V(\tau)$ be the volume of the region $\mathcal{P}(\tau)$ and let $\partial\mathcal{P}(\tau)$ be the boundary of $\mathcal{P}(\tau)$. In general, neither $V_1(\tau)$ nor $V_2(\tau)$, $(\tau \in \overline{I})$, is material with respect to the motion $\check{\boldsymbol{\chi}}$. Thus, the subbody $\mathcal{S}_1$ that instantaneously occupies $\overline{\mathcal{P}}_1$ at time t will not, in general, be mapped by $\check{\boldsymbol{\chi}}$ into $V_1(\tau)$; instead, $V_1(\tau)$ will contain either more or less of the matter in $\mathcal{S}_1$, depending on how the singular surface is moving relative to the continuum. However, $\mathcal{P}(\tau)$ is material with respect to $\check{\boldsymbol{\chi}}$, since it is the image of the subbody $\mathcal{S}$ in the motion $\check{\boldsymbol{\chi}}$.

We define a sampling motion $\boldsymbol{g}$ on $\overline{\mathcal{P}}$ by homeomorphically mapping the points in $\overline{\mathcal{P}}_1$ into $V_1(\tau)$, for each $\tau \in \overline{I}$, and by homeomorphically mapping the points in $\overline{\mathcal{P}}_2$ into $V_2(\tau)$. The points on $\Sigma(t)$ are mapped by $\boldsymbol{g}$ into $\Sigma(\tau)$. The following relations hold for all $\tau \in \overline{I}$:

$$\begin{gathered} V_i(\tau) = \boldsymbol{g}(\overline{\mathcal{P}}_i), \qquad A_i(\tau) = \boldsymbol{g}(\partial\overline{\mathcal{P}}_i), \qquad (i = 1, 2) \\ V(\tau) = V_1(\tau) + V_2(\tau), \\ \boldsymbol{g}(\Sigma(t), \tau) = \Sigma(\tau). \end{gathered} \tag{61}$$

Each family of volumes, $V_i(\tau)$, $(i = 1, 2;\ \tau \in \overline{I})$, is material with respect to the sampling motion $\boldsymbol{g}$. The family $V(\tau)$, $(\tau \in \overline{I})$, is material with respect to $\boldsymbol{g}$, and as we have seen, is also material with respect to $\check{\boldsymbol{\chi}}$. It follows from $(61)_3$ that, for all $\tau \in \overline{I}$, at the surface of discontinuity, the velocity field $\boldsymbol{u}^*$ in the sampling motion will be equal to the velocity $\boldsymbol{u}$ of the surface of discontinuity:

$$\boldsymbol{u}^* = \boldsymbol{u} \quad \text{on} \quad \Sigma(\tau), \qquad (\tau \in \overline{I}). \tag{62}$$

We further suppose that the sampling motion $\boldsymbol{g}$ is chosen in such a way that its velocity field satisfies the condition[22]

$$\boldsymbol{u}^* \cdot \boldsymbol{n} = \boldsymbol{v} \cdot \boldsymbol{n} \quad \text{on} \quad \partial\overline{\mathcal{P}}, \qquad (\tau \in \overline{I}). \tag{63}$$

We allow for a jump in the particle velocity at $\Sigma(\tau)$.

Using an obvious notation for the amount of the property ϕ contained in the sampling volumes $V_1(\tau)$ and $V_2(\tau)$, as in (53) let

$$\Phi_i(\tau) = \int_{V_i(\tau)} \phi \, \mathrm{d}V_i(\tau), \qquad (i = 1, 2). \tag{64}$$

For all $\tau \in \overline{I}$,

$$\Phi(\tau) = \Phi_1(\tau) + \Phi_2(\tau). \tag{65}$$

Now, if we could show that $\Phi_1(\tau)$ and $\Phi_2(\tau)$ are differentiable functions of τ on $\overline{I}$, then we could deduce from (65) that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(\tau) = \frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau}(\tau) + \frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau}(\tau), \tag{66}$$

for all $\tau \in \overline{I}$. The differentiability of $\Phi_1(\tau)$ and $\Phi_2(\tau)$ may be established by the following argument.

