Vorticity and surf zone currents

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Abstract

Water wave breaking is of considerable importance in the transfer of momentum from the waves to currents. Near shore lines most of the water motions are dominated by breaking waves. Recent work on the generation of vorticity by breaking waves and bores in the surf zone on beaches within the shallow water approximation has shown that non-uniformity in the strength of bores is an important source of potential vorticity. This is illustrated by discussing the generation of longshore currents and illustrated with a numerical example of vorticity generation due to a non-uniform bed. These demonstrate that the horizontal excursion of vorticity transported by the incident waves may be a significant factor in interpreting velocity measurements at a fixed site.

Introduction

The role of horizontal eddies in surf zone currents, and the generation of their vorticity are discussed in Peregrine (1995, 1998). The latter paper gives a quantitative measure of vorticity generation by bores, which is briefly recounted below. The aim of this paper is to look at some of the implications of this vorticity generation.

Note: the vorticity that is being discussed here is not the vorticity caused directly by the breaking of the wave and the subsequent organised and turbulent motions on the scale of

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the wave crest. Rather, we are concerned with vorticity on a larger scale, at the scale of the crest length or wave length of the wave, such as is important in describing mean currents generated by breakers.

The usual way to model the transfer of momentum from waves to currents is to average over the wave motion, assuming the waves to be sufficiently regular for this to be appropriate. The resulting equations have a momentum transfer term, that is known as radiation stress following its development by Longuet-Higgins & Stewart (1964). These averaged equations include mean currents and mean pressures: a standard way of studying fluid motion. However, an alternative way of analysing any flow is to consider its vorticity. Peregrine (1995) points out that surf zone flows on beaches of gentle slope are almost two-dimensional flows in the horizontal, and that little attention had been given to the properties of such flows such as are used in the geophysical fluid dynamics for atmospheres and oceans. This has been followed up by numerical modellers including vorticity plots in their results which have clearly shown discrete eddies even when radiation stress is used to drive the currents (Allen, Newberger & Holman,1996; Özkan-Haller & Kirby, 1998; Sancho & Svendsen, 1998).

After summarising the main result on vorticity generation from Peregrine (1998) and noting the importance of potential vorticity in these flows, we present a comparison between the usual radiation stress description of the generation of longshore currents and the view that is obtained by considering potential vorticity generation. Then data from a numerical computation of a normally incident wave meeting a non-uniform bar is shown to generate eddies. The accuracy of this inviscid computation is verified by the maintenance of their potential vorticity as they are convected by the wave motion.

Vorticity generation by bores in shallow water

The simplest dynamic model for waves breaking on a beach of gentle slope is for shallow water. The equations for finite amplitude shallow water waves, e.g. see Stoker (1957), permit waves to steepen until the necessary approximation of gentle surface slopes is no longer valid. In practice, if the variation of surface elevation is large enough compared with the depth, waves break shortly after they steepen significantly. Thus, if details of the breaking and the associated turbulent motions are on shorter length and time scales than are of immediate interest, the breaking event can be modelled as the development of a surface and current discontinuity in the shallow water equations. Such discontinuities are bores and are dynamically consistent if mass and momentum are conserved.

For the development of wave generated currents it is these longer scales that are most relevant. Modelling of waves on a beach with the shallow water equations plus bores has developed from early numerical models (Keller, Levine & Whitham, 1960; Hibberd & Peregrine; 1979) to more effective examples (e.g. Kobayashi & Wurjanto, 1992; Watson, Peregrine & Toro, 1992). These examples are of one-dimensional models, and only now are models with two horizontal dimensions which include bores being used in this area. However, comparisons with one-dimensional experiments described in Barnes, Peregrine & Watson (1994) show that although fine details of wave breaking are poorly modelled the overall generation of currents is well described.

Kelvin's circulation theorem can be derived for the shallow water equations, as can the conservation of potential vorticity following material particles. Potential vorticity is defined by (vorticity)/(total waterdepth). Note that conservation of potential vorticity implies a change in vorticity for any water that changes its depth. This is a dynamically important feature and generates or absorbs vorticity, but the major point for discussion here is the generation of potential vorticity.

