Nonlinear Distribution of Neashore Free Surface and Velocity

Nobuhito MORI * and Nobuhisa KOBAYASHI [†]

ABSTRACT

The Edgeworth expansion with the measured skewness and kurtosis is shown to be capable to describe the measured nonlinear distributions of the surface elevation and horizontal velocity in the shoaling and surf zones. The moments involved in the energetics-based model are expressed in terms of the skewness and kurtosis and shown to be in agreement with available data. Stokes wave theory is applied to obtain a relationship between skewness and kurtosis. The relationship is adjusted empirically because of the limitation of Stokes wave theory in the shoaling and surf zones. The relative simple relationship for the higher moments obtained here may be applied for cross-shore sediment transport analysis.

INTRODUCTION

Nonlinear waves in nearshore regions are important in estimating sediment transport and designing coastal structures. The estimation of an extreme wave crest is an important factor in predicting the deck elevation of an offshore structure. The wave profile in the surf zone is significantly skewed because of wave breaking and bottom topography effects. Time dependent numerical models based on extended Boussinesq equations have been shown to be capable of predicting the nonlinear profile of the surface elevation outside the surf zone(*e.g.*, Nwogu 1993). Time dependent numerical models, however, require large computation time to calculate the wave profiles.

On the other hand, time-averaged models for random waves are more efficient computationally at the expense of the loss of the detailed temporal information (e.g., Battjes and Janssen 1978; Thornton and Guza 1983; Mase and

^{*}Hydraulics Dept., Abiko Research Laboratory, Central Research Institute of Electric Power Industry(CRIEPI), Abiko 1646, Chiba 270-11, JAPAN (mori@criepi.denken.or.jp)

[†]Center for Applied Coastal Research, University of Delaware, Newark, DE, 19716, USA (nk@coastal.udel.edu)

Kobayashi 1991). However, time-averaged models may not be accurate enough, because random wave are expressed as the superposition of regular waves or by a representative wave. Guza and Thornton (1985) has pointed out that both randomness and nonlinearity are necessary to predict the moments of the crossshore fluid velocity on a beach. Hence, the purpose of this study is to develop a probabilistic model of the surface elevation and cross-shore velocity in the nearshore including both random and nonlinear effects.

Many studies have were performed to describe the nonlinear distribution of the free surface elevation. Ochi and Ahn (1994) and Kobayashi et al. (1998) used Siegert solution and the exponential gamma distribution, respectively, to describe the distribution of the surface elevation and velocity. These distributions agree fairly with the measured skewed distributions. However, it is difficult to obtain the relationships among the various moments of the distribution analytically. In contrast, attempts were made to describe the nonlinear distribution (Longuet-Higgins 1963; Haung and Long 1980; Bitner 1980). These methods based the Gram-Charlier or Edgeworth approximation can be expanded as a function of skewness and kurtosis. Then, it is much easier express the various moments in terms of the skewness and kurtosis, although these approximations give negative density values for certain skewed distributions.

In the following, the probability density distributions and moments of the measured free surface elevation and horizontal velocity in a large wave flume are compared with the Edgeworth expansion and the measured skewness and kurtosis. Stokes wave theory is applied to derive a relationship between the skewness and kurtosis.

EXPERIMENTS

The experimental data used here was reported in Japanese by Shimizu *et al.*(1996). The experiment was conducted in a large wave flume that was 205m long, 3.4m wide, 6m high. The water depth in the flume was 4m. A sand beach of a 1:30 slope was installed at the end of the wave tank. Water surface displacements at 17 locations in Fig.1 were measured using capacitance type wave gages located in the still water depth range of 0.1-4.0m. Fluid velocities were measured with six electro-magnetic current meters. The current meters were set 15cm above the bottom. The current meters $C_1 - C_6$ in Fig.1 were located in the still water depths 2.25, 1.92, 1.58, 1.25, 0.92 and 0.48m, respectively. The measurements with a sampling frequency of 20Hz were performed for the duration of 819s.

Random waves based on the JONSWAP spectrum and random phases were generated using linear wave theory with a computer-controlled piston-type wave paddle. Two cases were reported by Shimizu *et al.*(1996) as summarized in Table 1 where $H_{1/3}$ and $T_{1/3}$ are the significant wave height and period above the horizontal bottom. In Table 1, the wave amplitude $a_0=H_{1/3}/2$, the water



Figure 1 Experimental setup and locations of wave gages and current meters.



Table 1 Experimental conditions for two cases.