The volumes $V_1(\tau)$ and $V_2(\tau)$, $(\tau \in \overline{I})$, are material with respect to the sampling motion $\boldsymbol{g}$. Further, neither of these two families encloses a surface of discontinuity. The result (54) may therefore be applied to the sampling motion of the points in $\overline{\mathcal{P}}_1$, and separately, also to the sampling motion of the points in $\overline{\mathcal{P}}_2$. Thus, for the points in $\overline{\mathcal{P}}_1$, (54) furnishes

$$\left.\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau}(\tau)\right|_{\tau=t} = \int_{\overline{\mathcal{P}}_1} \frac{\partial\widetilde{\phi}}{\partial t} \, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}'} \phi \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}a + \int_{\Sigma(t)} \phi_1 \, u_n \, \mathrm{d}a, \tag{67}$$

where use has been made of (51), (62) and (63).[23] This expression gives the rate of change of Φ_1 that is being experienced by points of $\overline{\mathcal{P}}_1$ in the sampling motion $\boldsymbol{g}$, in which the singular surface $\Sigma(t)$ has velocity $\boldsymbol{u}$ and points on the outer boundary $\partial\overline{\mathcal{P}}'$ have normal velocity $\boldsymbol{v}\cdot\boldsymbol{n}$. It is <u>not</u>, in general, equal to the material derivative $\dot{\Phi}_1$ of Φ_1.[24]

Likewise, we may apply (54) to the sampling motion of the points in $\overline{\mathcal{P}}_2$, taking into account that at points where $\partial\overline{\mathcal{P}}_2$ coincides with $\Sigma(t)$, its outward unit normal is $-\boldsymbol{n}(q^1, q^2, t)$:

$$\left.\frac{\mathrm{d}\Phi_2}{\mathrm{d}\tau}(\tau)\right|_{\tau=t} = \int_{\overline{\mathcal{P}}_2} \frac{\partial\widetilde{\phi}}{\partial t} \, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}''} \phi \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}a - \int_{\Sigma(t)} \phi_2 \, u_n \, \mathrm{d}a. \tag{68}$$

Hence, with the aid of (66) and (39), we deduce that

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(\tau)\right|_{\tau=t} = \int_{\overline{\mathcal{P}}} \frac{\partial\phi}{\partial t} \, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}a - \int_{\Sigma(t)} [\![\phi]\!] \, u_n \, \mathrm{d}a. \tag{69}$$

It has been noted previously that, whereas the singular surface $\Sigma(\tau)$ divides the subbody $\mathcal{S}$ into different subbodies as τ changes, nonetheless, their union is always equal to the same subbody $\mathcal{S}$. The amount of the property ϕ that is possessed by $\mathcal{S}$ at time τ, $(\tau \in \overline{I})$, is $\Phi(\tau)$, given by (65). Therefore, the quantity $\mathrm{d}\Phi(\tau)/\mathrm{d}\tau$ is equal to the rate at which the function $\check{\Phi}(\mathcal{S}, \tau)$ changes as $\mathcal{S}$ is transported through the field by its motion $\check{\boldsymbol{\chi}}$,

[22]In general, the tangential components of $\boldsymbol{u}^*$ and $\boldsymbol{v}$ on $\partial\overline{\mathcal{P}}$ will differ from one another, i.e., there may be relative sliding.

[23]Henceforth, the decompositions in (55) will be used freely, without explicit mention.

[24]Recall that the material derivative of Φ_1 measures the time-rate of change of Φ_1 as one rides along with the subbody $\mathcal{S}$ while it undergoes the motion $\check{\boldsymbol{\chi}}$; the corresponding velocity field is $\boldsymbol{v}$, which may jump at the singular surface. The material derivatives of Φ_i $(i = 1, 2)$ will appear prominently in subsequent equations.

while the surface of discontinuity $\Sigma(\tau)$ simultaneously sweeps across the field. In other words, it is the material derivative of $\Phi(\tau)$. Hence, at time t,

$$\left.\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}(\tau)\right|_{\tau=t} = \dot{\Phi}. \tag{70}$$

