NUMERICAL SIMULATIONS OF DIRECTIONALLY SPREAD SHOALING SURFACE GRAVITY WAVES

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ABSTRACT

The study aims at investigating the non-linear triad interaction process affecting shoaling surface gravity wave fields in shallow water areas. Attention is specifically paid to analyse the effects of these second-order non-linearities on the directional distribution of incident waves. A stochastic approach was chosen to model the triad interaction process. Three source terms have been implemented in the spectral wave model TOMAWAC developed at the Laboratoire National d'Hydraulique (LNH). The model results are compared to the laboratory data of Nwogu (1994).

1. INTRODUCTION:

This work focusses to the study of the non-linear interactions between triplets of waves which occur in the nearshore zone. These non-linearities, of lower order than the four wave interactions, are usually considered as the main exchange mechanism for wave energy of an irregular wave train towards sub- (long waves) and super-harmonics (bound higher order waves) (e.g. Freilich and Guza, 1984). The strength of these near-resonant interactions is governed by the phase mismatch between the bound and free wavenumbers. As for decreasing water depth the waves tend to become non-dispersive, the phenomenon is enhanced towards the shore. The effect of the transfer of energy associated with near-resonant wave interactions is not only distortions of the frequency spectrum, but also modifications of the directional spreading of energy. Elgar et al. (1993) observed that directionally bi-modal wave spectra can give rise to a new directional peak. This implies that both collinear and non-collinear triad interactions are important.

Boussinesq equations have been extensively used to establish evolution equations for the amplitudes and phases of unidirectional waves propagating over a mildly sloping bottom. The models (e.g. Freilich and Guza (1984), Madsen and Sorensen (1993)) are able to correctly reproduce the generation of harmonics. The quality of the results incited us to extend the deterministic model of Madsen and Sorensen (1993) to bidimensional situations. The model is presented in part 2.

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Alternatively, stochastic (phase-averaged) models are more useful to predict the evolution of directional wave spectra. The shallow water three-waves interaction process is simulated by including a source term in the energy balance equation. From the works of Zakharov et al. (1992), Eldeberky et al. (1996) proposed a directionally coupled source term. Some misbehaviours of this non-linear model were pointed out by the authors. We studied more particularly the problem of energy conservation and the ability to generate new directional peaks (which was not demonstrated in Eldeberky, 1996). For practical applications, Eldeberky et al. (1996) recommended the use of the parametrized LTA (Lumped Triad Approximation) model proposed by Eldeberky and Battjes (1995) and Eldeberky (1996). This model has been applied in the wave propagation model SWAN developed at Delft University of Technology (Ris, 1997). The LTA model is characterised by the restriction to self-self interactions leading to a great computational efficiency. From the 2D spectral deterministic equation presented in §2, we developed a new stochastic model. To solve the classical problem of closure, we used the works of Holloway (1980) who assumes that the sum over fourth-order moments draws some contribution from third-order moments through a coupling coefficient interpreted as a broadening of the resonance condition. The three formulations for phase-averaged models are presented in part 3. They are implemented in the spectral wave model TOMAWAC developed at the LNH (Benoit et al., 1996) and they have been compared to the experimental laboratory results presented by Nwogu (1994) (§4).

2. DETERMINISTIC APPROACH:

Deterministic Boussinesq (SB) model:

Starting from their extended Boussinesq equations, Madsen and Sorensen (1993) derived a set of deterministic evolution equations for the amplitudes and phases of unidirectional waves propagating over a mildly sloping bottom. We extended the model to bidimensional situations by use of the 2D Boussinesq equations of Madsen and Sorensen (1992):

$$\frac{\partial \zeta}{\partial t} + \vec{\nabla}.\vec{P} = 0 \tag{1}$$

$$\begin{aligned}
&\frac{\partial \vec{P}}{\partial t} + \left[\frac{\vec{P}}{h+\zeta}\left(\vec{\nabla}.\vec{P}\right) + \left(\vec{P}.\vec{\nabla}\right)\frac{\vec{P}}{h+\zeta}\right] + g(h+\zeta)\vec{\nabla}\zeta - \left(B+\frac{1}{3}\right)h^2\vec{\nabla}\left(\vec{\nabla}.\frac{\partial \vec{P}}{\partial t}\right) \\
&\quad -\frac{h}{6}\left(\vec{\nabla}.\frac{\partial \vec{P}}{\partial t}\right)\vec{\nabla}h - \frac{h}{6}\left(\vec{\nabla}h.\vec{\nabla}\right)\left(\frac{\partial \vec{P}}{\partial t}\right) - \frac{h}{6}\vec{\nabla}h \otimes \left(\vec{\nabla}\otimes\frac{\partial \vec{P}}{\partial t}\right) \\
&\quad -Bgh^2\left(\vec{\nabla}h.\vec{\nabla}\right)\left(\vec{\nabla}\zeta\right) - Bgh^2\left(\vec{\nabla}.\vec{\nabla}\zeta\right)\vec{\nabla}h - Bgh^3\vec{\nabla}\left(\vec{\nabla}.\vec{\nabla}\zeta\right) = 0
\end{aligned} \tag{2}$$

where ζ is the free surface elevation, $\vec{P}$ is the depth-integrated velocity and h is the water depth. These equations include a linear parameter B which improves the dispersion properties and the shoaling mechanisms associated with the model. The optimal value $B=1/15$ was obtained by the authors by a fit to the reference linear shoaling coefficient predicted by the Stokes first-order theory. The standard form of the Boussinesq equations can be recovered by setting $B=0$.

