

Dispersion of Wave Action

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Abstract

The dispersion of particles beneath a random sea, produced by the associated random variations in the Stokes drift is considered. In particular, a sea state described by a Pierson-Moskowitz spectrum is considered and the diffusion rate on and below the surface is examined. For such a random sea the diffusion rate at the surface can be related to the wind speed. This expression is extended to consider dispersion rates below the surface and to relates these to the wind speed. The theory is then applied to study the dispersion of particles denser than water as they fall through a random sea.

Introduction

An accurate prediction of the dispersion of pollutants in the sea is essential in the planning of contingency exercises and also in risk analysis. Computer models have been developed to simulate the spread of pollutants in the sea but these generally ignore the effects of wave action in dispersing the material which, on a scale of up to a few kilometers, can have a large role to play (Schott *et al.* 1978, Herterich and Hasselmann 1982). Recent oil spill disasters such as the sinking of the Braer off the Shetland Islands have highlighted the fact that the dispersion of pollutants is highly dependent on the prevailing wave climate. Discharges

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from chemical and sewage works can also cause pollution. In such cases an understanding of the dispersion of the pollutants, on and below the sea surface, is clearly desirable. In this paper we consider the dispersion of such pollutants due to random wave motion described by a Pierson-Moskowitz (Pierson and Moskowitz 1964) surface spectrum. The Pierson-Moskowitz spectrum is a JONSWAP spectrum (Hasselmann *et al.* 1973) describing a fully developed sea and is given by

$$f(\omega) = \alpha g^2 \omega^{-5} \exp \left[-\frac{5}{4} \left(\frac{\omega_m}{\omega} \right)^4 \right], \quad (1)$$

where ω_m is the peak frequency, g is the acceleration due to gravity, ω is the frequency and $\alpha = 0.0081$. Other processes related to dispersion include turbulence and ocean currents. These act on a larger scale than the wave dispersion considered here and are discussed elsewhere (Craig and Banner 1994, Sanderson and Pal 1990).

Diffusion at the Surface of a Random Sea

The diffusion of tracers at the sea surface due to wave action has been investigated by Herterich and Hasselmann (1982) who consider the random fluctuations in the Stokes drift velocity,

$$\langle \mathbf{u}_s \rangle = 2 \int F_\eta(\mathbf{k}) \omega \mathbf{k} e^{2kz} d^2k \quad (2)$$

where angle brackets denote ensemble averaging, ω is the frequency, z is the vertical distance, \mathbf{k} is the wavenumber and the two-dimensional wave spectrum $F_\eta(\mathbf{k})$ is normalized to the mean-square surface displacement of the wave. Herterich and Hasselmann (1982) show that, provided the surface spectrum can be expressed as $F_\eta(\theta, \omega) = f_\eta(\omega) S(\theta)$ where $S(\theta)$ is a directional spreading function, the dispersion of particles is governed by a mean advection velocity and a diffusion tensor D_{ij} which is associated with the random wave-surface. The mean advection velocity is simply the mean Stokes-drift velocity. For a single particle the diffusion tensor is given by

$$\begin{aligned} & \begin{pmatrix} D_{xx} & D_{xy} \\ D_{yx} & D_{yy} \end{pmatrix} \\ &= \frac{\pi}{4g^2} \int_0^\infty \omega^6 f_\eta^2(\omega) \int_{-\pi}^\pi \int_{-\pi}^\pi S(\theta_1) S(\theta_2) [1 + \cos(\theta_1 - \theta_2)]^2 M_{ij} d\theta_1 d\theta_2 d\omega \end{aligned} \quad (3)$$

where, provided $S(\theta)$ is symmetric about $\theta = 0$, we have $M_{xx} = (\cos \theta_1 + \cos \theta_2)^2$, $M_{yy} = (\sin \theta_1 + \sin \theta_2)^2$, $D_{xy} = 0$ and $D_{yx} = 0$.