It follows from (69) and (70) that

$$\dot{\Phi} = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}, t) = \int_{\overline{\mathcal{P}}} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}a - \int_{\Sigma(t)} [\![\phi]\!]\,u_n\,\mathrm{d}a. \tag{71}$$

In this important expression of the transport theorem, the material derivative of Φ is seen to be equal to the sum of three integrals: (1) the integral of $\partial\widetilde{\phi}/\partial t$ over the fixed region $\overline{\mathcal{P}}$ that the subbody $\mathcal{S} \subseteq \mathcal{B}$ occupies at the instant t; (2) the flux of $\phi\boldsymbol{v}$ across the boundary $\partial\overline{\mathcal{P}}$ of $\overline{\mathcal{P}}$; and (3) the integral, taken over the surface of discontinuity at time t, of the negative of the jump of ϕ multiplied by the speed of displacement u_n of the singular surface. See Thomas (1949) and Section 192 of Truesdell and Toupin (1960).

Next, we observe that the transport theorem may be applied in the form (47) to the subbody $\mathcal{S}_1$ that instantaneously occupies the fixed region $\partial\overline{\mathcal{P}}_1$ at the instant t, and likewise for the subbody $\mathcal{S}_2$. Thus, using an obvious notation, we deduce from (47) that the material derivative of $\breve{\Phi}(\mathcal{S}_i, t)$, $(i = 1, 2)$, is given by

$$\dot{\Phi}_i = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_i, t) = \int_{\overline{\mathcal{P}}_i} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}_i} \phi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}a, \qquad (i = 1, 2). \tag{72}$$

Hence,

$$\dot{\Phi}_1 = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_1, t) = \int_{\overline{\mathcal{P}}_1} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}'} \phi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}a + \int_{\Sigma(t)} \phi_1\,\boldsymbol{v}_1\cdot\boldsymbol{n}\,\mathrm{d}a. \tag{73a}$$

Likewise, for the subbody $\mathcal{S}_2$,

$$\dot{\Phi}_2 = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_2, t) = \int_{\overline{\mathcal{P}}_2} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}''} \phi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}a - \int_{\Sigma(t)} \phi_2\,\boldsymbol{v}_2\cdot\boldsymbol{n}\,\mathrm{d}a. \tag{73b}$$

Therefore,

$$\begin{aligned}\dot{\Phi}_1 + \dot{\Phi}_2 &= \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_1, t) + \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_2, t) \\ &= \int_{\overline{\mathcal{P}}} \frac{\partial \widetilde{\phi}}{\partial t}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \phi\,\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}a - \int_{\Sigma(t)} [\![\phi\,\boldsymbol{v}]\!]\cdot\boldsymbol{n}\,\mathrm{d}a. \end{aligned} \tag{74}$$

It follows from (71) and (74) that

$$\dot{\Phi} = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}, t) = \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_1, t) + \frac{\partial \breve{\Phi}}{\partial t}(\mathcal{S}_2, t) + \int_{\Sigma(t)} \left\{[\![\phi\boldsymbol{v}]\!]\cdot\boldsymbol{n} - [\![\phi]\!]\,u_n\right\}\,\mathrm{d}a, \tag{75a}$$

Or, more succinctly,

$$\begin{aligned}\dot{\Phi} &= \dot{\Phi}_1 + \dot{\Phi}_2 + \int_{\Sigma(t)} [\![\phi\,(\boldsymbol{v}\cdot\boldsymbol{n} - u_n)]\!]\,\mathrm{d}a \\ &= \dot{\Phi}_1 + \dot{\Phi}_2 + \int_{\Sigma(t)} [\![\phi\,w_n]\!]\,\mathrm{d}a, \end{aligned} \tag{75b}$$

where (41) and (42a) have been utilized. This expression of the transport theorem involves the three material derivatives $\dot{\Phi}$, $\dot{\Phi}_1$, and $\dot{\Phi}_2$, pertaining to the three subbodies in (57), and the integral over the surface of discontinuity of the jump of ϕw_n, where w_n is the normal relative velocity. It holds for each $t \in I$. As will be seen in Section 5, this form of the transport theorem is very convenient for the derivation of jump conditions from the integral balance equations.

