WKB APPROXIMATION TO THE MODIFIED MILD-SLOPE EQUATION

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WKB approximation for water wave scattering by rapidly varying topography is obtained from a modified mild-slope equation of the general form by Porter (2003). The present WKB solution is reduced to the previous study where shallow water conditions are present. WKB models from the transformed mild-slope equation, without the described bottom curvature modification, show better performance than those by the original developed mild-slope equation. The underlying significance of the present equation is discussed in the context of linear wave scattering. The selected figures representing our results further characterize main feature of this study.

Keywords: modified mild-slope equation; WKB approximation; transformed mild-slope equations

1. INTRODUCTION

For a slowly varying depth, such as where the mild slope assumption is satisfied, the WKB approximation has been shown to produce adequate results in estimating wave scattering. Kajiura (1961) utilized the WKB method to investigate the scattering of linear shallow water waves, which was further developed by Mei (1989) to a more systematic interpretation. The WKB method gives an analytical solution from which we can get a better understanding of wave scattering processes. In the present study, this method is adapted and extended from those previous, in order to be applicable to wave scattering on waters of a rapidly varying depth.

Mei et al. (2005) applied the WKB method to Laplace equation with linearized boundary conditions, so as to obtain a surface wave evolution equation for periodic sandbars, a study which was originally initiated by Mei (1985) using multiple-scale expansion. He concentrated on the wave scattering near the resonance condition, and exhibited the Bragg resonant mechanism through these analytic methods. However in this study, we present modified mild-slope equation which applies this concept of wave scattering of sandbars to the WKB method.

The mild-slope equation and its variants are obtained from a single term approximation of velocity potential, which is composed of an unknown horizontal wave motion multiplied by a known vertical dependency. The vertical distribution of wave motion is represented by an eigenfunction of propagating mode, in water of constant depth. Applying the Galerkin method to the depth removes the vertical dependency of wave motion, and thusly, one of the dimensional problems is eliminated. This simplified approximate equation can account for wave transformation due to a slowly varying depth.

Various extensions of the mild-slope equation have been developed, which are aimed either at shorter irregularities, such as sand bars, or at steeper slopes. The simplest extension, of Chamberlain and Porter (1995), is to keep all terms which have been discarded in the mild-slope equation of Berkhoff (1972), and is referred to as the modified mild-slope equation (MMSE).

When the shallow water condition is invoked, MMSE reduces to the linear shallow water equation, and it also gives accurate results for wave scattering, by the bottom topography consisting of a rapidly varying small-amplitude component, superimposed on a slowly varying component. In this type of bedform, it has been reported that the original mild-slope equation cannot produce adequate approximations. Chamberlain and Porter (1995) have shown that MMSE is capable of describing scattering properties of singly sinusoidal beds, for which the mild-slope equation fails to give an adequate estimation. And they also showed that MMSE contains the extended mild-slope equation of Kirby (1986) as a special case. Its capability to produce accurate predictions for wave scattering over a rapid varying bottom is attributed to these additional terms.

In MMSE, there is a bottom curvature term $\nabla^2 h$, though it is not defined at locations where the bottom slope is discontinuous. By direct integration of the equation, Porter (2003) showed that a slope discontinuity in the free surface profile was induced by a bottom slope discontinuity. In order to get rid of this cumbersome term from the equation, he presented an alternative form, by using a change of variable. He also proposed a generalized equation which contains not only the transformed equation, but also the standard equation. We derive WKB approximation from this generalized equation so that different solutions can be obtained by choosing appropriate set of variables.

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Further significant advances have been made in approximations associated with the mild-slope equation, and have been proposed to include additional eigenfunctions of evanescent modes in studies by Massel (1993) and Porter and Staziker (1995). As the number of terms in the evanescent modes increases, the solution shows improved accuracy, because of a modification of the propagation mode resulting from the modal coupling. However Athanassoulis and Belibassakis (1999) showed that the mild-slope equation and its variants are inherently flawed, in that the eigenfunctions cannot satisfy the exact bottom conditions where the bed is not flat. If a sloping bed mode is added to the conventional expansion set, further improvement on the coupled mode approximation can be achieved (Athanassoulis and Belibassakis 1999; Chamberlain and Porter 2006). However, a simpler model is required in order to gain better physical understanding of scattering processes caused by variations in bathymetry, and hence, the modified mild-slope equation is suitable.