The extended Boussinesq equations are combined into one single equation. For this, the partial derivative in time of the continuity equation is substracted to the partial derivative in space of the momentum equation. This leads to:

$$L = M + N \tag{3}$$

where:

$$L = \zeta_{tt} - gh\vec{\nabla}.\vec{\nabla}\zeta + Bgh^3\vec{\nabla}.\left(\vec{\nabla}\left(\vec{\nabla}.\vec{\nabla}\zeta\right)\right) - \left(B + \frac{1}{3}\right)h^2\vec{\nabla}.\left(\vec{\nabla}\zeta_{tt}\right) \tag{4}$$

$$M = g\vec{\nabla}h.\vec{\nabla}\zeta + (2B+1)h\vec{\nabla}h.\vec{\nabla}\zeta_{tt} - 5Bgh^2\vec{\nabla}h.\vec{\nabla}\left(\vec{\nabla}.\vec{\nabla}\zeta\right) + \frac{h}{6}\vec{\nabla}h.\left(\vec{\nabla}\otimes\left(\vec{\nabla}\otimes\vec{P}_t\right)\right) \tag{5}$$

$$N = \vec{\nabla}.\left[\frac{\vec{P}}{h+\zeta}\left(\vec{\nabla}.\vec{P}\right) + \left(\vec{P}.\vec{\nabla}\right)\frac{\vec{P}}{h+\zeta}\right] + g\vec{\nabla}\zeta.\vec{\nabla}\zeta + g\zeta\vec{\nabla}.\vec{\nabla}\zeta \tag{6}$$

L contains the lowest order and dispersive terms, M represents the effects of slowly varying depths on the wave propagation and N models the non-linear effects due to wave coupling. Now, the equation is transformed into the spectral domain. ζ is written as a discrete Fourier series:

$$\zeta(x,y,t) = \sum_{p=-\infty}^{\infty} A_p(x,y)\exp\left[i\left(\omega_p t - \psi_p(x,y)\right)\right] \tag{7}$$

in which p refers to a directional wave component. A_p is the complex Fourier amplitude ($A_{-p} = A_p^*$), ω_p is the angular frequency ($\omega_{-p} = -\omega_p$) and ψ_p is the linear wave phase, linked to the wavenumber by the relation $\nabla\psi_p = \vec{k}_p(x,y)$. $\vec{P}$ is linked to ζ through the continuity equation, so:

$$\vec{P}(x,y,t) \approx \sum_{p=-\infty}^{\infty} \vec{c}_p A_p(x,y)\, e^{\left[i\left(\omega_p t - \psi_p(x,y)\right)\right]} \tag{8}$$

where $\vec{c}$ is the wave phase celerity. The Fourier expressions of ζ and $\vec{P}$ are inserted in the equation (3). We follow for this the guideline presented by Madsen and Sorensen (1993) for 1D situations. First derivatives of A_p, k_p and h are assumed to be small, and products of derivatives and higher derivatives of these quantities are neglected. After algebraic manipulations, we get,

- at lowest order, the linear dispersion relation:

$$\frac{\omega_p^2 h}{g} = \frac{\left(k_p h\right)^2 + B\left(k_p h\right)^4}{1 + \left(B + \frac{1}{3}\right)\left(k_p h\right)^2} \tag{9}$$

- at second order, the coupled differential equations:

$$\frac{\partial A_p}{\partial t} + \vec{C}g_p.\vec{\nabla}A_p + \frac{1}{2}\frac{Cg_p}{k_p}A_p\vec{\nabla}.\vec{k}_p + \frac{S_{3,p} - S_{4,p}}{2S_{1,p}}A_p = \frac{ig}{2S_{1,p}}N_p \tag{10}$$

where $\vec{C}g$ is the group velocity:

$$\vec{C}g_p = \frac{S_{2,p}}{S_{1,p}}\vec{k}_p \tag{11}$$

$$S_{1,p} = \omega_p\left[1 + (B + 1/3)\, h^2 k_p^2\right] \tag{12}$$

$$S_{2,p}=\left[gh+2Bgh^3k_p^2-\left(B+\frac{1}{3}\right)\omega_p^2h^2\right] \quad (13)$$

$$S_{3,p}=4Bgh^3\frac{-k_p^2\left[g+3Bgh^2k_p^2-2(B+1/3)h\omega_p^2\right]}{2\left[gh+2Bgh^3k_p^2-(B+1/3)h^2\omega_p^2\right]}\vec{k}_p.\vec{\nabla}h \quad (14)$$

$$S_{4,p}=-(\vec{k}_p.\vec{\nabla}h)\left[g+5Bgh^2k_p^2-(2B+1)\omega_p^2h\right] \quad (15)$$

$$N_p=\sum_{m=1}^{\infty}2R_{-m,p+m}\,A_m^*A_{p+m}\,e^{-i(-\psi_p-\psi_m+\psi_{p+m})}+\sum_{m=1}^{p-1}R_{p-m,m}\,A_mA_{p-m}\,e^{-i(-\psi_p+\psi_m+\psi_{p-m})} \quad (16)$$

The left hand side of (10) represents the propagation terms, whereas the right hand side governs the non-linear triads interactions. N models the exchange of energy towards sub and super-harmonics through the coupling coefficient, R, which controls the strength of the interactions. Its expression is given by:

$$R_{p,m}=\left[\frac{1}{2}\left(k_p^2+k_m^2+2\vec{k}_p.\vec{k}_m\right)+\frac{1}{gh}\left(\frac{\omega_p\omega_m}{\left(k_pk_m\right)^2}\left[\left(k_p^2+k_m^2\right)\vec{k}_p.\vec{k}_m+\left(\left(k_pk_m\right)^2+\left(\vec{k}_p.\vec{k}_m\right)^2\right)\right]\right)\right] \quad (17)$$