Derivation of Kelvin's circulation theorem and of the material conservation of potential vorticity from the shallow water equations requires that the flows be represented by continuous and differentiable functions. The development of bores introduces a complicating feature: the discontinuity of velocity and depth that represents a bore gives a rate of change of circulation in any material circuit that cuts through the bore unless it has another section through the bore in the opposite direction at a point where the bore has the same properties. It is clear that velocity, and hence circulation along a material line, is changing most rapidly at bores. By considering the effect of a bore on a material circuit over a small time interval Peregrine (1998) derives the rate of change of circulation due to one intersection with the bore. It turns out to have a magnitude equal to the rate of energy dissipation in the bore at that point, divided by the water density. This neat result has the nice feature that to a large extent the rate of dissipation is related to the visible strength of a bore in terms of the intensity of splashing and air entrainment.

However, we need to discuss the large-scale generation of potential vorticity. Peregrine (1998) derives a formula for the generation of vorticity by considering the change in vertical vorticity, Ω, that occurs in an infinitesimal material circuit as it passes through a bore. The result, at a bore that has an increase in water depth from h_1 to h_2, is $\Delta\Omega_A + \Delta\Omega_D$. Here $\Delta\Omega_A$ is the change in vorticity due to the vertical stretching of fluid elements:

$$\Delta\Omega_A = \left(\frac{h_2}{h_1} - 1\right)\Omega,$$

where Ω is the original vorticity of the fluid element. This part represents no change in the pre-existing potential vorticity. On the other hand, the motion of the bore over running water in front of it gives the change of vorticity

$$\Delta\Omega_D = \left[\frac{2h_2}{gh_1(h_1+h_2)}\right]^{\frac{1}{2}} \frac{dE_D}{dy}, \quad \text{where} \quad E_D = \frac{g(h_2-h_1)^3}{4h_1h_2}$$

is the dissipation rate at the bore divided by fluid density. This does represent generation of potential vorticity:

$$\left[\frac{2}{gh_1h_2(h_1+h_2)^3}\right]^{\frac{1}{2}}\frac{dE_D}{dy}.$$

Pratt (1983) also derived this result from manipulation of the shallow water equations and bore relations.

As may be seen, a bore with uniform properties, such that E_D is constant, conserves potential vorticity. On the other hand if the bore varies due to non-uniformities in E_D, then new potential vorticity is generated. Non-uniformities in E_D come from variations in the water depths h_1 and h_2, either because of non-uniformities in the bed, or because of variations along the wave crest. The effect of bed non-uniformities is discussed here. Pcregrine (1999) discusses the effect of crest variations for finite length breaking wave crests.

Longshore currents

Figure 1 is a sketch of regular waves incident at an angle on a gently sloping beach which is uniform in the longshore direction; the slightly irregular lines indicate the bores and runup. The x-direction is on shore, and the y-direction along shore. The standard approach to considering the generation of longshore currents, from Longuet-Higgins (1970), is to average over the waves and consider the momentum flux or radiation stress. A short straight line is drawn to illustrate this. Through such a shore-parallel surface the flux of x momentum carried shorewards by the waves, S_{xx}, is reducing towards the shore as the bores dissipate wave energy. The resulting gradient of momentum flux is balanced, in the simplest case by a correponding gradient of the mean surface giving a set up towards the shoreline.

On the other hand, the flux of y momentum carried shorewards by the waves, S_{yx}, is also reducing, but with no pressure gradient to balance it. Since the momentum must be conserved it thus contributes to a longshore current. In fact a balance is reached, either by direct bed friction, or, more likely, by the horizontal eddy viscosity which is enhanced by the long life of almost two-dimensional eddies which are generated by instabilities in the flow, bed irregularities, and/or irregularities in the incident waves.

The vorticity view of the longshore current generation depends on vorticity generation. Again this results from the shoreward diminution of the waves, but now we focus on the bores. Each bore starts abruptly as a wave breaks so that it is at its strongest when created and decays to zero where it meets the shoreline. Thus E_D is continually diminishing shorewards and its gradient leads to vorticity generation as described above. This vorticity is the gradient of the longshore current profile.