2. MODIFIED MILD-SLOPE EQUATION

The modified mild-slope equation for a monochromatic wave with angular frequency ω can be written in terms of free surface elevation $\eta(x, y)$, having time dependency $\exp(-i\omega t)$ factored out of the form

$$\nabla \cdot \left(P_0 \nabla \eta \right) + P_0 k^2 \left(1 + \tilde{q} \right) \eta = 0 \tag{1}$$

where wavenumber k, variables P_0 and \tilde{q} are functions of depth h(x, y). Wavenumber satisfies the dispersion equation.

$$\omega^2 = gk \tanh kh \,. \tag{2}$$

Parameters in Eq. 1 are produced by the following integrations of the prescribed vertical depth dependency f

$$f = \frac{\cosh k \left(h+z\right)}{\cosh kh} \tag{3}$$

and are defined, based on local depth, by

$$\begin{cases} P_0 = \int_{-h}^0 f^2 dz; \quad P_1 = \int_{-h}^0 f \frac{\partial f}{\partial h} dz \\ P_2 = \int_{-h}^0 f \frac{\partial^2 f}{\partial h^2} dz + f \frac{\partial f}{\partial h} \Big|_{z=-h}; \quad \tilde{q} = \frac{P_2 \left| \nabla h \right|^2 + P_1 \nabla^2 h}{P_0 k^2} \end{cases}$$
(4)

Explicit formulas for these variables can be generated in a straightforward manner, and are provided by Chamberlain and Porter (1995).

Although the bottom curvature term $\nabla^2 h$ is small enough to be neglected for slowly varying topography, it cannot be negligible for rapidly varying bed profiles. As shown in Porter (2003), the term $\nabla^2 h$ is bigger than that of ∇h for a rapidly varying bed. Furthermore, when there is a slope discontinuity in the bed profile, this term cannot be defined, leads to difficulties in the numerical calculations. It should be noted that many bed types considered in the previous studies have specific types of slope discontinuity. For example, the bed profile in which a finite length of a sinusoidal bed is superimposed on an otherwise constant depth has slope discontinuity at two points. If a bed form has slope discontinuity, the curvature term presents such a degree of complexity within the calculation, that its consideration is quite disproportional to that of the original equation. Porter (2003) provided a transformed equation which removed the term from the original modified mild-slope equation.

To investigate the effect of the curvature term on the WKB solution, a general form of modified mild-slope equation, which was also proposed by Porter (2003), is used. As shown in Table 1, Eq. 5 contains both the original equation and transformed one, through the selective choice of appropriate variables.

$$\nabla \cdot \left(P \nabla \chi \right) + P k^2 \left(1 + q \right) \chi = 0 \tag{5}$$

Table 1. Variables in the general modified mild-slope equation.				
	χ	Р	q	remarks
Smith & Sprinks (1975)	η	P_0	$\tilde{q} = \frac{1}{P_0 k^2} \left(P_2 \left \nabla h \right ^2 + P_1 \nabla^2 h \right)$	
Porter (2003)	ζ	$\frac{1}{k^2}$	$\hat{q} = \frac{1}{P_0 k^2} \left(\frac{P_1^2}{P_0} - \left\ \frac{\partial f}{\partial h} \right\ ^2 \right) \left \nabla h \right ^2$	$\eta = \frac{\zeta}{k\sqrt{P_0}}$

After deriving the WKB approximation from Eq. 5, a performance test of the original and transformed forms of the modified mile-slope equation for sinusoidal beds is carried out.

3. WKB APPROXIMATION TO THE MMSE

When the WKB method is applied to the modified mild-slope equation, the previous studies can be extended to scattered waves on water of intermediate depth. We introduce a small parameter $\mu = O(\nabla h/kh) \ll 1$, which is based on the conventional mild-slope assumption that the typical wavelength is much less than the horizontal length scale of depth variation. For simplicity, we consider a horizontally one-dimensional wave scattering problem, whereupon a slow coordinate $\overline{x} = \mu x$ can be defined, which is exclusively used in the WKB analysis.