3. STOCHASTIC APPROACHES:

Stochastic Parametrized Zakharov (SPZ) model:

The model proposed by Eldeberky et al. (1996) is based on the so-called Zakharov equation for resonant three-wave interactions (Zakharov, 1968; Zakharov et al., 1992):

$$\frac{\partial a_{\vec{k}_3}}{\partial t}=-i\omega_{\vec{k}_3}a_{\vec{k}_3}-i\int\int\left[R_{312}a_{\vec{k}_1}a_{\vec{k}_2}\delta_{3-1-2}+2R_{132}a_{\vec{k}_1}a_{\vec{k}_2}^*\delta_{1-3-2}\right]d\vec{k}_1d\vec{k}_2 \quad (18)$$

δ_{3-1-2} is shorthand for $\delta\left(\vec{k}_3-\vec{k}_1-\vec{k}_2\right)$ where δ stands for the delta Kronecker symbol. R_{312} is a coupling coefficient depending on the physics of the interacting waves. For surface gravity waves in intermediate water depth, the expression is given by (Stiassnie and Shemer, 1984; Eldeberky et al., 1996):

$$\begin{aligned}R_{312}=\frac{g^{1/2}}{8\pi\sqrt{2}}\Big\{&\left[\vec{k}_3.\vec{k}_1-(\omega_3\omega_1/g^2)\right](\omega_2/\omega_1\omega_3)^{1/2}\\&+\left[\vec{k}_3.\vec{k}_2-(\omega_3\omega_2/g^2)\right](\omega_1/\omega_2\omega_3)^{1/2}\\&+\left[\vec{k}_1.\vec{k}_2+(\omega_2\omega_1/g^2)\right](\omega_3/\omega_1\omega_2)^{1/2}\Big\}\end{aligned} \quad (19)$$

A statistical formulation of the three-wave interaction process, expressed in term of the correlation functions of the wave field, can be obtained from (18) (Zakharov et al., 1992). Because of non-linearities, the statistical description consists in a series of interconnected equations where each moment evolves according to the next higher-order moment. The problem is closed by assuming that the fourth-order moments can be expressed as a function of the lower order moments. For resonant wave-wave interactions, Zakharov et al. (1992) used the quasi-gaussian hypothesis. A broadening of the resonance condition is included in the model by the assumption that the fourth-order cumulant depends on the third-order moment through a coupling coefficient Ω which represents a frequency uncertainty among the three interacting waves (Holloway, 1980). A small (but finite) value for Ω is used (Eldeberky et al., 1996). Defining the wave action density $n_{\vec{k}}$ by:

$$\left\langle a_{\vec{k}} a^*_{\vec{k}'} \right\rangle = 4\,\pi^2\, n_{\vec{k}}\, \delta\left(\vec{k} - \vec{k}'\right) \qquad (20)$$

the wave action evolution equation can be expressed as (Zakharov et al., 1992):

$$\frac{dn_{\vec{k}_3}}{dt} = 4 \int\int d\vec{k}_1 d\vec{k}_2 \left[R^2_{312} N_{312}\, \mu_{3-1-2} \delta_{3-1-2} - 2R^2_{132} N_{132}\, \mu_{3-1+2} \delta_{3-1+2} \right] \qquad (21)$$

where N_{312} depends on the spectral action density functions (n_j) of the interacting waves according to:

$$N_{312} = n_1 n_2 - n_3 n_1 - n_3 n_2 \qquad (22)$$

In Holloway's approach, μ_{3-1-2} acts as a spectral filter through the frequency mismatch ($\omega_3 - \omega_1 - \omega_2$) and through the parameter Ω characterising non-resonant interactions:

$$\mu_{3-1-2} = \frac{\Omega}{\left(\omega_3 - \omega_1 - \omega_2\right)^2 + \Omega^2} \qquad (23)$$

For the resonant conditions: $\vec{k}_3 - \vec{k}_1 - \vec{k}_2 = 0$ and $\omega_3 - \omega_1 - \omega_2 = 0$, the weak wave theory of Zakharov et al. (1992) assumes that $\Omega \to 0$, implying:

$$\mu_{3-1-2} = \pi\, \delta\left(\omega_3 - \omega_1 - \omega_2\right) \qquad (24)$$

Eldeberky et al. (1996) showed by numerical simulations that the spectral source term for triad interactions results in an artificial energy decay/gain. This non-conservative feature was explained by the specification of the filter bandwidth Ω. As done by Hasselmann and Hasselmann (1985) for quadruplets interactions, we exploited the basic symmetry of the triplets interactions to study more accurately the problem of energy conservation. The interaction coefficient R_{312}, as well as N_{312} and μ_{312} are invariant with respect to permutations between waves 1 and 2. So, for the collision:

$$\vec{k}_3 - \vec{k}_1 - \vec{k}_2 = 0 \qquad (25)$$

the changes Δn_i in wave action per unit time for the three wavenumbers can be expressed by:

$$\begin{Bmatrix} \Delta n_{\vec{k}_1} \\ \Delta n_{\vec{k}_2} \\ \Delta n_{\vec{k}_3} \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ +1 \end{Bmatrix} 4 R^2_{312}\, N_{312}\, \mu_{312}\, \delta\left(\vec{k}_3 - \vec{k}_2 - \vec{k}_1\right) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \qquad (26)$$

leading to:

$$\begin{Bmatrix} \Delta F_{\vec{k}_1} \\ \Delta F_{\vec{k}_2} \\ \Delta F_{\vec{k}_3} \end{Bmatrix} = \begin{Bmatrix} -f_1 \\ -f_2 \\ +f_3 \end{Bmatrix} 8\pi\, R^2_{312}\, N_{312}\, \mu_{312}\, \delta\left(\vec{k}_3 - \vec{k}_2 - \vec{k}_1\right) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 \qquad (27)$$