The most interesting aspect of this viewpoint comes when we look at the offshore end of the bore where it is created by the initial breaking of a wave crest. Here the bore

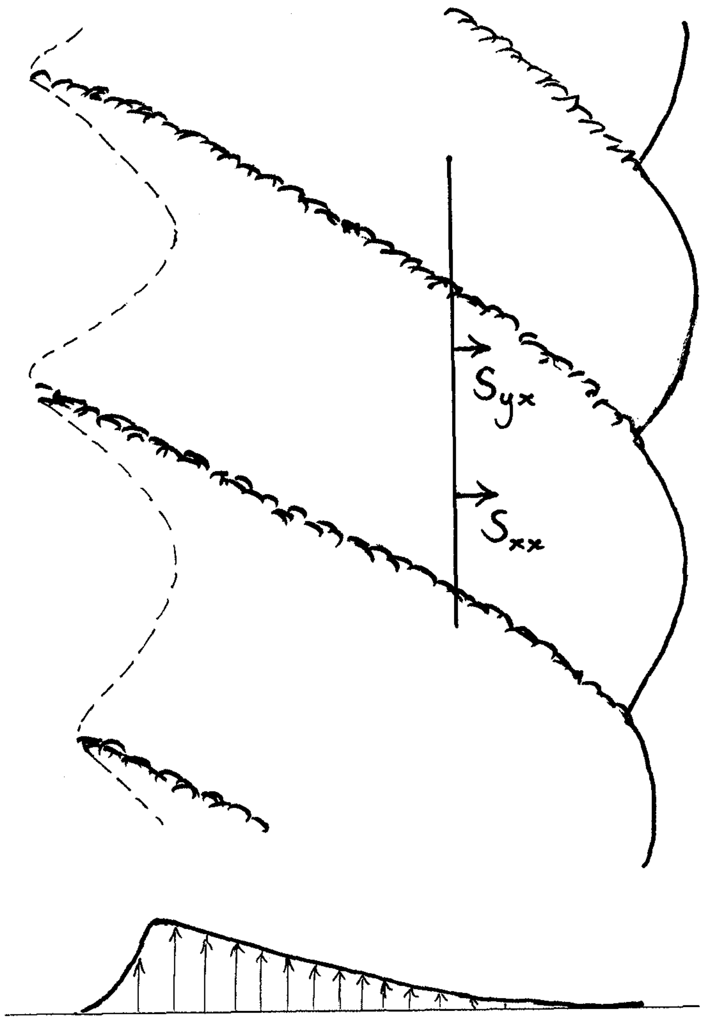

Figure 1. Thc irregular heavy lines sketch the positions of bores due to regular obliquely incident waves approaching a shoreline on the right. The flux of both x- and y-momentum past a shore-parallel line is indicated by arrows. The dashed line is an estimate of the edge of the region of vorticity as it is displaced by the wave motion. The lower profile is a sketch of a longshore current profile in the absence of horizontal mixing.

dissipation has its greatest value and also its greatest gradient, that is from zero offshore from breaking up to its maximum at breaking. This results in generation of vorticity of the opposite sense to that of the rest of the surf zone, as is needed for the longshore current to diminish to zero offshore. This draws attention to the localised generation of vorticity, and hence to a feature that appears to have been overlooked in discussions about the offshore profile of longshore currents. As may be seen in the computational example described in the next section, any vorticity that is generated is convected with the water motions. In particular the horizontal displacements of the incident wave motion are not generally negligible.

As an example consider linear long wave theory on water of constant depth h. For a sinusoidal wave of height H, the total horizontal displacement of the water is $HL/2\pi h$, where L is the wavelength. Note L/h is large for long waves, and H is not small when a wave breaks. The trajectory of the vorticity due to this displacement is sketched with a dashed line in figure 1. Thus any velocity sensor at a fixed position near the breaking point is affected by the vorticity for only part of the wave period. This means that even for perfectly regular waves with no horizontal eddy viscosity, the measured velocity profile could not show the very steep reduction to zero current offshore that the wave averaged approach predicts. This feature of the problem has yet to be studied quantitatively, but a sketch of the velocity profile that would be measured is included in figure 1.

Eddies generated by a non-uniform bar

Vorticity can be generated at a bore because the dissipation, E_D , varies due to variation in the level of the bed. As an illustration we present results from computation of a uniform wave incident normally on a plane beach with a bar. Bed non-uniformity is introduced by having a dip in the elevation of the bar. The contours of the bed and the numerical domain of integration are shown in figure 2. The domain is taken to be periodic in the long-shore direction. A single large wave of elevation propagating towards the shore constitutes the initial condition.