Anticipating progressive monochromatic waves, we seek η in the form of a WKB expansion in powers of μ .

$$\chi(\overline{x}) = \left[\chi_0(\overline{x}) + i\mu\chi_1(\overline{x}) + \cdots\right] e^{iS(\overline{x})/\mu}$$
(6)

Substituting Eq. 6 into Eq. 5, and collecting terms of like powers of μ in the resulting equation, we obtain

$$O(\mu^{0}): -\left(\frac{dS}{d\overline{x}}\right)^{2} + k^{2}(1+q) = 0$$

$$O(\mu^{1}): 2\frac{dS}{d\overline{x}}\frac{d\chi_{0}}{d\overline{x}} + \frac{d^{2}S}{d\overline{x}^{2}}\chi_{0} + \frac{1}{P}\frac{dP}{d\overline{x}}\frac{dS}{d\overline{x}}\chi_{0} = 0$$
(7)

From Eq. 7, the solutions of eikonal and transport equations can be obtained.

$$S(\bar{x}) = \int^{\bar{x}} k \sqrt{1+q} d\xi,$$

$$\chi_0(\bar{x}) = \frac{c_1}{\sqrt{Pk} (1+q)^{1/4}} e^{iS/\mu} + \frac{c_2}{\sqrt{Pk} (1+q)^{1/4}} e^{-iS/\mu}$$
(8)

Both solutions clearly show that wave phase and amplitude are affected by this additional term q, only found in the modified mild-slope equation. This leading order solution of the transport equation reflects that of ray theory, when reflection is negligible. Specifically, for a slowly varying bed profile, we have $\tilde{q} \approx 0$ and $\sqrt{P_0 k} \propto \sqrt{C_g}$. Hence the amplitude variation from the original MMSE follows that of the ray theory.

If the fractional depth change within a wavelength is more pronounced, it can be anticipated that appreciable amounts of wave reflection by the topography occurs. In order to take reflection into

consideration in the WKB analysis, the constants in solution Eq. 8 should be replaced with the unknown functions E(x) and F(x), as in the previous studies

$$\chi(x) = \frac{E(x)}{\sqrt{Pk}(1+q)^{1/4}} e^{iS/\mu} + \frac{F(x)}{\sqrt{Pk}(1+q)^{1/4}} e^{-iS/\mu}$$
(9)

Because of these additional unknowns, two equations must be specified to solve them.

To this end, let us define a weighted mass flux in terms of the original variables.

$$Q = \int_{-h}^{0} f \frac{\partial \phi}{\partial x} dz \tag{10}$$

Since Eq. 10 becomes $Q \rightarrow (d\phi/dx)h = uh$ in shallow water, it inherently implies mass flux. Substituting velocity potential $\phi = -(ig/\omega)\eta f$, based on locally constant depth, into Eq. 10 and neglecting small term gives

$$Q = -\frac{ig}{\omega} \frac{\partial \eta}{\partial x} P_0 \quad \Rightarrow \quad Q = -\frac{ig}{\omega} \frac{\partial \chi}{\partial x} P \tag{11}$$

After multiplying $-i\omega$ both sides of Eq. 11, we can readily recognize that it turns out to be the momentum equation for the present MMSE. Substitution of the second equation in Eq. 11 into the horizontally one-dimensional version of Eq. 5 leads to the other equation, which is a mass equation in the present context.

$$i\omega\chi = \frac{\omega^2}{gPk^2\left(1+q\right)}\frac{dQ}{dx}$$
(12)

From Eqs. 11 and 9, the mass flux to the leading order can be expressed as

$$Q(x) = \frac{g\sqrt{Pk}(1+q)^{1/4}}{\omega} \left[E(x)e^{iS/\mu} - F(x)e^{-iS/\mu} \right]$$
(13)

After differentiation of Eqs. 9 and 13 with respect to x, substituting the resulting equations into Eqs. 11 and 12 produces a pair of coupled linear equations.