(26) and (27) show that, for resonant three wave interactions $f_3 = f_1 + f_2$, only the conservation of energy applies, as expected (Komen et al., 1994). Non-conservation of energy occurs for near-resonant interactions when the linear dispersion relationship is used to link the wavenumber to the frequency since $f^{lin}(\vec{k}_3)<f^{lin}(\vec{k}_1)+f^{lin}(\vec{k}_2)$. The outcome is a decay of energy during the generation of bound super-harmonics and an increase in energy during the generation of bound sub-harmonics.

For its implementation in a spectral wave model, the SPZ source term is expressed in terms of frequency-directional variance density function $F(f,\theta)$ (Eldeberky et al., 1996):

$$S(\omega_3,\theta_3)=16\pi^2 g\int_0^{\infty}\int_0^{2\pi}\frac{c_2\,Cg_2}{\omega_1\omega_2^2}\left(T_{312}-2T_{132}\right)d\omega_1 d\theta_1 \qquad (28)$$

where

$$T_{312}=R_{312}^2\,\mu_{312}\left[\frac{\omega_3}{Cg_3}F_1F_2-\frac{\omega_1}{Cg_1}F_3F_2-\frac{\omega_2}{Cg_2}F_1F_3\right] \qquad (29)$$

T_{312} and T_{132} represent the sum ($\vec{k}_3=\vec{k}_1+\vec{k}_2$) and difference ($\vec{k}_3=\vec{k}_1-\vec{k}_2$) interactions. The formulation is directionally coupled and allows for both collinear and non-collinear interactions.

<u>LTA model:</u>

The parametrized LTA model proposed by Eldeberky and Battjes (1995) and Eldeberky (1996) is based on the evolution equation for complex Fourier amplitudes presented by Madsen and Sorensen (1993). The statistical nature of a wave field allows to transform the equation in term of discrete spectrum of energy. The result is a set of interconnected equations where each moment evolves in terms of the next higher order moment. Assumptions are made on the third-order moment (bispectrum) to close the system. The magnitude of bispectrum (bicoherence) is expressed only in terms of second-order moment (quasi-gaussian hypothesis) whereas a parametrical formulation is given for the phase β (biphase). In order to reduce the computational effort, the triad interactions phenomenon is restricted to self-self interactions. Finally, the net source term is given by (Eldeberky, 1996):

$$\begin{aligned}
&S_{nl}(f_p)=S_{nl}^{+}(f_p)+S_{nl}^{-}(f_p)\\
&S_{nl}^{+}(f_p)=\alpha\, c_p\, Cg_p\, R_{(p/2,p/2)}^2 \sin\left|\beta_{p/2,p/2}\right|\left[F^2(f_{p/2})-2F(f_p)F(f_{p/2})\right] \qquad (30)\\
&S_{nl}^{-}(f_p)=-2\,S_{nl}^{+}(f_{2p})
\end{aligned}$$

$S_{nl}^{\pm}$ represents the positive and negative contributions to the self-interactions; c_p and Cg_p are the phase and group velocities for the p harmonic, R is the sum interaction coefficient derived from the deterministic evolution equation and α is a tuning parameter.

<u>Stochastic Parametrized Boussinesq (SPB) model:</u>

By using the statistical closure hypothesis proposed by Holloway (1980), we built up a stochastic model based on the 2D evolution equation for complex Fourier amplitudes presented in §2. We define the complex wave amplitude C_p as:

$$C_p=A_p\,e^{-i\psi_p} \qquad (31)$$

So, the evolution equation for C_p reads, from (10):

$$\frac{\partial C_p}{\partial t}+\vec{C}g_p.\vec{\nabla}C_p+\frac{1}{2}\frac{Cg_p}{k_p}\vec{\nabla}.\vec{k}_p C_p+\frac{S_{3,p}-S_{4,p}}{2S_{1,p}}C_p=-\,i\frac{S_{2,p}}{S_{1,p}}\left(\vec{k}_p.\vec{k}_p\right)C_p + \frac{ig}{2S_{1,p}}\sum_{m=-\infty}^{\infty}R_{m,p-m}C_mC_{p-m} \quad (32)$$

Since:

$$\frac{\partial\left\langle C_pC_p^*\right\rangle}{\partial t}=\left\langle C_p\frac{\partial C_p^*}{\partial t}\right\rangle+\left\langle C_p^*\frac{\partial C_p}{\partial t}\right\rangle \quad (33)$$

and,

$$\vec{\nabla}\left\langle C_pC_p^*\right\rangle=\left\langle C_p\vec{\nabla}C_p^*\right\rangle+\left\langle C_p^*\vec{\nabla}C_p\right\rangle \quad (34)$$

by defining the discrete variance density F_p as:

$$F_p=\left\langle C_pC_p^*\right\rangle \quad (35)$$

the evolution equation for F_p writes:

$$\frac{\partial F_p}{\partial t}+\vec{C}g_p.\vec{\nabla}F_p+\frac{Cg_p}{k_p}\left(\vec{\nabla}.\vec{k}_p\right)F_p+\frac{S_{3,p}-S_{4,p}}{S_{1,p}}F_p=-\frac{g}{S_{1,p}}\sum_{m=-\infty}^{\infty}R_{m,p-m}\mathrm{Im}\left(B_{m,p-m}\right) \quad (36)$$