The wave and current motions are modelled by the finite-amplitude shallow-water equations in the dimensionless form:

$$u_t + uu_x + vu_y + h_x = d_x,$$

$$v_t + uv_x + vv_y + h_y = d_y,$$

$$h_t + (hu)_x + (hv)_y = 0.$$

where the coordinates (x, y, t), depth to bed, $d(x,y)$, total water depth, $h(x,y,t)$ and horizontal velocity components, (u,v) are related to the corresponding dimensional variables, starred, by

$$x = x^* / D,\ y = y^* / D,\ t = t^* (\alpha g / D)^{1/2},\ d = d^* / \alpha D,\ h = h^* / D,\ u = u^* /(\alpha g D)^{1/2},\ v = v^* /(\alpha g D)^{1/2}.$$

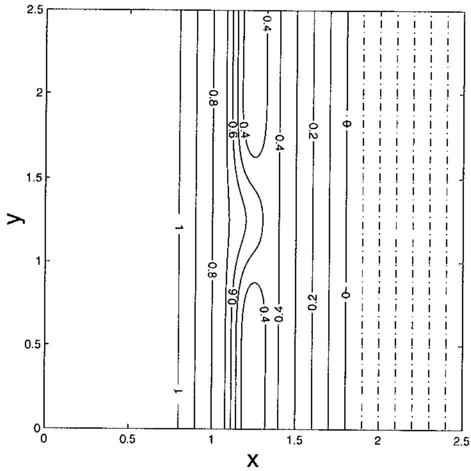

Figure 2: Contour plot of the bed, $d(x,y)$. Those above the initial still water level are shown with dashed lines. Contour interval: 0.1.

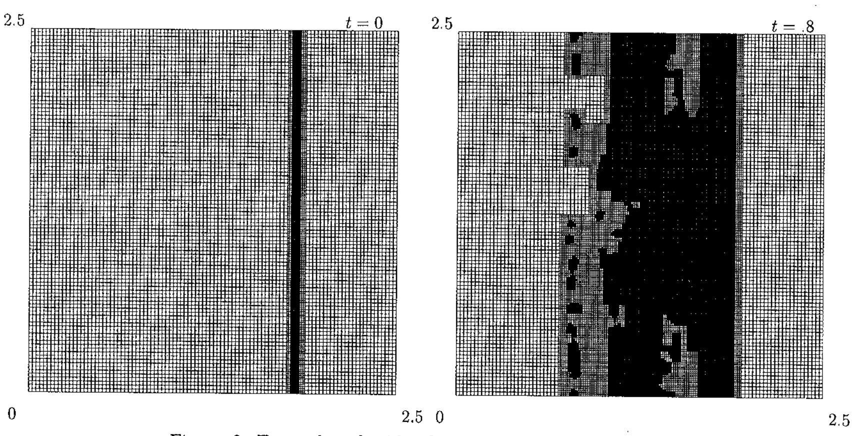

Figure 3: Examples of grid refinement at $t = 0$ and $t = 0.8$.

Here D and α are a reference depth and beach slope respectively: e.g. for a beach of constant slope rising from a horizontal bed, D could be chosen equal to the depth of undisturbed water over the horizontal bed, and α equal to the bed slope. This distortion of variables is used to scale out the beach slope, as is possible when no friction terms are included, see Brocchini & Peregrine (1996) or Hibberd & Peregrine (1979) where a slightly different scaling has the same effect.

The computational domain has a bed that is horizontal and constant unit depth for $x < 0.8$, at which point the bed rises with x with a slope 1. The bar is between $x = 1.05$ and 1.383, formed by adding a the positive sinusoid of amplitude 0.125. The dip is formed by multiplying the bar height by a suitable factor varying in the alongshore direction. Thus the bed is given by:

$$d = 1-(x-0.8)-0.125f(y)\{1+\sin[6\pi(x-1.05)-\pi/2]\}$$

where $$f(y) = 1+\cos(2\pi y/L)\left[\exp\{-(y-L/2)^2/a^2\}+\exp\{-(y-3L/2)^2/a^2\}\right].$$

The width of the domain is $L = 2.5$, and $a = 0.3$. A plot of bed contours is given in figure 2.

The initial conditions are to have water at rest except over the horizontal bed, $0 < x < 0.8$ where

$$u(x,0) = 0.3\,[1+\sin(2\pi x/0.8 - \pi/2)]$$
$$h(x,0) = \tfrac{1}{4}u(x,0)^2 + u(x,0).$$

The relationship between u and h is chosen from the properties of simple unidirectional waves to give only onshore propagation initially.