5. Derivation of jump conditions

As a result of the developments in Section 4, we are now in a position to deal with the material derivatives that appear in the balance laws and entropy inequality (Section 2), for the case when a subbody $\mathcal{S} \subseteq \mathcal{B}$ contains a singular surface.

At values of $(\boldsymbol{x}, t)$ not lying on a singular surface, the smoothness assumptions that were made in Section 2 lead to Cauchy's lemma

$$\boldsymbol{t}(\boldsymbol{x}, t, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) \tag{76}$$

for the stress vector.[25] Similarly, for the heat flux scalar function, it can be shown that

$$h(\boldsymbol{x}, t, -\boldsymbol{n}) = -h(\boldsymbol{x}, t, \boldsymbol{n}). \tag{77}$$

At a singular surface $\Sigma(t)$, the conditions (76) and (77) hold in the limit; in particular,

$$\boldsymbol{t}_2(\boldsymbol{x}, t, -\boldsymbol{n}) = -\boldsymbol{t}_2(\boldsymbol{x}, t, \boldsymbol{n}), \qquad h_2(\boldsymbol{x}, t, -\boldsymbol{n}) = -h_2(\boldsymbol{x}, t, \boldsymbol{n}). \tag{78}$$

5.1 Conservation of mass

For any subbody $\mathcal{S} \subseteq \mathcal{B}$, the mass of $\mathcal{S}$ is given by (21), and also by (22). Mass conservation may be stated in the alternative form (23). Similarly, for each of the two subbodies in (56), we have

$$m_i = \check{m}(\mathcal{S}_i), \qquad (i = 1, 2) \tag{79}$$

and

$$\dot{m}_i = 0, \qquad (i = 1, 2). \tag{80}$$

For the case of mass, we choose the property ϕ to be the mass density ρ, and the transport theorem (75b) then yields

$$\dot{m} = \dot{m}_1 + \dot{m}_2 + \int_{\Sigma(t)} [\![\rho\, w_n]\!]\, \mathrm{d}a. \tag{81}$$

It follows immediately from (23), (80), and (81) that

$$\int_{\Sigma(t)} [\![\rho\, w_n]\!]\, \mathrm{d}a \;=\; 0. \tag{82}$$

This is the jump condition that expresses the law of conservation of mass at a singular surface, for all $t \in I$.

5.2 Balance of linear momentum

In view of (24) and (25d), for any subbody $\mathcal{S} \subseteq \mathcal{B}$, the balance of linear momentum (28a) may be stated as

$$\dot{\boldsymbol{G}} = \int_{\overline{\mathcal{P}}} \rho\, \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \boldsymbol{t}\, \mathrm{d}a, \tag{83}$$

[25] See, e.g., Noll (1959) and Gurtin (1981). Continuity in the unit vector $\boldsymbol{n}$ also follows.

where $\overline{\mathcal{P}}$ is the control volume in Figure 2.

The subbodies $\mathcal{S}_i$ in (56) have linear momenta

$$\boldsymbol{G}_i = \breve{\boldsymbol{G}}(\mathcal{S}_i, t), \qquad (i = 1, 2) \tag{84}$$

where the notation in (24) is again being employed. For the subbody $\mathcal{S}_1$, (28a) may be written as

$$\dot{\boldsymbol{G}}_1 = \int_{\overline{\mathcal{P}}_1} \rho\, \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}'} \boldsymbol{t}\, \mathrm{d}a + \int_{\Sigma(t)} \boldsymbol{t}_1\, \mathrm{d}a, \tag{85}$$

Similarly, for the subbody $\mathcal{S}_2$,

$$\dot{\boldsymbol{G}}_2 = \int_{\overline{\mathcal{P}}_2} \rho\, \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}''} \boldsymbol{t}\, \mathrm{d}a - \int_{\Sigma(t)} \boldsymbol{t}_2\, \mathrm{d}a, \tag{86}$$

where $(78)_1$ has also been invoked. Consequently,

$$\dot{\boldsymbol{G}}_1 + \dot{\boldsymbol{G}}_2 = \int_{\overline{\mathcal{P}}} \rho\, \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \boldsymbol{t}\, \mathrm{d}a - \int_{\Sigma(t)} [\![\boldsymbol{t}]\!]\, \mathrm{d}a. \tag{87}$$