where $B_{m,p-m}=\left\langle C_p^*C_mC_{p-m}\right\rangle$ is the bispectrum. When $B=1/15$, Madsen et Sorensen (1992) have shown that the shoaling term corresponding to their extended Boussinesq equations fits quite satisfactorily the shoaling term of the first order Stokes theory (percentage errors less than 6% for $h/\lambda_0<0.3$). So, we write (36) in the following form:

$$\frac{\partial F_p}{\partial t}+\vec{\nabla}.\left(\vec{C}g_pF_p\right)=-\frac{g}{S_{1,p}}\sum_{m=-\infty}^{\infty}R_{m,p-m}\,\mathrm{Im}\left(B_{m,p-m}\right) \quad (37)$$

where $\vec{C}g$ refers here to the group celerity of the first order Stokes theory.

To derive an expression for the bispectrum, we assume stationary conditions. In the limit $h_x\to 0$, (32) is reduced to:

$$\vec{k}_p.\vec{\nabla}\left(C_p\right)+i\left(\vec{k}_p.\vec{k}_p\right)C_p=\frac{ig}{2S_{2,p}}\sum_{m=-\infty}^{\infty}R_{m,p-m}C_mC_{p-m} \quad (38)$$

The complex amplitudes, C_p, are expanded in a perturbation series with respect to ε:

$$C_p=\varepsilon C_p^{(1)}+\varepsilon^2C_p^{(2)}+\ldots \quad (39)$$

and (38) has a simple steady solution:

$$C_p^{(2)}=\frac{ig}{2S_{2,p}}\sum_{m=-\infty}^{\infty}R_{m,p-m}C_m^{(1)}C_{p-m}^{(1)}\,\Delta\left(\vec{k},\vec{x}\right) \quad (40)$$

with $\Delta\left(\vec{k},\vec{x}\right)=\dfrac{1-e^{-i\int\vec{k}.dx}}{i\left(\vec{k}.\vec{k}_p\right)}$ (41)

and $\vec{k}=\vec{k}_p-\vec{k}_{p-m}-\vec{k}_m$ (42)

At lowest order, the bispectrum vanishes (gaussian wave field):

$$\left\langle C_p^{*(1)} C_{p-m}^{(1)} C_m^{(1)} \right\rangle = 0 \qquad (43)$$

Here, $\left\langle C_p^* C_{p-m} C_m \right\rangle$ is decomposed as:

$$\left\langle C_p^* C_{p-m} C_m \right\rangle = \left\langle C_p^{*(1)} C_{p-m}^{(1)} C_m^{(2)} \right\rangle + \left\langle C_p^{*(1)} C_{p-m}^{(2)} C_m^{(1)} \right\rangle + \left\langle C_p^{*(2)} C_{p-m}^{(1)} C_m^{(1)} \right\rangle \qquad (44)$$

Substitution of the steady contribution of (40) into (44) leads to:

$$\begin{aligned}\left\langle C_p^* C_{p-m} C_m \right\rangle = & \frac{ig}{2S_{2,m} k_m} \sum_{q=-\infty}^{\infty} R_{m-q,q} \left\langle C_q^{(1)} C_{m-q}^{(1)} C_p^{*(1)} C_{p-m}^{(1)} \right\rangle \frac{1}{i\left|\vec{k}_m - \vec{k}_{m-q} - \vec{k}_q\right|} \\ & + \frac{ig}{2S_{2,p-m} k_{p-m}} \sum_{q=-\infty}^{\infty} R_{p-m-q,q} \left\langle C_q^{(1)} C_{p-m-q}^{(1)} C_p^{*(1)} C_m^{(1)} \right\rangle \frac{1}{i\left|\vec{k}_{p-m} - \vec{k}_{p-m-q} - \vec{k}_q\right|} \\ & + \frac{ig}{2S_{2,p} k_p} \sum_{q=-\infty}^{\infty} R_{p-q,q} \left\langle C_q^{*(1)} C_{p-q}^{*(1)} C_{p-m}^{(1)} C_m^{(1)} \right\rangle \frac{1}{i\left|\vec{k}_p - \vec{k}_{p-q} - \vec{k}_q\right|}\end{aligned} \qquad (45)$$

where $\langle C_q^{(1)} C_{m-q}^{(1)} C_p^{*(1)} C_{p-m}^{(1)} \rangle$ is the fourth order moment, often referred as the trispectrum. We assume that the sum over the quadruple correlations gives a contribution on the triple correlations (Holloway, 1980). So, (45) transforms to:

$$\begin{aligned}\left(\vec{k}_p - \vec{k}_{p-m} - \vec{k}_m\right)\left\langle C_p^* C_{p-m} C_m \right\rangle = & \frac{gR_{m-p,p}}{S_{2,m} k_m} \left\langle C_p^{*(1)} C_p^{(1)} \right\rangle \left\langle C_{p-m}^{*(1)} C_{p-m}^{(1)} \right\rangle \\ & + \frac{gR_{p,-m}}{S_{2,p-m} k_{p-m}} \left\langle C_p^{*(1)} C_p^{(1)} \right\rangle \left\langle C_m^{*(1)} C_m^{(1)} \right\rangle \\ & - \frac{gR_{p-m,m}}{S_{2,p} k_p} \left\langle C_{p-m}^{*(1)} C_{p-m}^{(1)} \right\rangle \left\langle C_m^{*(1)} C_m^{(1)} \right\rangle \\ & + iK \left\langle C_p^* C_{p-m} C_m \right\rangle\end{aligned} \qquad (46)$$

where K is the tuning parameter of the model (fixed value). Only the imaginary part of the bispectrum is required to close the evolution equation for the variance density, so:

$$\mathrm{Im}\left(B_{m,p-m}\right) = \frac{g\,K}{K^2 + \Delta k^2} \left[\frac{R_{m-p,p}}{S_{2,m} k_m} F_p F_{p-m} + \frac{R_{-m,p}}{S_{2,p-m} k_{p-m}} F_p F_m - \frac{R_{p-m,m}}{S_{2,p} k_p} F_{p-m} F_m \right] \qquad (47)$$

with $\Delta k^2 = \left|\vec{k}_p - \vec{k}_{p-m} - \vec{k}_m\right|^2$.