The numerical scheme used is one due to Quirk, "AMRITA", which has adaptive mesh refinement, with a Roe type solver, conserving mass and momentum, and a total variation diminishing scheme that can accurately represent the initiation and propagation of the discontinuities in the solutions that represent bores. It is very similar to the gas dynamic code incorporated in the same program package that is described in Quirk & Karni (1996). The computation reported here includes no dissipation other than that which occurs at bores. The mesh refinement criterion are specified by the user. Refinement is clearly needed at bores, but our preliminary studies of accuracy showed that it is also needed at the shoreline and in regions with vorticity. An important feature for checking is that potential vorticity is conserved following a fluid particle. Figure 3 shows the numerical mesh at the initial time, where only the shore line has refinement since we start with a smooth wave, and as the wave meets the shoreline where much refinement is needed for the more complex flow.

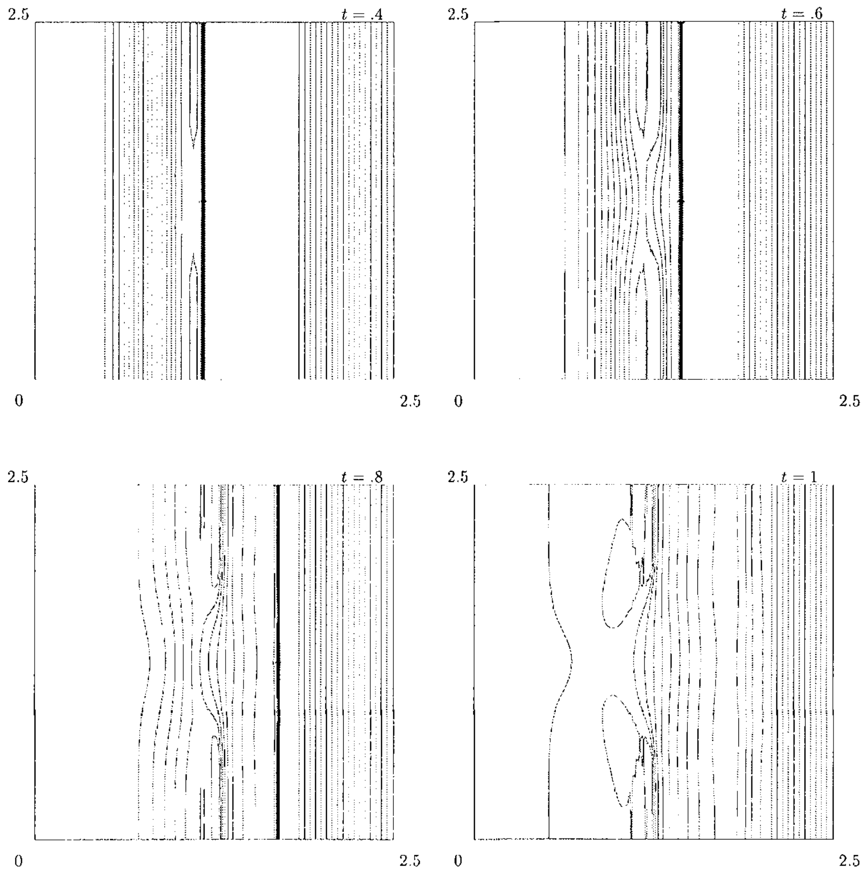

Figure 4: Contours of water surface height, $h - d$, with 25 contours in each of the intervals $[-1.0, -0.04]$ and $[0.04, 1.0]$. Contours of the exposed beach are also included.