$$\begin{cases} \frac{dE}{dx}e^{iS/\mu} - \frac{dF}{dx}e^{-iS/\mu} = -\mu\frac{d}{d\overline{x}}\frac{\ln\left(Pk\sqrt{1+q}\right)}{2}\left(Ee^{iS/\mu} - Fe^{-iS/\mu}\right) \\ \frac{dE}{dx}e^{iS/\mu} + \frac{dF}{dx}e^{-iS/\mu} = \mu\frac{d}{d\overline{x}}\frac{\ln\left(Pk\sqrt{1+q}\right)}{2}\left(Ee^{iS/\mu} + Fe^{-iS/\mu}\right) \end{cases}$$
(14)

For brevity in the following derivation, we now introduce a complex function G and its complex conjugate G^* .

$$G = \frac{1}{2} \frac{d \ln Pk \sqrt{1+q}}{d \overline{x}} e^{-2iS/\mu}$$
(15)

Solving the simultaneous equations with respect to the derivatives in Eq. 14 returns them in a simplified form.

$$\frac{dE}{dx} = \mu GF, \quad \frac{dF}{dx} = \mu G^* E \tag{16}$$

If the shallow water assumption $kh \to 0$ is invoked, and the original variables are employed in the expression as well, we have $P_0 \to h$ and $\tilde{q} \to 0$. As a result, Eq. 14 expressed in the original variables is reduced to that of Mei (1989). It can also be observed that for a region of constant depth, both q and G vanish, which is an interpretation that scattered waves are generated only if sufficient depth variation exists.

We now introduce the perturbation expansions to the unknowns E and F.

$$E = E_0 + \mu E_1 + \cdots, \quad F = F_0 + \mu F_1 + \cdots$$
 (17)

Substituting them into Eq. 16 and separating equal orders of μ , we have

$$\frac{dE_0}{dx} = 0, \ \frac{dF_0}{dx} = 0; \ \frac{dE_n}{dx} = GF_{n-1}, \ \frac{dF_n}{dx} = G*E_{n-1} \quad (n = 1, 2, \cdots)$$
(18)

Hence, the unknown functions for each order E_n and F_n can be solved with the assistance of the given boundary conditions. The associated boundary conditions are described in the following section.

4. BOUNDARY CONDITIONS AND SOLUTION

For a bedform of a smooth profile, and connecting two horizontal planes, we pose a wave of unit amplitude coming from the left. In the neighborhood of the boundary points, depth is constant. This results in q = 0.

Waves in these constant regions can be written by

$$\eta(x) = \begin{cases} e^{ik_1(x-x_1)} + Re^{-ik_1(x-x_1)}, & x \le x_1 \\ Te^{ik_2(x-x_2)}, & x \ge x_2 \end{cases}$$
(19)

where R and T are complex reflection and transmission coefficients to be determined. Boundary points x_1 and x_2 denote locations of the up-wave and down-wave boundaries, which are sufficiently far away from the depth varying region $(0 \le x \le L)$, as evanescent modes are not taken into consideration.

The associated boundary conditions with the one-dimensional version of Eq. 5 follow directly from manipulating Eq. 19, as shown by Porter (2003).

$$\frac{d\eta(x_{1})}{dx} + ik_{1}\eta(x_{1}) = 2ik_{1}, \quad \frac{d\eta(x_{2})}{dx} - ik_{2}\eta(x_{2}) = 0$$

$$\frac{d\eta(x_{1})}{dx} - ik_{1}\eta(x_{1}) = -2ik_{1}R, \quad \frac{d\eta(x_{2})}{dx} + ik_{2}\eta(x_{2}) = 2ik_{2}T$$
(20)

The boundary condition for the original equation can be readily obtained if Eq. 9, expressed in terms of the original variables, is substituted into Eq. 20.

$$\frac{E(x_{1})}{\left(\sqrt{P_{0}k}\right)_{1}} = 1, \quad \frac{F(x_{1})}{\left(\sqrt{P_{0}k}\right)_{1}} = R;$$

$$\frac{F(x_{2})e^{-iS_{2}/\mu}}{\left(\sqrt{P_{0}k}\right)_{2}} = 0, \quad \frac{E(x_{2})e^{iS_{2}/\mu}}{\left(\sqrt{P_{0}k}\right)_{2}} = T$$
(21)

where the phase of the boundaries is set to $S_1 = 0$ and $S_2 = S(\bar{x}_2)$. Using surface elevation relationship $\eta = \zeta / (k \sqrt{P_0})$ and Eq. 9, it can be easily shown that Eq. 21 is still relevant to the transformed equation, if the phase S_2 is computed by its relevant equation.