Substitution of equation (47) into (37) leads to:

$$\begin{aligned}&\frac{\partial F_p}{\partial t} + \vec{\nabla}.\left(\vec{C}g_p F_p\right) = \\ &- \frac{g^2}{S_{1,p}} \sum_{m=-\infty}^{\infty} \frac{K}{K^2 + \Delta k^2} R_{m,p-m} \left[\frac{R_{m-p,p}}{S_{2,m} k_m} F_p F_{p-m} + \frac{R_{-m,p}}{S_{2,p-m} k_{p-m}} F_p F_m - \frac{R_{m,p-m}}{S_{2,p} k_p} F_{p-m} F_m \right]\end{aligned} \qquad (48)$$

For its implementation in the TOMAWAC spectral wave model, the SPB source term (right hand side of (48)) is expressed in terms of the continuous frequency-directional variance density function $F(f,\theta)$. The source term may be written as:

$$\begin{aligned} S(f_3,\theta_{\vec{k}_3}) = & \int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} df_1 df_2 d\theta_1 d\theta_2 T_{312}^{SPB}\,\delta\left(\theta_{\vec{k}_3} - \theta_{\vec{k}_1+\vec{k}_2}\right)\delta\left(f_3 - f_1 - f_2\right) \\ & -2\int_0^{\infty}\int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} df_1 df_2 d\theta_1 d\theta_2 T_{132}^{SPB}\,\delta\left(\theta_{\vec{k}_3} - \theta_{\vec{k}_1-\vec{k}_2}\right)\delta\left(f_3 - f_1 + f_2\right) \end{aligned} \quad (49)$$

where T_{312}^{SPB} and T_{132}^{SPB} represent respectively the sum and difference triad interactions. The formulation is directionally coupled and allows for both collinear and non-collinear interactions.

4. NUMERICAL SIMULATIONS:

The three non-linear source terms presented in §3 have been implemented in the spectral wave model TOMAWAC, developed at the Laboratoire National d'Hydraulique (Benoit et al., 1996). Numerical simulations are compared to the experimental laboratory results presented by Nwogu (1994), for the propagation of crossing seas over a constant slope.

Laboratory Basin description (Nwogu, 1994; Nwogu, 1993)
The three-dimensional wave basin (30 m wide, 20 m long and 3 m deep) is located at the Hydraulics Laboratory, National Research Council of Canada. It is equipped with a segmented directional wave generator. Reflections at the basin sidewalls are reduced by wave energy absorbers made of perforated metal sheets. A bathymetric profile was constructed in the basin. It consists in a constant slope beach (1:25) with an impermeable concrete cover. The free surface elevation was measured along the centreline of the basin with a linear array of 23 water level gauges.

Wave conditions imposed at the wave generator
The incident laboratory wave spectrum at the wave generator ($h = 0.56$ m) is bimodal: it is composed of two sea states characterising local sea and swell components (figure 1). The frequency distributions of the sea states are described by a JONSWAP spectrum characterised by the peak period T_p, the significant wave height H_{m0} and the peak enhancement factor γ. The directional distributions are given by the cosine function:

$$D(\theta) = \cos^{2s}(\theta - \theta_0) \quad (50)$$

Table 1 sums up the wave parameters of the target spectrum.

Swell					Local sea				
H_{m0}(m)	T_p(s)	γ	s	θ_0(°)	H_{m0}(m)	T_p(s)	γ	s	θ_0(°)
0.068	2.5	10	22	22.5	0.062	1.5	3.3	6	-22.5

Table 1. Spectral wave parameters characterising the initial spectrum.

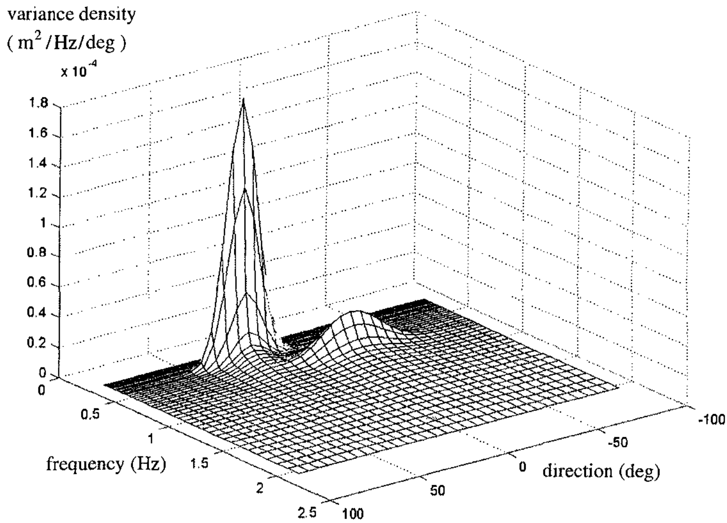

Figure 1 : Directional wave spectrum imposed at the wave generator.