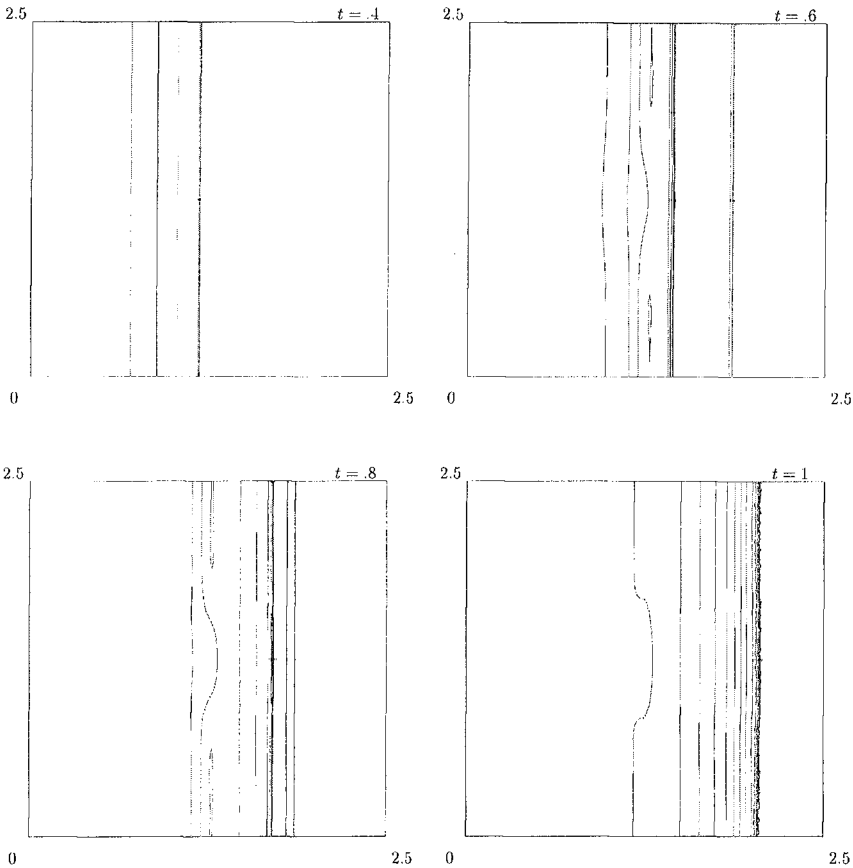

Figure 5: Onshore velocity component, u, with 12 contours in $[-2.5, -0.2]$ and $[0.2, 2.5]$.

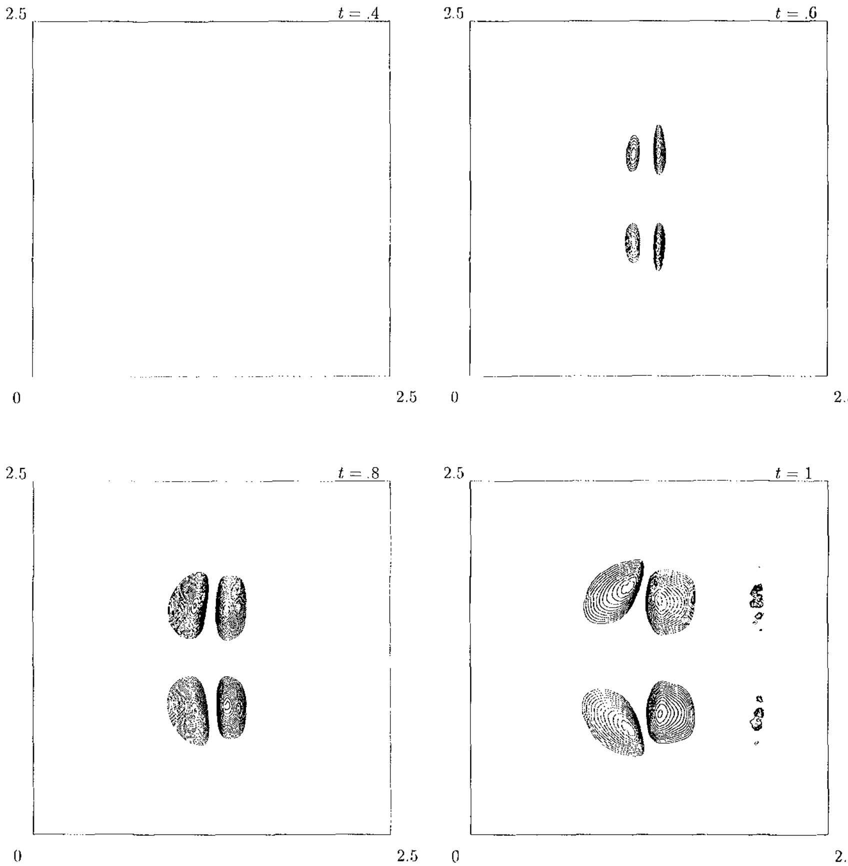

Figure 6: Alongshore velocity component, v, with 20 contours in $[-0.03, -0.015]$ and $[0.015, 0.03]$.

The primary results are shown as contour plots of the surface elevation, and the onshore and alongshore velocity components in figures 4, 5 and 6. As may be seen the bore, the following wave's elevation and the onshore water velocity all become lower over the dip in the bar than they are elsewhere. The alongshore velocity component would be zero if the bar were uniform, so its values give a clear view of the disturbance caused by the dip in the bar.