Diffusion Below the Surface of a Random Sea

The derivation of equation (3) is not reliant on the particle being on the surface and can equally be applied at any depth provided the horizontal displacement spectrum is known. To find this spectrum we model the surface, η , of a random sea by

$$\eta(x, y, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\Phi(k_x, k_y, \omega) e^{i(k_x x + k_y y - \omega t)} \tag{4}$$

where k_x and k_y are the x - and y -components of the two-dimensional wave number and $\{d\Phi\}$ is a stochastic, typically complex, function representing an innovation process which incorporates all the random features of the sea surface. Assuming that the fluid is irrotational and so can be described in terms of a potential, ϕ , which is separable: $\phi(x, y, x, t) = a(z)b(x, y, t)$, then we can solve the Laplace equation for the surface, equation (4), to find the displacements $\xi_j(x, y, z, t)$. This is done in a manner analogous to the standard solution for a monotonic surface wave ($\eta = Ae^{i(kx-\omega t)}$) (Lighthill 1978) subject to the standard boundary conditions: no vertical motion at the bed; the vertical velocity at the surface is given by $\partial\eta/\partial t$; and the Bernoulli equation is satisfied at the surface. With these boundary conditions the displacements are found as

$$\xi_j(x, y, z, t) = \int_0^{\infty} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{a_j}{\sinh(kh)} d\Phi(k, \theta, \omega) e^{i(kx \cos \theta + ky \sin \theta - \omega t)}, \tag{5}$$

where

$$\begin{aligned} a_x &= i \cos \theta \cosh[k(z + h)], \\ a_y &= i \sin \theta \cosh[k(z + h)], \\ a_z &= \sinh[k(z + h)], \end{aligned} \tag{6}$$

h is the mean water depth and $j = x, y, z$. Invoking the properties of $\{d\Phi\}$ we can express

$$f_z(\omega; z) = \left(\frac{A_z}{\sinh(Kh)} \right)^2 f_\eta(\omega), \tag{7}$$

where f_z is the vertical displacement spectra and $A_z = a_z|_{k=K}$ where K is a solution of $\omega^2 = gK \tanh(Kh)$. Thus, replacing $f_\eta(\omega)$ with $f_z(\omega; z)$ in equation (3), we can calculate the diffusion tensor D_{ij} at any depth for any appropriate surface spectrum. We note here that the diffusion arises from considering the higher-order Stokes-drift term while the expression for $f_z(\omega; z)$ is only considered in the linear limit. This is a valid approach since higher-order terms will only have a small affect on the form of the displacement spectra below the surface and hence on the calculated diffusion terms.

For the Pierson-Moskowitz surface spectrum the diffusion tensor at the surface can be written $D_{ij} \propto \omega_m^{-3} \bar{D}_{ij}$ where \bar{D}_{ij} is a non-dimensional function of $S(\theta)$. The peak frequency, ω_m , is given by $\omega_m = 0.280\pi g/U$ where U is the wind speed measured 19.5 m above the sea height. This means that once the surface diffusion coefficients have been calculated for one velocity, for a given S , they can be found for any other velocity using the scaling relationship $D_{ij}(U_1) = (U_2/U_1)^3 D_{ij}(U_2)$. In three dimensions the same surface relation

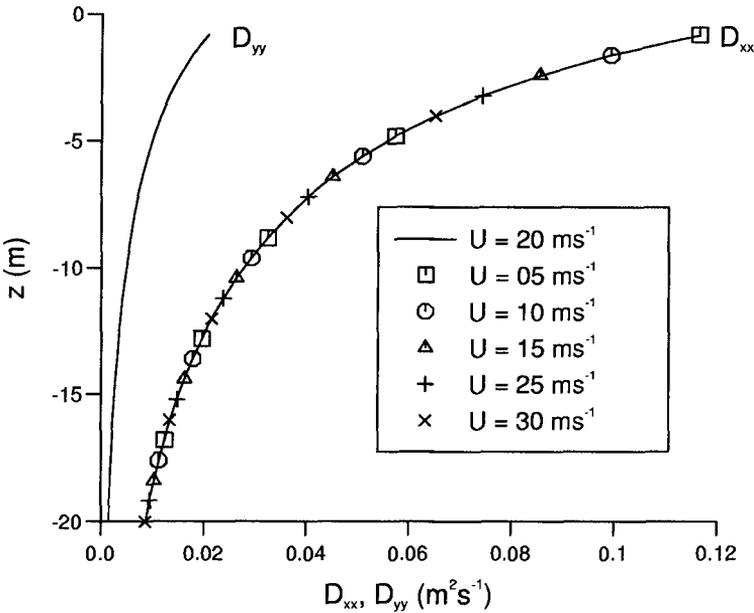


Figure 1: The value of D_{xx} and D_{yy} calculated for a Pierson-Meskowitz surface spectrum with $S(\theta) = 2 \cos^2(\theta)/\pi$ for a wind speed $U = 20 \text{ ms}^{-1}$ and a water depth of 100 m. Also shown are results calculated for wind speeds $U = 25, 15, 10$ and 5 ms^{-1} after they have been scaled by equation (9) to align them with the results for $U = 20 \text{ ms}^{-1}$.