Numerical results

The SPZ source term was activated in the TOMAWAC simulation, with $\Omega=0.05$. The directional wave spectrum obtained in the shallow water area ($h = 0.18$ m) is presented on figure 2. It shows that the variance density spectrum is affected by refraction, shoaling and non-linear triad interactions. Non-linear wave-wave interactions strongly affect the frequency-directional shape of the spectrum. By collinear wave-wave interactions, energy is transferred from the swell spectral peak towards its two higher harmonics and by non-collinear interactions, energy is transferred from the two primary spectral peaks (swell and local sea) towards the component corresponding to the vector sum of the primary peak wavenumbers. Figure 3 shows a comparison between the non-linear SPZ simulation and measurements for the frequency spectrum. The results obtained with a linear simulation are also presented to point-up the effects of wave-wave interactions on the spectral shape. Figure 3 clearly demonstrates that the non-linear triads interactions cannot be neglected in the shoaling zone. As shown by Eldeberky et al. (1996), the new spectral peaks generated using the SPZ model are shifted towards lower frequencies as compared to measurements. Since the interaction condition between the three interacting waves is imposed through $\delta(\vec{k}_3 - \vec{k}_1 - \vec{k}_2)$, the negative curvature of the linear relationship implies that if $\vec{k}_3 = \vec{k}_1 + \vec{k}_2$, then $f_3 \leq f_1 + f_2$ (Eldeberky et al., 1996).

The simulation using the LTA source term was performed with $\alpha=0.5$. The numerically simulated shallow water spectrum is presented on figure 4. Since the LTA model is restricted to self-self (collinear) interactions, the simulated shallow water spectrum only shows two new spectral peaks at frequencies $2f_1$ and $2f_2$, where f_1 and f_2 refer here to

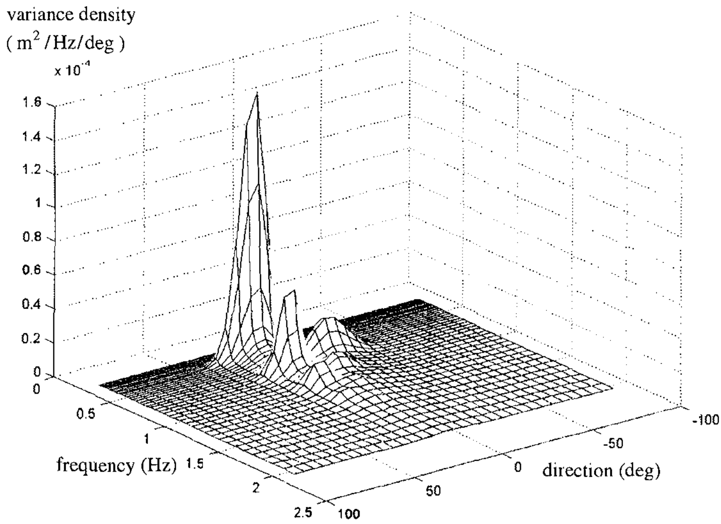

Figure 2: Directional wave spectrum in shallow water ($h=0.18m$).
Non-linear simulation using the SPZ source term.

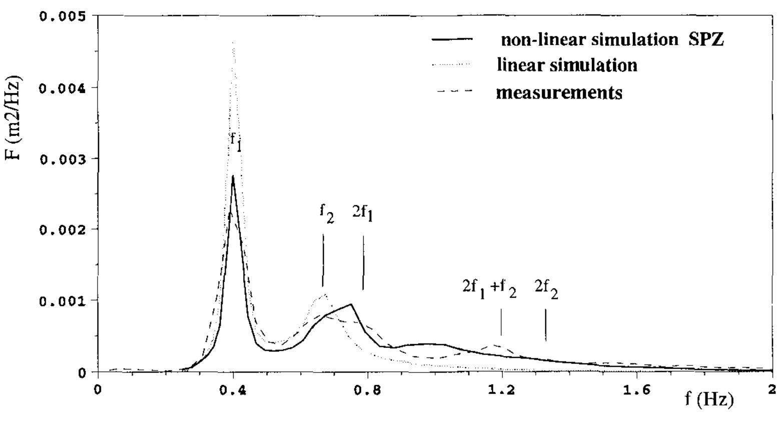

Figure 3: Frequency wave spectrum in shallow water ($h=0.18m$).

the peak frequencies of the swell and local sea respectively. Figure 5 shows the frequency wave spectrum in shallow water. The energy at $2f_1$ is strongly overestimated by the LTA model, when $\alpha=0.5$. The transfer of energy towards super-harmonics decreases with α. But, since the LTA model is energy flux conservative and is restricted to self-self interaction, the choice of α requires a balance between an overestimation of the primary peaks or an overestimation of the harmonics.

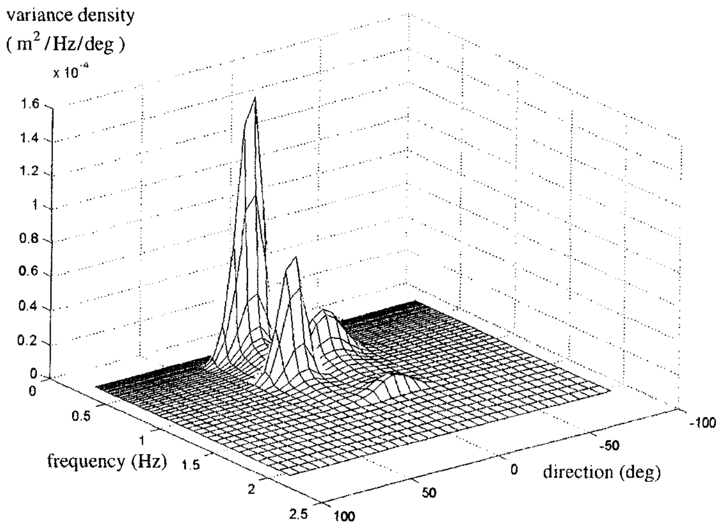

Figure 4: Directional wave spectrum in shallow water ($h=0.18m$). Non-linear simulation using the LTA source term.