We now choose ϕ to be $\rho\, \boldsymbol{v}$ and apply the transport theorem (75b) to deduce that

$$\dot{\boldsymbol{G}} = \dot{\boldsymbol{G}}_1 + \dot{\boldsymbol{G}}_2 + \int_{\Sigma(t)} [\![\rho\, \boldsymbol{v}\, w_n]\!]\, \mathrm{d}a. \tag{88}$$

It then follows from (83), (87), and (88) that

$$\int_{\Sigma(t)} [\![\rho\, \boldsymbol{v}\, w_n - \boldsymbol{t}]\!]\, \mathrm{d}a \;=\; \boldsymbol{0}. \tag{89}$$

This is the expression for the balance of linear momentum at a singular surface. It holds for all $t \in I$.

5.3 Balance of angular momentum

The jump condition for angular momentum may be obtained by following the same steps as in the preceding subsection. Thus, in view of (28b), for any subbody $\mathcal{S} \subseteq \mathcal{B}$, the balance of angular momentum about O may be stated as

$$\dot{\boldsymbol{H}}_{\mathrm{O}} = \int_{\overline{\mathcal{P}}} \rho\, \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \boldsymbol{x} \times \boldsymbol{t}\, \mathrm{d}a. \tag{90}$$

The subbodies $\mathcal{S}_i$ in (56) have angular momenta

$$\boldsymbol{H}_{\mathrm{O}i} = \breve{\boldsymbol{H}}_{\mathrm{O}}(\mathcal{S}_i, t), \qquad (i = 1, 2) \tag{91}$$

where the notation in (26) is used. For the subbody $\mathcal{S}_1$, (28b) becomes

$$\dot{\boldsymbol{H}}_{\mathrm{O}1} = \int_{\overline{\mathcal{P}}_1} \rho\, \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}'} \boldsymbol{x} \times \boldsymbol{t}\, \mathrm{d}a + \int_{\Sigma(t)} \boldsymbol{x} \times \boldsymbol{t}_1\, \mathrm{d}a. \tag{92}$$

Similarly, for the subbody $\mathcal{S}_2$,

$$\dot{\boldsymbol{H}}_{\mathrm{O}2} = \int_{\overline{\mathcal{P}}_2} \rho\, \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}''} \boldsymbol{x} \times \boldsymbol{t}\, \mathrm{d}a - \int_{\Sigma(t)} \boldsymbol{x} \times \boldsymbol{t}_2\, \mathrm{d}a, \tag{93}$$

Hence,

$$\dot{\boldsymbol{H}}_{\mathrm{O}1} + \dot{\boldsymbol{H}}_{\mathrm{O}2} = \int_{\overline{\mathcal{P}}} \rho\, \boldsymbol{x} \times \boldsymbol{b}\, \mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \boldsymbol{x} \times \boldsymbol{t}\, \mathrm{d}a - \int_{\Sigma(t)} \boldsymbol{x} \times [\![\boldsymbol{t}]\!]\, \mathrm{d}a. \tag{94}$$

This time, we choose ϕ to be $\rho\,\boldsymbol{x} \times \boldsymbol{v}$ and apply the transport theorem (75b) to deduce that

$$\dot{\boldsymbol{H}}_{\mathrm{O}} = \dot{\boldsymbol{H}}_{\mathrm{O1}} + \dot{\boldsymbol{H}}_{\mathrm{O2}} + \int_{\Sigma(t)} \boldsymbol{x} \times [\![\rho\,\boldsymbol{v}\,w_n]\!]\,\mathrm{d}a. \tag{95}$$

It then follows from (90), (94), and (95) that

$$\int_{\Sigma(t)} \boldsymbol{x} \times [\![\rho\,\boldsymbol{v}\,w_n - \boldsymbol{t}]\!]\,\mathrm{d}a \;=\; \boldsymbol{0}, \tag{96}$$

which is the expression for the balance of angular momentum at a singular surface, for all $t \in I$.