Figure 2 Measured cross-shore variations of rms values of horizontal fluid velocity u and surface elevation η .

depth $h_0=4.0$ m on the horizontal bottom, and k is the linear wave number based on $T_{1/3}$ and h_0 . The value of ka_0 and kh_0 in Table 1 indicate wave steepness and nondimensional water depth at the wave paddle. Case 2 was more nonlinear than case 1.

The surface displacement η and the horizontal fluid velocity u near the bottom are analyzed in the following. Fig.2 shows the spatial distributions of the root-mean-square(rms) values of the horizontal fluid velocity u and the surface elevation η . Solid lines with open symbols indicate the rms values of u and dashed lines with filled symbols indicate the rms values of η . Circles \bigcirc are

for case 1 and triangles \triangle are for case 2. Fig.2 illustrates the difference in the width of the surf zone for case 1 and 2. The *rms* values of *u* increased landward, whereas the *rms* values of η decreased monotonically landward.

The moments μ_3 , μ_4 , μ_3^* and μ_4^\bullet for $f=\eta$ or u are defined as follows:

$$\mu_n = \frac{1}{N} \sum_{i=1}^N \left(\frac{f_i - \bar{f}}{f_{rms}} \right)^n \tag{1}$$

$$\mu_n^* = \frac{1}{N} \sum_{i=1}^{N} \left| \frac{f_i - \bar{f}}{f_{rms}} \right|^n \tag{2}$$

$$\mu_n^{\bullet} = \frac{1}{N} \sum_{i=1}^N \left(\frac{f_i - \bar{f}}{f_{rms}} \right) \left| \frac{f_i - \bar{f}}{f_{rms}} \right|^{n-1}$$
(3)

where \bar{f} is the mean value of f, f_{rms} is the rms value of f, and N is the number of data point. Eq.(1) with n=3 and 4 gives μ_3 =skewness and μ_4 =kurtosis. The moments are calculated after removing the high-frequency components with frequency larger than 4 times of the peak frequency to reduce the statistical sensitivity. The third order absolute moment μ_3^* and the fourth order signed moment μ_4^\bullet for the velocity are related to the energetics-based sediment transport model(Bailard 1981). The measured third and fourth moments are shown in Fig.3. The measured moments of η show systematic trends. The values of the moments of η are close to the Gaussian values, $\mu_3=0$, $\mu_4=3$, $\mu_3^*=1.6$ and $\mu_4^*=0$, offshore. These values then increase landward before their decrease toward the shoreline. On the contrary, the measured moments of u are below the values of the Gaussian offshore. μ_4 and μ_3^* are reduced first and then increase landward, whereas μ_3 and μ_4^\bullet increase monotonically.

In summary, the spatial variations of the moments of the horizontal fluid velocity u and the surface elevation η deviate from the Gaussian values near and inside the surf zone. The measured moments of u are close to the Gaussian values but the moments of η exceed the Gaussian values significantly inside the surf zone. These results are consistent with the small scale tests on a 1:16 slope by Kobayashi et al. (1998).

STATISTICAL MODELING OF SURFACE AND VELOCITY MOMENTS

The data shown in Fig.3 indicates strong nonlinearity of η in the shoaling and surf zones. The horizontal velocities near the bottom have weaker nonlinearity. Hence, the Edgeworth expansion(Kendall and A.Stuart 1963) is applied to describe the probability density function(PDF) of the surface elevation and horizontal velocity. The Edgeworth expansion is adequate to describe random and weak nonlinear stochastic processes. The Edgeworth expansion is derived in the same manner as the Gram-Charlier expansion but the terms of the series are expressed in terms of cumulants.



(a) Measured cross-shore variations of skewness μ_3



(b) Measured cross-shore variations of kurtosis μ_4

Figure 3 Measured cross-shore variations of third and fourth of moments of horizontal fluid velocity u and surface elevation η .

The PDF p(x)dx of the normalized statistical variables x with zero mean and its standard deviation of unity can be described as(Kendall and A.Stuart 1963)

$$p(x)dx = \sum_{r=0}^{\infty} c_r H_r(x) G(x) dx$$
(4)

with

$$G(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{5}$$

where G(x) is the Gaussian distribution, $H_r(x)$ is the Chebyshev-Hermite poly-



(c) Measured cross-shore variations of third order absolute moment μ_3^*



(d) Measured cross-shore variations of fourth order signed moment μ_4^{\bullet}

Figure 3 Continued

nomial and c_r is the *r*th order coefficient of the Gram-Charlier expansion. Introducing the characteristic function, the Edgeworth expansion of type A is given by (*e.g.*, Longuet-Higgins 1963)