must hold for $D(U, z = 0)$. Below the surface we note that the z -dependence appears only in the integral

$$\int_0^\infty \omega^6 f^2(\omega; z) d\omega \tag{8}$$

as a product kz which in deep water can be expressed $\omega^2 z/g$. In this integral the function $f(\omega; z)$ is negligible except close to ω_m so we find, to a good approximation, the following scaling relation for different values of U :

$$D(U_2; z) = \left[\frac{U_2}{U_1} \right]^3 D \left(U_1; \left[\frac{U_1}{U_2} \right]^2 z \right). \tag{9}$$

This allows us to predict the value of D at different depths and different wind speeds from a knowledge of D for any U . Figure 1 shows D_{xx} and D_{yy} plotted against z for a wind speed $U = 20 \text{ ms}^{-1}$ in water of depth 100 m. Also shown are the values of D_{xx} calculated

for $U = 5, 10, 15$ and 25 ms^{-1} after they have been scaled according to equation (9) to align them with the $U = 20$ result. The good agreement validates the use of equation (9). Also shown in figure 1 is the difference between D_{xx} and D_{yy} for the spreading function $S(\theta) = 2 \cos^2(\theta)/\pi$, where D_{xx} describes diffusion in the mean wind direction and D_{yy} in the direction perpendicular to the mean wind. As expected $D_{xx} > D_{yy}$ for all values of z . In fact, for any z the ratio D_{xx}/D_{yy} is a function of $S(\theta)$ and independent of z . Thus the good agreement of D_{xx} between the results for different wind speeds is also true for D_{yy} .

Diffusion of Particles Denser Than Water in a Random Sea

The above theory allows us to calculate diffusion rates for particles on the sea surface or at a fixed depth below the surface. We now wish to consider particles which do not remain at a fixed depth. To demonstrate how this can be done we consider, as an example, a particle denser than water falling through the sea under gravity. A particle with density $\rho > \rho_w$, the density of water, initially at the sea surface, will sink to the bottom. As it does so it will diffuse in the horizontal directions according to the diffusion tensor $D_{ij}(z(t))$ where $z(t)$ is the particle's vertical position as a function of time. Neglecting any vertical motion due to the wave action, since there is no vertical drift or diffusion (Herterich and Hasselmann 1982), the vertical velocity w of a spherical particle of radius a can be found by considering the forces acting on it. Gravity is acting with force $-4\pi a^3 \rho g/3$, there is a buoyancy force $4\pi a^3 \rho_w/3$ and there is a drag force $-6\pi \mu a w$, where μ is the fluid viscosity. This gives

$$w(t) = A \frac{\rho_w - \rho}{\rho} g (1 - e^{-t/A}) \tag{10}$$

where

$$A = \frac{2 \rho a^2}{9 \mu} \tag{11}$$

The vertical position of the particle is therefore given by

$$z(t) = A \frac{\rho_w - \rho}{\rho} g t + A^2 \frac{\rho_w - \rho}{\rho} g (e^{-t/A} - 1) \tag{12}$$

for $t < t^*$ where $z(t^*) = -h$. Thus we can calculate the vertical position $z(t)$ and the ensemble average of the square of the difference between the components of the particles horizontal position at time t and it's ensemble position at time t , $\langle (x_i(t) - \langle x_i(t) \rangle)^2 \rangle$ and $\langle (y_i(t) - \langle y_i(t) \rangle)^2 \rangle$, which are given by

$$\begin{aligned} & \left(\frac{\langle (x(t) - \langle x(t) \rangle)^2 \rangle}{\langle (y(t) - \langle y(t) \rangle)^2 \rangle} \right) = \frac{\pi}{2g^2} \int_0^t dt \int_0^\infty d\omega \int_{-\pi}^\pi d\theta_1 \\ & \times \int_{-\pi}^\pi d\theta_2 \omega^6 \left\{ \frac{\sinh\{k[z(t_1) + d]\}}{\sinh(kd)} \right\}^4 f_\eta^2(\omega) S(\theta_1) S(\theta_2) \\ & \times [1 + \cos(\theta_1 - \theta_2)]^2 \left(\frac{(\cos \theta_1 + \cos \theta_2)^2}{(\sin \theta_1 + \sin \theta_2)^2} \right). \end{aligned} \tag{13}$$