5.4 Balance of energy

The jump condition for energy can be obtained in a similar way. By virtue of (31), the balance of energy for any subbody $\mathcal{S} \subseteq \mathcal{B}$ may be written as

$$\dot{E} = \int_{\overline{\mathcal{P}}} \rho\,\{\boldsymbol{b}\cdot\boldsymbol{v} + r\}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \{\boldsymbol{t}\cdot\boldsymbol{v} - h\}\,\mathrm{d}a. \tag{97}$$

The subbodies $\mathcal{S}_i$ in (56) have energies

$$E_i = \breve{E}(\mathcal{S}_i, t), \qquad (i = 1, 2) \tag{98}$$

where the notation in (29) is used. For the subbody $\mathcal{S}_1$, (31) becomes

$$\dot{E}_1 = \int_{\overline{\mathcal{P}}_1} \rho\,\{\boldsymbol{b}\cdot\boldsymbol{v} + r\}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}'} \{\boldsymbol{t}\cdot\boldsymbol{v} - h\}\,\mathrm{d}a + \int_{\Sigma(t)} \{\boldsymbol{t}_1\cdot\boldsymbol{v}_1 - h_1\}\,\mathrm{d}a. \tag{99}$$

Similarly, for the subbody $\mathcal{S}_2$,

$$\dot{E}_2 = \int_{\overline{\mathcal{P}}_2} \rho\,\{\boldsymbol{b}\cdot\boldsymbol{v} + r\}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}''} \{\boldsymbol{t}\cdot\boldsymbol{v} - h\}\,\mathrm{d}a - \int_{\Sigma(t)} \{\boldsymbol{t}_2\cdot\boldsymbol{v}_2 - h_2\}\,\mathrm{d}a, \tag{100}$$

where the conditions in (78) have also been utilized. Hence,

$$\dot{E}_1 + \dot{E}_2 = \int_{\overline{\mathcal{P}}} \rho\,\{\boldsymbol{b}\cdot\boldsymbol{v} + r\}\,\mathrm{d}v + \int_{\partial\overline{\mathcal{P}}} \{\boldsymbol{t}\cdot\boldsymbol{v} - h\}\,\mathrm{d}a - \int_{\Sigma(t)} [\![\boldsymbol{t}\cdot\boldsymbol{v} - h]\!]\,\mathrm{d}a. \tag{101}$$

Next, we choose ϕ in the transport theorem (75b) to be $\rho\,\{\varepsilon + \frac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{v}\}$ to deduce that

$$\dot{E} = \dot{E}_1 + \dot{E}_2 + \int_{\Sigma(t)} \left[\!\!\left[\rho\,(\varepsilon + \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v})\,w_n\right]\!\!\right]\mathrm{d}a. \tag{102}$$

It follows from (97), (101), and (102) that

$$\int_{\Sigma(t)} \left[\!\!\left[\rho\,(\varepsilon + \frac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{v})\,w_n - \boldsymbol{t}\cdot\boldsymbol{v} + h\right]\!\!\right]\mathrm{d}a = 0. \tag{103}$$

This is the expression for the balance of energy at a singular surface. It holds for all $t \in I$.

5.5 Entropy production inequality

The Clausius-Duhem inequality (34) for a subbody $\mathcal{S} \subseteq \mathcal{B}$ may be written as

$$\dot{\mathcal{H}} \geq \int_{\overline{\mathcal{P}}} \rho \, \frac{r}{\theta} \, \mathrm{d}v - \int_{\partial \overline{\mathcal{P}}} \frac{h}{\theta} \, \mathrm{d}a. \tag{104}$$

The subbodies $\mathcal{S}_i$ in (56) have entropies

$$\mathcal{H}_i = \check{\mathcal{H}}(\mathcal{S}_i, t), \qquad (i = 1, 2) \tag{105}$$

where the notation in (32) has been used. For the subbody $\mathcal{S}_1$, the inequality (34) furnishes

$$\dot{\mathcal{H}}_1 \geq \int_{\overline{\mathcal{P}}_1} \rho \, \frac{r}{\theta} \, \mathrm{d}v - \int_{\partial \overline{\mathcal{P}}'} \frac{h}{\theta} \, \mathrm{d}a - \int_{\Sigma(t)} \frac{h_1}{\theta_1} \, \mathrm{d}a. \tag{106}$$