$$p(x)dx = G(x)\left\{1 + \frac{\kappa_3}{6}H_3(x) + \left[\frac{\kappa_4}{24}H_4(x) + \frac{\kappa_3^2}{72}H_6(x)\right] + \left[\frac{\kappa_5}{120}H_5(x) + \frac{\kappa_3\kappa_4}{144}H_7(x)\right] + \cdots\right\}dx$$
(6)

where κ_r is the *r*th order cumulant. The cumulants with r=3-6 are related to

the rth order moments μ_r as follows:

$$\left. \begin{array}{l} \kappa_{3} = \mu_{3} \\ \kappa_{4} = \mu_{4} - 3 \\ \kappa_{5} = \mu_{5} - 10\mu_{3} \\ \kappa_{6} = \mu_{6} - 15\mu_{4} - 10\mu_{3}^{2} + 30 \end{array} \right\}$$

$$(7)$$

where the mean value μ_1 of x is equal to zero($\kappa_1=0$) and the standard deviation of x is unity($\kappa_2=1$). Therefore, μ_3 is skewness and μ_4 is kurtosis.

It must be noted that an asymptotic expansion does not have monotonic convergence for high order corrections. There are many studies about the truncation of (6)(Longuet-Higgins 1963; Haung and Long 1980; Ochi and Wang 1984; Mori and Yasuda 1996). They have shown that the first three terms of (6) are sufficient for describing the nonlinear property of the PDF of the surface elevation. Hence, the first three terms in (6) are used to represent the PDF of x denoting the normalized the surface elevation and horizontal velocity

$$p(x)dx = G(x)\left\{1 + \frac{\kappa_3}{6}H_3(x) + \left[\frac{\kappa_4}{24}H_4(x) + \frac{\kappa_3^2}{72}H_6(x)\right]\right\}dx$$
(8)

which requires $\kappa_3 = \mu_3 =$ skewness and $\kappa_4 = (\mu_4 - 3)$ with $\mu_4 =$ kurtosis. The truncation of high order terms in (6) gives the following relationships based on $\kappa_5 = 0$ and $\kappa_6 = 0$ for the high order moments:

$$\mu_5 = 10\mu_3 \tag{9}$$

$$\mu_6 = 10\mu_3^2 + 15\mu_4 - 30 \tag{10}$$

The other higher moments related to the energetics-based sediment transport model(Bailard 1981) can be calculated using (8)

$$\mu_3^* = \int_{-\infty}^{\infty} |x|^3 p(x) dx = \frac{1}{6\sqrt{2\pi}} (3\mu_4 - \mu_3^2 + 15)$$
(11)

$$\mu_5^* = \int_{-\infty}^{\infty} |x|^5 p(x) dx = \frac{2}{3\sqrt{2\pi}} (15\mu_4 + 5\mu_3^2 - 21)$$
(12)

$$\mu_{4}^{\bullet} = \int_{-\infty}^{\infty} x |x|^{3} p(x) dx = \frac{8}{\sqrt{2\pi}} \mu_{3}$$
(13)

Fig.4 show the comparisons of the measured PDF of u and η with (8) for case 2 where the measured values of μ_3 and μ_4 are used in (8). The PDF of u is skewed negatively offshore but positively in the surf zone. The measured spatial variation of μ_3 shown in Fig.3a indicates the corresponding sign shift. This may be important for the cross-shore sediment transport. On the other hand, the PDF of η is always skewed positively as expected from nonlinear wave theory. The nonlinear PDF given by (8) agrees better with the data than the linear Gaussian distribution partly because of the additional input of μ_3 and μ_4 .



Figure 4 Measured PDF of u and η for case 2 in comparison with the theory.



Figure 5 Comparison between measured μ_5 , μ_3^* , μ_5^* and μ_4^\bullet , and calculated μ_5' , $\mu_3^{*'}$, $\mu_5^{*'}$ and μ_4^\bullet for measured μ_3 and μ_4 .

To check the validity of (9)-(13), the comparisons between the measured moments μ_n and the calculated moments μ'_n indicated by the prime are shown in Fig.5. Eq.(9) gives a simple relationship between μ_3 and μ_5 . The correlation coefficient of in Fig.5a is 0.98, indicating good agreement between the measured μ_5 and the calculated μ'_5 . The sixth moments μ_6 given by (10) for u and η show similar results(not shown). Fig.5b-d also show good agreement between the measured moments(μ_3^* , μ_5^* and μ_4^*) and the calculated moments($\mu_3^{*\prime}$, $\mu_5^{*\prime}$ and $\mu_4^{\bullet\prime}$) using (11)-(13). The PDF of u and η in the shoaling and surf zones can be described by the Edgeworth expansion given by (8). As a consequence, the associated moments can be estimated by (9)-(13).