From Eq. 21, the reflection coefficient can be determined by unknown variable $F(x_1)$ and the transmission coefficient by $E(x_2)$. To find these variables, substituting the perturbation expansion Eq. 17 into Eq. 21 yields boundary conditions for Eq. 18.

$$\begin{cases} E_0(x_1) = \left(\sqrt{P_0 k}\right)_1, & E_n(x_1) = 0 & (n = 1, 2, \cdots) \\ F_n(x_2) = 0 & (n = 0, 1, \cdots) \end{cases}$$
(22)

Integrating Eq. 18 aided by the appropriate boundary conditions in Eq. 22 gives solutions of the unknown amplitude functions.

$$\begin{cases} E_n(x) = \int_{x_1}^x G(\xi) F_{n-1}(\xi) d\xi, \\ F_n(x) = -\int_x^{x_2} G^*(\xi) E_{n-1}(\xi) d\xi \end{cases}$$
(23)

Noting that $F_0(x) = 0$ from Eqs. 18 and 22, part of Eq. 23 become

$$E_{2n+1}(x) = F_{2n}(x) = 0 \quad (n = 1, 2, \cdots).$$
⁽²⁴⁾

In summary, Eqs. 23 and 24 are reduced to

r

$$\begin{cases} E_n(x) = \int_{x_1}^x G(\xi) F_{n-1}(\xi) d\xi, \\ F_n(x) = -\int_x^{x_2} G^*(\xi) E_{n-1}(\xi) d\xi \end{cases}$$
(25)

Finally introducing perturbation expansions to R and T

$$R = R_0 + \mu R_1 + \cdots, \quad T = T_0 + \mu T_1 + \cdots.$$
 (26)

and utilizing the boundary conditions computed from Eq. 25, Eq. 21 yields the physical coefficients of interest.

$$\begin{cases} R_{2n} = 0, \quad R_{2n+1} = \frac{F_{2n+1}(x_1)}{\sqrt{(P_0 k)_1}} \\ T_{2n} = \frac{E_{2n}(x_2)e^{iS_2/\mu}}{\sqrt{(P_0 k)_2}}, \quad T_{2n+1} = 0 \end{cases}$$
(27)

To $O(\mu^0)$, there are no contributions to the reflection coefficient component, while the transmission component is given by

$$T_0 = \frac{\sqrt{(P_0 k)_1}}{\sqrt{(P_0 k)_2}} e^{iS_2/\mu} \,. \tag{28}$$

We remark that T_0 also denotes ratio of square root of group velocity C_g at x_1 to that at x_2 times the phase $e^{iS_2/\mu}$ evaluated at x_2 . We now see that this is nothing but the transmission coefficient computed by geometric optics approximation.

On the other hand, at $O(\mu^1)$ the transmission component is equal to zero, whereas the reflection component is given by

$$R_{1} = \frac{F_{1}(x_{1})}{\sqrt{(P_{0}k)_{1}}} = -\int_{x_{1}}^{x_{2}} \frac{1}{2} \frac{d\ln\left(Pk\sqrt{1+q}\right)}{d\xi} \exp\left(\frac{2i}{\mu}\int_{x_{1}}^{\xi} k\sqrt{1+q}\,dt\right) d\xi \quad (29)$$

To gain some underlying physical significance of Eq. 29, let us denote the average wave number as \overline{k} over a period of bar length l, from which the bar wave number is given by $k_r = 2\pi/l$. Over one bar length, the phase may be written as $2\overline{k}l$ for the case of MSE (q=0). If the phase is equal to $2\pi m$ where m is an integer, the reflected wave phase is in phase with that of the incident wave, so that the reflection coefficient is reinforced. This resonance occurs whenever the condition $2\overline{k}l = 2\pi m$ is fulfilled, which can be rewritten as $2\overline{k}/k_r = m$. When a bed consists of multi periodic bars, multiple scattering occurs, which should be in turn summed, bringing about an enhancement of the reflection coefficient.