Similarly, for the subbody $\mathcal{S}_2$,

$$\dot{\mathcal{H}}_2 \geq \int_{\overline{\mathcal{P}}_2} \rho \, \frac{r}{\theta} \, \mathrm{d}v - \int_{\partial \overline{\mathcal{P}}''} \frac{h}{\theta} \, \mathrm{d}a + \int_{\Sigma(t)} \frac{h_2}{\theta_2} \, \mathrm{d}a, \tag{107}$$

where the condition $(78)_2$ has been utilized. We have allowed for a jump in temperature at the singular surface.

It follows from (106) and (107) that

$$\dot{\mathcal{H}}_1 + \dot{\mathcal{H}}_2 \geq \int_{\overline{\mathcal{P}}} \rho \, \frac{r}{\theta} \, \mathrm{d}v - \int_{\partial \overline{\mathcal{P}}} \frac{h}{\theta} \, \mathrm{d}a + \int_{\Sigma(t)} \left[\!\left[\frac{h}{\theta} \right]\!\right] \mathrm{d}a, \tag{108}$$

If we choose ϕ in the transport theorem (75b) to be $\rho \, \eta$, we find that

$$\dot{\mathcal{H}} = \dot{\mathcal{H}}_1 + \dot{\mathcal{H}}_2 + \int_{\Sigma(t)} [\![\rho \, \eta \, w_n]\!] \, \mathrm{d}a. \tag{109}$$

It follows from (108) and (109) that

$$\dot{\mathcal{H}} \geq \int_{\overline{\mathcal{P}}} \rho \, \frac{r}{\theta} \, \mathrm{d}v - \int_{\partial \overline{\mathcal{P}}} \frac{h}{\theta} \, \mathrm{d}a + \int_{\Sigma(t)} \left[\!\left[\rho \, \eta \, w_n + \frac{h}{\theta} \right]\!\right] \mathrm{d}a. \tag{110}$$

However, the pair of inequalities (104) and (110) do not imply any condition on the sign of the last integral in (110).

A "pillbox argument" can be used in conjunction with the expression (71) of the transport theorem to obtain an inequality on the integral of $[\![\rho \, \eta \, w_n + h/\theta]\!]$ taken over the singular surface. To this end, imagine the region $\overline{\mathcal{P}}$ in Figure 2 to be a thin cylindrical volume, projecting a distance $\delta/2$ in front and in back of the singular surface $\Sigma(t)$. In view of (71), (32), and (33d), the material derivative of the entropy of the subbody $\mathcal{S}$, which instantaneously occupies the region $\overline{\mathcal{P}}$, is

$$\dot{\mathcal{H}} = \int_{\overline{\mathcal{P}}} \frac{\partial (\rho \eta)}{\partial t} \, \mathrm{d}v + \int_{\partial \overline{\mathcal{P}}} \rho \, \eta \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}a - \int_{\Sigma(t)} [\![\rho \, \eta]\!] \, u_n \, \mathrm{d}a. \tag{111}$$

Substituting this equation in (104), we obtain

$$\int_{\overline{\mathcal{P}}} \frac{\partial (\rho \eta)}{\partial t} \, \mathrm{d}v - \int_{\overline{\mathcal{P}}} \rho \, \frac{r}{\theta} \, \mathrm{d}v + \int_{\partial \overline{\mathcal{P}}} \rho \, \eta \, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}a - \int_{\Sigma(t)} [\![\rho \, \eta]\!] \, u_n \, \mathrm{d}a + \int_{\partial \overline{\mathcal{P}}} \frac{h}{\theta} \, \mathrm{d}a \geq 0. \tag{112}$$

We now take the limit as δ tends to zero; the volume of the region $\overline{\mathcal{P}}$ tends to zero, while the boundary $\partial\overline{\mathcal{P}}$ approaches the fixed singular surface $\Sigma(t)$. Assuming that the integrands in the volume integrals are all bounded, we deduce from (112) that[26]

$$\int_{\Sigma(t)} [\![\rho\,\eta\,\boldsymbol{v}\cdot\boldsymbol{n}]\!]\,\mathrm{d}a - \int_{\Sigma(t)} [\![\rho\,\eta]\!]\,u_n\,\mathrm{d}a + \int_{\Sigma(t)} \left[\!\!\left[\frac{h}{\theta}\right]\!\!\right]\mathrm{d}a \geq 0. \tag{113}$$