Eq.(8) requires both skewness μ_3 and kurtosis μ_4 as input. An attempt is made to express μ_4 in terms of μ_3 .

RELATIONSHIP BETWEEN SKEWNESS AND KURTOSIS

Tayfun (1980), Srokosz and Longuet-Higgins (1986) and Winterstein et al. (1991) derived the PDF of the surface elevation based on the assumption that random waves can be expressed as a summation of Stokes 2nd or 3rd waves. This assumption considers only the self wave interaction components, although there are random wave-wave interaction components. This approach is easy to apply and calculate the moments in comparison with the fully nonlinear random interaction method (e.g., Sharma and Dean 1979). Admittedly, Stokes wave theory may not be valid in shallow water and the derived relationship will be interpreted in view of this limitation. The 3rd order Stokes wave in finite water depth h is given by:

$$k\eta = \frac{1}{2}(ak)^2 D_1 + ak\cos\theta + \frac{1}{2}(ak)^2 D_2\cos 2\theta + \frac{3}{8}(ak)^3 D_3\cos 3\theta \qquad (14)$$

where a is the amplitude, k is the wave number, θ is the phase with $\theta = (kx - \omega t + \varepsilon)$ in which ω is the angular frequency, and the phase ε is assumed to random. Eq.(14) neglects the phase shift of the second and third harmonics which may be important for the wave profile pitched landward in the surf zone. D_i is the function of the nondimensional water depth kh:

$$D_{1} = \coth kh$$

$$D_{2} = \coth kh \left(1 + \frac{3}{2\sinh^{2}kh}\right)$$

$$D_{3} = 1 + \frac{1}{\sinh^{2}kh} \left(3 + \frac{3}{\sinh^{2}kh} + \frac{9}{8\sinh^{4}kh}\right)$$
(15)

It is assumed that the PDF of the first order component $a \cos \theta$ is the Gaussian and that the amplitude a and the phase ε are independent of each other with ε being distributed uniformally between 0 to 2π . These assumptions yields the joint PDF of a and θ as

$$f(a,\theta) = \frac{a}{2\pi\sigma^2} \exp\left(-\frac{a^2}{2\sigma^2}\right),\tag{16}$$

where σ is the *rms* value of the first order component($a\cos\theta$). The moments λ with $n=0, 1, \cdots$ can be calculated using (14) and (16):

$$\lambda_n = \int_{a=0}^{\infty} \int_0^{2\pi} [\eta(a,\theta)]^n f(a,\theta) \, da \, d\theta \tag{17}$$

which yields

$$\lambda_0 = \sigma D_1 \alpha \tag{18}$$

$$\lambda_2 = \sigma^2 \left[1 + (D_1^2 + D_2^2)\alpha^2 + O(\alpha^4) \right]$$
(19)

$$\lambda_3 = \sigma^3 \left[3(D_1 + D_2)\alpha + \frac{1}{2}(4D_1^3 + 12D_1D_2^2 + 27D_2D_3)\alpha^3 + O(\alpha^4) \right]$$
(20)

$$\lambda_4 = \sigma^4 \left[3 + 3(6D_1^2 + 6D_2^2 + 8D_1D_2 + 3D_3)\alpha^2 + O(\alpha^4) \right]$$
(21)



Figure 6 Measured and predicted μ_4 as a function of measured μ_3 .

where α is the wave steepness ak of the linear component. Dividing λ_3 by $\lambda_2^{3/2}$, the leading skewness contribution is given by

$$\mu_3 = 3\alpha (D_1 + D_2) \tag{22}$$

The low frequency component involving D_1 in (14) was retained by Vinje (1989) but was neglected by Tayfun (1980) and Winterstein *et al.*(1991). Dividing λ_4 by λ_2^2 , neglecting D_1 and D_3 and substituting (22), the leading kurtosis contribution obtained

$$\mu_4 = 3 + \left(\frac{4}{3}\mu_3\right)^2 \tag{23}$$

The effect of the water depth D_2 on μ_4 is included in (23) through μ_3 given by (22) with $D_1=1$.