If the restriction of the mild-slope equation (q = 0) is imposed in shallow water, Eq. 29 reduces it to the reflection component of shallow water approximation given by Mei (1989). It can be easily anticipated that higher order contributions might prove to be more accurate, due to multiple scattering by topography.

5. RESULTS

In this section we describe numerical procedure to compute scattering properties derived in the previous sections. For convenience, we introduce sequential functions $\beta_n(x)$ in the following equations

$$\begin{cases} \beta_0(x) = 1, \\ \beta_n(x) = \int_{x_1}^x G(\xi) \beta_{n-1}^*(\xi) d\xi \quad (n = 1, 2, \cdots) \end{cases}$$
(30)

Using Eq. 30 gives the simplified expression of the recursion formulae Eq. 25.

$$\begin{cases} E_{2n}(x) = \sum_{m=1}^{n} F_{2(n-m)+1}(x_1) \beta_{2m-1}(x) + E_0 \beta_{2n}(x), \\ F_{2n-1}(x) = \sum_{m=1}^{n} F_{2(n-m)+1}(x_1) \beta_{2(m-1)}^*(x) + E_0 \beta_{2n-1}^*(x) \end{cases}$$
(*n* = 1, 2, ···) (31)

To evaluate variables in Eq. 31, quantity $F_{2n-1}(x_1)$ must be determined in advance. Noting that $F_n(x_2) = 0$, from Eq. 22, it can be rewritten as

$$F_{2n-1}(x_1) = -\sum_{i=1}^{n-1} F_{2(n-i)-1}(x_1) \beta_{2i}^*(x_2) - E_0 \beta_{2n-1}^*(x_2) \quad (n = 1, 2, \cdots) \quad (32)$$

From Eq. 31, we can compute $E_n(x_2)$ too.

$$E_{2n}(x_2) = \sum_{m=1}^{n} F_{2(n-m)+1}(x_1) \beta_{2m-1}(x_2) + E_0 \beta_{2n}(x_2) \quad (n = 1, 2, \cdots) \quad (33)$$

Finally, reflection and transmission coefficients are obtained by substituting Eqs. 32 and 33 into Eq. 27. For computation of the complex function $\beta_n(x)$ in Eq. 30, the trapezoidal rule is used. To compute the overall R and T in Eq. 26, the small parameter is set to $\mu = 1$, as in Mei (1989). And, the bottom curvature for the standard MMSE is obtained by using central difference in this computation.

Wave scattering by sinusoidal beds has been well documented. These beds consist of a finite patch of small-amplitude sinusoidal ripples in an otherwise horizontal bed. First, we examine the performance of four different WKB approximations in relation to singly sinusoidal ripples: standard and transformed versions of both MSE and MMSE.

For the purpose of comparison with existing results, we consider the depth studied by Davies and Heathershaw (1984).

$$h(x) = \begin{cases} h_0 + A\sin(k_r x), & (0 \le x \le L) \\ h_0, & (x < 0, \quad x > L) \end{cases}$$
(34)

where h_0 is a constant mean depth, A is amplitude of the bar, and $L = 2n\pi/k_r$. There is a sequence of n ripples in the bed profile. The results are presented in the form of graphs of reflection coefficient |R| plotted against $2k/k_r$. The abscissa denotes twice the ratio of bar wavelength and incident wavelength. In all results presented here, reflection coefficient is summed up to R_{17} , which turns out to be a virtually convergent value as long as convergent result is obtained.

Fig. 1 shows WKB results for the case n = 2 and $A/h_0 = 0.32$. For a better visual comparison of the figures, laboratory data obtained by Davies and Heathershaw (1984) are not presented in all figures, as MMSE predictions are in agreement with data for the singly sinusoidal ripple.



Figure 1. Comparison of computed reflection coefficients for singly sinusoidal ripple with n=2, $A/h_0 = 0.32$ and $h_0/L = 6.41$.

All three models, except the standard MSE, show very close agreement, and all models correctly position the first resonance peak near $2k/k_r = 1$. Hence, it is clear that the result of the transformed MSE shows almost the same thing as that of the transformed MMSE. This may imply that the bottom slope term q in Eq. 5 is small enough to be considered negligible, which is in alignment with results given by Porter (2003).