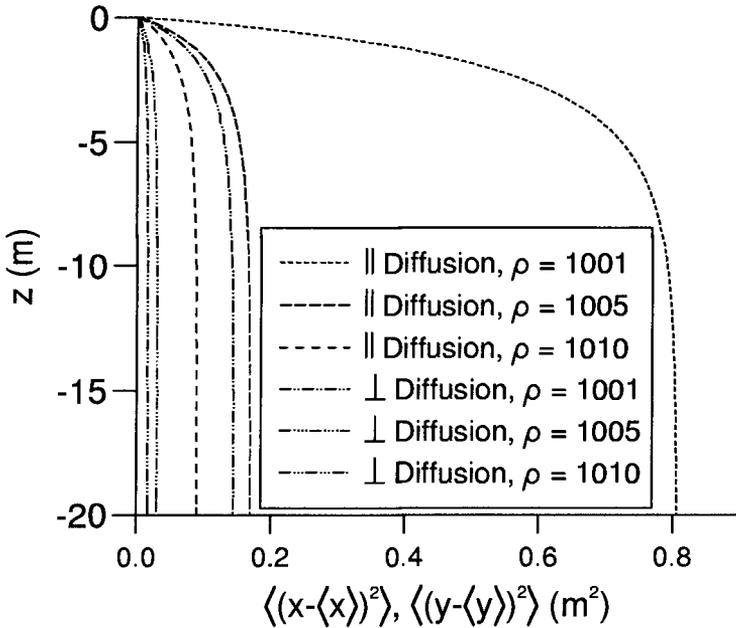


Figure 2: The value of $\langle (x_i(t) - \langle x_i(t) \rangle)^2 \rangle$ for diffusion parallel to the mean wind direction and $\langle (y_i(t) - \langle y_i(t) \rangle)^2 \rangle$ perpendicular to the mean wind direction for a single particle with radius $a = 1$ mm falling through the ocean plotted as a function of the vertical position for a random sea state described by a Pierson-Moskowitz spectrum with a wind speed $U = 10 \text{ ms}^{-1}$ and spreading function $S(\theta) = 2 \cos^2(\theta)/\pi$. The particle densities considered were 1001, 1005 and 1010 kgm^{-3} and the density of the ocean was taken to be 1000 kgm^{-3} .

Figure 2 shows the value of $\langle (x(t) - \langle x(t) \rangle)^2 \rangle$ and $\langle (y(t) - \langle y(t) \rangle)^2 \rangle$ plotted against the vertical position of the particle for a particle with density 100.1%, 100.5% and 101% of the density of water and radius 1 mm. The surface spectrum is a Pierson-Moskowitz spectrum with $U = 10 \text{ ms}^{-1}$, the spreading function used was $S(\theta) = 2 \cos^2(\theta)/\pi$. The water depth is 100 m but only the top 20 m are shown since there is only negligible horizontal motion below this depth and the particle falls vertically to the ocean bed. The graph shows that the majority of the dispersion occurs in the top 10 m (for our choice of wind speed). This is to be expected since the horizontal wave velocities are maximum at the surface and decay exponentially below it. Clearly, the slower the particle falls through the top 10 m of the sea, the longer it is subject to the larger random velocities and so the larger its potential to be dispersed. Thus the closer the particle density is to the density of water, the slower it sinks and the larger its potential to be diffused by the wave motion.

Conclusion and Discussion

The dispersion of particles due to random wave motion have been studied. Consideration has been given to buoyant particles on the surface, neutrally buoyant particles suspended in the sea and particles moving through the fluid under the action of gravity. It has been seen that diffusion due to random waves can have a significant affect on the dispersion of particles or pollutants in the ocean. An approximate scaling relationship has been found which relates the diffusion tensor at different depths and different wind speeds. The validity of this relationship has been tested by calculating the diffusion tensor explicitly at different values of U and z and comparing them to the values obtained using the scaling relationship. Of the three cases considered, buoyant particles on the surface, neutrally buoyant particles suspended in the sea and particles dropping through the sea under gravity, particles on the sea surface were subject to the greatest dispersion. Particles suspended (at a constant depth) beneath the surface can also be dispersed by a significant amount if they are close to the surface (within the top 10 m for the values considered here). Particles falling through the ocean are only subject to significant dispersion, due to surface waves, during the time they are in the top 0-10 m of the ocean. Since they are only in this region for a limited time any dispersion is also limited. Below this region particles are virtually unaffected by the surface waves, however, other influences such as currents (Sanderson and Pal 1990) and internal waves (Sanderson and Okumbo 1988) may produce dispersion.

Acknowledgements

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