Potential vorticity at four different times is displayed in figure 7. Since vorticity is obtained from the primary variables by differentiation an increase in numerical noise is likely, and occurs. We have not smoothed the results, some of the noise is closely related to the steep front of the bore and although it mainly translates with the bore some small mesh-scale disturbances appear to be left behind. However, the generated vorticity shows up clearly. In the first frame, at $t = 0.4$, the bore is in the process of passing over the bar and the effect of the dip in generating vorticity at its sides where E_D has a significant gradient is clear. Only part of the vorticity generation has occurred at this time. Note figure 4 shows little discernible deviation in the line of the bore front at any time despite the gradient of its strength induced by the dip in the bar.

By time $t = 0.6$ the bore has passed over the bar and two narrow regions of vorticity have been generated. The bore's position can be seen by the thin spurious contour line just in front of these new eddies. The subsequent frames at later times show how the vorticity is convected shorewards by the forward motion of the water in the wave. The area over which the vorticity is spread increases substantially. This is because the vorticity moves with the water into a shallower region where the water has to spread out. As can be seen the values of the potential vorticity appear to be well conserved, e.g. the peak value hardly changes.

Conclusion

The interpretation of surf zone currents and their generation by considerations of vorticity and potential vorticity are illustrated and discussed here. They give a picture of the currents which is more directly related to the dissipative structure in surf zone waves where almost all the dissipation occurs in breaking and bores. We demonstrate the generation of discrete eddies by relatively small non-uniformities in the bed profile. Also evident in the numerical solutions is a large horizontal convection of the eddies by the wave motion following the bore. This horizontal motion is also relevant to the interpretation of the generation of longshore current profiles by regular waves, and appears to be a matter that should be borne in mind when interpreting velocity measurements from fixed velocity sensors.

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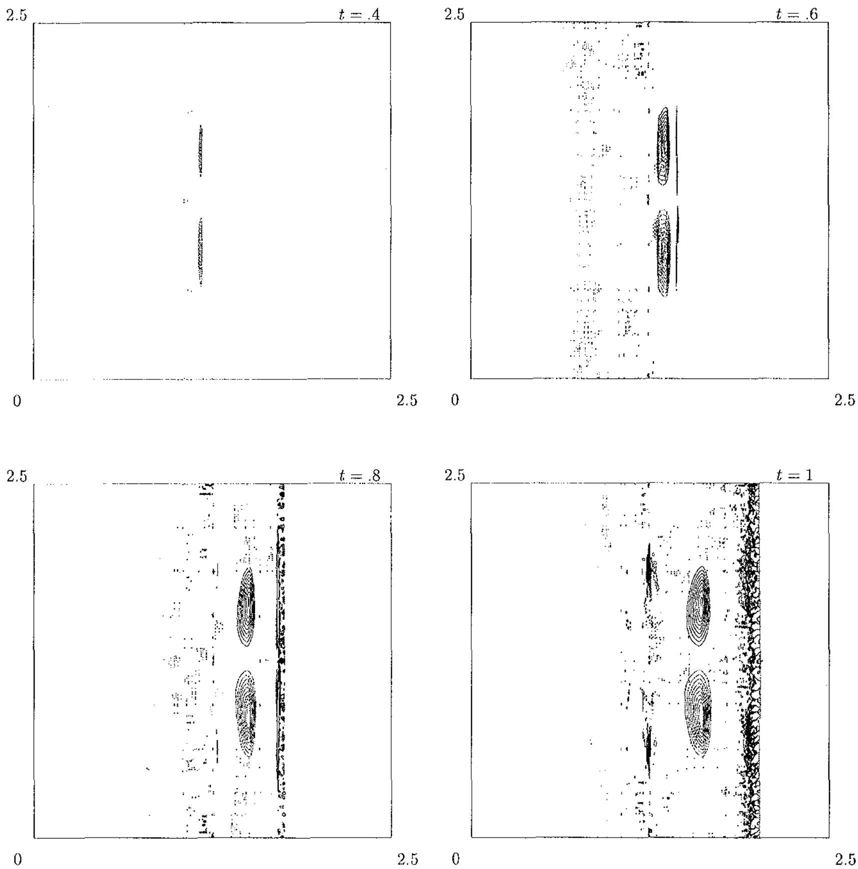

Figure 7: Contours of potential vorticity with 9 contours in $[-0.3, -0.06]$ and $[0.06, 0.3]$.

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