In Fig. 2, WKB results for the case n = 4 and $A/h_0 = 0.32$ are shown. In this case, all results except that of the standard MSE are, again, in very close agreement. Each graph shows the correct position of the first-order resonance, and we can observe that the maximum value of the reflection coefficient is significantly enhanced as the number of ripples increase. An almost negligible difference between the transformed MSE and MMSE, in this case, indicates that the bottom slope term contributes little to the scattering properties. All graphs shown in Fig. 2 detect significant second-order resonance near $2k/k_r = 2$. A more accurate model devised by Porter and Porter (2003) shows that a very small value of reflection coefficient is found at this location, as if the second-order resonance does not arise. Hence they concluded that the MMSE exaggerated the second-order resonance.



Figure 2. Comparison of computed reflection coefficients for singly sinusoidal ripple with n = 4, $A/h_0 = 0.32$ and $h_0/L = 6.41$.

Fig. 3 shows graphs for ten ripples in the bed. The water depth in this case is deeper than in previous cases, so that the ratio of ripple amplitude to water depth is given by $A/h_0 = 0.16$. The standard MSE does not produce adequate prediction as previous studies were pointed out. Another remarkable feature in the context of present numerical experiments is that the standard MMSE shows a divergent result. The transformed MSE and MMSE produce very close results, which perform much better than standard methods, and compare very favorably with previous studies (O'Hare and Davies 1993; Chamberlain and Porter 1995; Porter 2003). However, the reflection coefficient computed by the transformed models shows over-prediction near $2k/k_r = 2$ in comparison with the result presented by Porter and Porter (2003).



Figure 3. Comparison of computed reflection coefficients for singly sinusoidal ripple with n = 10, $A/h_0 = 0.16$ and $h_0/L = 3.19$.

Since the singly sinusoidal periodic topography includes the feature of a small bed slope, it has been reported that the MMSE compares very favorably with the accurate result by other methods, except for values close to $2k/k_r = 2$, as shown above. As mentioned before, the transformed equations incorporate the effect of bottom curvature through change of variables. In addition, the term of correction in the transformed MMSE is of the second order in the bed slope, so that it produces a virtually negligible contribution to the solution. In summary, for topography of a small bed slope, the transformed MMSE.

The doubly sinusoidal periodic topography considered by Guazzelli et al. (1992) has an extra complexity within the bed shape, as well as a steeper bed slope, which significantly reduces the accuracy of the reflection coefficient, especially for models based on propagating modes only: For example, MMSE and the successive-application-matrix model by O'Hare and Davies (1993). We attempted to apply the present WKB based models to these topographies, but none of them produced convergent results.

6. CONCLUSIONS

WKB approximation was applied to the mild-slope equations and wave scattering by sinusoidal beds was further investigated. We present a WKB solution from the general form of MMSE by Porter (2003), in which the bottom curvature term of standard MMSE is eliminated, but its effect is incorporated into the transformed equations. By introducing two unknown functions for wave scattering, and imposing an appropriate mass and momentum equation for them, a pair of coupled differential equations are constructed which can be reduced to the previous ones when a shallow water condition is invoked. The boundary conditions subject to the differential equations are specified.

Substituting WKB expansion to the differential equations and solving them, gives the sequential WKB solutions. The performance of four different approximations, which are directly obtained from the present WKB solution by an appropriate choice of variables, is compared on the basis of computed reflection coefficients: a standard mild-slope equation and modified mild-slope equation as well as their transformed versions.

The present WKB models produce predictions of wave scattering very similar to previous results, so long as the WKB series converges. In cases of doubly sinusoidal ripples, where bed slope is steep and bed shape is complex, all of the present WKB models yield divergent series. WKB models based on transformed equations give more an accurate solution compared to those by standard equations. For singly sinusoidal ripples, very close results are obtained between WKB models based on transformed MSE and MMSE. Because the transformed MMSE has the extra correction term of bed slope,

compared with the corresponding MSE, this implies that the extra term gives negligible contribution to the result of the singly sinusoidal ripples.

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