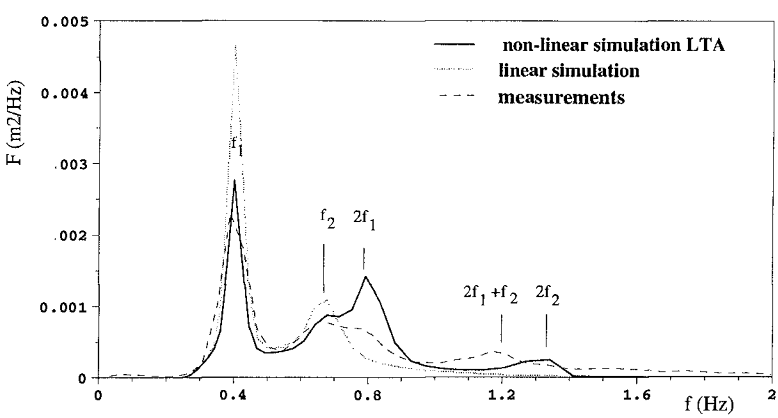

Figure 5: Frequency wave spectrum in shallow water ($h=0.18m$).

The results obtained with the SPB model are presented on figure 6. Collinear and non-collinear wave-wave interactions are both modelled using the SPB source term. Energy is transferred from the swell spectral peak towards its first and second harmonics. The swell also interacts with the local sea and generates components at the vector sum of the local sea and swell components. A comparison of the model to measurements is presented on figure 7. A relatively good fit to measurements is obtained with the SPB source term.

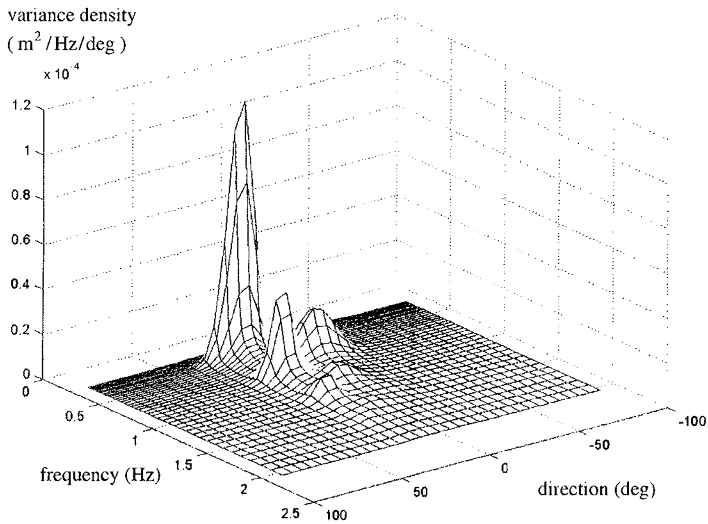

Figure 6: Directional wave spectrum in shallow water (h=0.18m).
Non-linear simulation using the SPB source term.

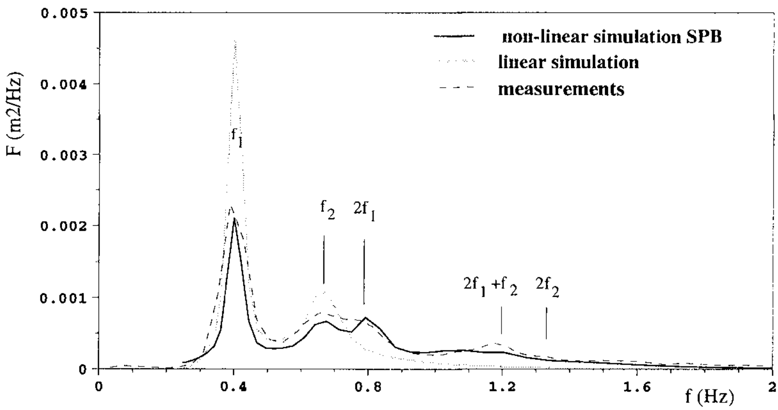

Figure 7: Frequency wave spectrum in shallow water (h=0.18m).

5. CONCLUSIONS:

The spectral deterministic model of Madsen and Sorensen (1993) has been extended to bidimensional situations where combined refraction, shoaling and triads interactions act simultaneously to modify the frequency and directional spreading of the ocean surface energy field. A stochastic source term (SPB) is developed from the deterministic equations obtained. The model is directionally coupled and implies that the energy transfer among spectral components is governed by collinear and non-collinear near-resonant triad interactions. The consequcnces are not only distortions of the frequency spectrum, but also alterations of the directional spreading of wave energy. This source term has been compared to two other spectral source terms: the LTA and SPZ models. The LTA model is able to catch the main features of energy transfers from the main spectral peaks towards their higher harmonics. It cannot simulate all the new spectral peaks occurring in crossing seas, but it is characterised by a great computational efficiency. The SPZ model and the SPB are directionally coupled and are able to transfer energy to a third component at the sum vector of the primary waves. The SPB model gives the best results since the frequency of the new components corresponds exactly to the sum frequency of the primary waves, whereas with the SPZ model, a shift towards lower frequency is observed (Eldeberky et al., 1996).

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