With the aid of (41) and (42a) , it follows from (113) that

$$\int_{\Sigma(t)} \left[\!\!\left[\rho\,\eta\,w_n + \frac{h}{\theta}\right]\!\!\right]\mathrm{d}a \geq 0. \tag{114}$$

Thus, the Clausius-Duhem inequality (34) implies the jump inequality (114) at a singular surface.

6. Summary of equations in pointwise form

For values of $(\boldsymbol{x}, t)$ not lying on a singular surface, the smoothness assumptions that were made in Section 2 lead, by standard arguments, to the existence of the Cauchy stress tensor $\boldsymbol{T}$ and the heat flux vector $\boldsymbol{q}$, satisfying

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{T}(\boldsymbol{x}, t)\,\boldsymbol{n}, \qquad h(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{q}(\boldsymbol{x}, t)\,\boldsymbol{n}. \tag{115}$$

Also, the fields $\boldsymbol{T}(\boldsymbol{x}, t)$ and $\boldsymbol{q}(\boldsymbol{x}, t)$ are continuously differentiable. The usual field equations then follow by applying the integral balance equations of Section 2 to arbitrarily small subbodies:

$$\dot{\rho} + \rho\,\mathrm{div}\,\boldsymbol{v} = 0, \tag{116a}$$

$$\mathrm{div}\,\boldsymbol{T} + \rho\,\boldsymbol{b} = \rho\,\dot{\boldsymbol{v}}, \tag{116b}$$

$$\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{T}, \tag{116c}$$

$$\rho\,\dot{\varepsilon} = \boldsymbol{T}\cdot\boldsymbol{D} + \rho\,r - \mathrm{div}\,\boldsymbol{q}, \tag{116d}$$

where $\boldsymbol{D}$, the rate of deformation tensor, is the symmetric part of the spatial velocity gradient $\partial\widetilde{\boldsymbol{v}}/\partial\boldsymbol{x}$. Further, if the Clausius-Duhem inequality (34) is adopted, one has (Coleman and Noll 1963):

$$\rho\,\theta\,\dot{\eta} - \rho\,r + \mathrm{div}\,\boldsymbol{q} - \frac{1}{\theta}\,\boldsymbol{q}\cdot\boldsymbol{g} \geq 0, \tag{117}$$

where $\boldsymbol{g} = \partial\widetilde{\theta}/\partial\boldsymbol{x}$ is the spatial temperature gradient.

When a subbody contains a singular surface, the jump conditions derived in Section 5 must be enforced. If these conditions hold for every two-dimensional region lying on $\Sigma(t)$, and if all the jumps are continuous functions of the surface coordinates (q^1, q^2), it then follows from (82), (89), (96), (103), and (114), respectively, that the following jump conditions hold pointwise on $\Sigma(t)$ for all $t \in I$:

[26] Note that, for the pillbox, the outward unit normal on the 2-side is $\boldsymbol{n}$.

$$[\![\rho\, w_n]\!] = 0, \tag{118a}$$

$$[\![\rho\, \boldsymbol{v}\, w_n - \boldsymbol{t}]\!] = \boldsymbol{0}, \tag{118b}$$

$$\boldsymbol{x} \times [\![\rho\, \boldsymbol{v}\, w_n - \boldsymbol{t}]\!] = \boldsymbol{0}, \tag{118c}$$

$$\left[\!\!\left[\rho\,(\varepsilon + \frac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{v})\, w_n - \boldsymbol{t}\cdot\boldsymbol{v} + h\right]\!\!\right] = 0, \tag{118d}$$

$$\left[\!\!\left[\rho\,\eta\, w_n + \frac{h}{\theta}\right]\!\!\right] \geq 0. \tag{118e}$$

Clearly, when the jump condition (118b) for linear momentum is satisfied at a point, then the jump condition for angular momentum (118c) is automatically satisfied.

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