To examine the validity of (23), Fig.6a shows the comparison between the measured and calculated μ_4 as a function of the measured μ_3 . Eq.(23)(solid line) follows the trend of the data point but overpredicts the kurtosis μ_4 . The measured μ_4 corresponding to $\mu_3 \simeq 0$ is smaller than 3. It means that for zero skewness the kurtosis is smaller kurtosis than linear random wave theory. The field data of Ochi and Wang (1984) also showed similar tendency. Therefore, (23) is adjusted empirically as follows:

$$\mu_4 = \beta + \left(\frac{4}{3}\mu_3\right)^2 \tag{24}$$

where β is the empirical constant. Eq.(24) reduces to (23) for $\beta=3$. Eq.(24) with $\beta=2.3$ is shown as a dashed line in Fig.6. The value of $\beta=2.3$ is determined by a



Figure 7 Comparisons between measured μ_3^* and μ_5^* and calculated μ_3^{**} and μ_5^{**} for measured μ_3 .

least-square method using the laboratory data in Fig.6a. Additional two lines are plotted in Fig.6a. One is derived by Kobayashi *et al.*(1998) using the exponential gamma distribution and another is the empirical relationship obtained by Ochi and Wang (1984) using extensive field data on the free surface elevation. The exponential gamma distribution tend to overpredict μ_4 and is almost the same as (23). The empirical equation (24) and that by Ochi and Wang (1984) give better agreement with the data. Fig.6b shows the comparisons among (23), (24), the laboratory data, the empirical relation and the field data by Ochi and Wang (1984). Eq.(24) with the measured μ_3 underpredicts the field data but (23) overestimates the field data.

Substituting (24) into (11) and (12) gives μ_3^* and μ_5^* as a function of μ_3 only:

$$\mu_3^* = \frac{1}{18\sqrt{2\pi}} \left[13\mu_3^2 + 9(\beta+5) \right], \tag{25}$$

$$\mu_5^* = \frac{2}{9\sqrt{2\pi}} \left[95\mu_3^2 + 9(5\beta - 7) \right].$$
(26)

The measured μ_3^* and μ_5^* are compared with (25) and (26) in Fig.7. The predicted μ_3^* and μ_5^* as a function of the measured μ_3 are reasonable and the agreement is similar to Fig.5b and 5c.

COMPARISON WITH FIELD DATA

Finally, (9), (11) and (12) using the measured μ_3 and μ_4 are compared with the field data on cross-shore(c) and longshore(l) velocities measured by Guza and Thornton (1985) at Torreys Pines Beach, San Diego, California, during November 1978. The observed moments in Table 2 are spatially-averaged values. Table 2 also includes the predicted moments using (23), (25) and (26) with $\beta=3$ for the measured μ_3 only. Guza and Thornton (1985) analyzed their velocity data to estimate the sediment transport rates using the energetics model by Bailard (1981).

For the cross-shore velocities, the calculated moments μ_5 , μ_3^* and μ_5^* using the measured μ_3 and μ_4 are in good agreement with the data on Nov.20th but the agreement is worse for Nov.17th. This is due to the unexpected combination of μ_3 and μ_4 for the Nov.17th data(high skewness and low kurtosis). For the alongshore velocities, the measured skewness was almost zero and the Gaussian distribution with $\mu_3=0$ appears to be acceptable except for μ_5^* .

					,		
	Data		Model				
moment	Nov.17th	Nov.20th	Nov.17th		Nov.20th		Gaussian
			$\mu_3 \& \mu_4$	μ_3	$\mu_3 \& \mu_4$	μ_3	$\mu_3 = 0$
μ3-с	0.55	0.50	-		-	-	0
μ_3 -l	-0.04	0.01		_	-	-	0
μ_4 -c	2.86	3.50		3.54	-	3.44	3.0
μ_4 -l	3.41	3.44		3.00	_	3.00	3.0
μ_5 -c	4.95	5.39	5.5	5.5	5.0	5.0	0
μ_5 -l	-0.05	-0.52	-0.4	-0.4	0.1	0.1	0
μ_3^* -c	1.60	1.69	1.55	1.68	1.68	1.67	1.6
μ_3^* -l	1.68	1.67	1.68	1.60	1.68	1.60	1.6
μ_5^* -c	7.77	8.58	6.23	8.93	8.71	8.49	6.38
μ_5^* -l	8.06	8.56	8.02	6.40	8.14	6.38	6.38

Table 2 Comparison between velocity moments field data by Guza and Thornton(1985) and theory with $\beta=3$.

c for cross-shore velocities; l for longshore velocities

CONCLUSION

Nonlinear wave statistics of irregular waves are examined in the shoaling and surf zones on a beach. First, the probability density function of the surface elevation and velocity can be represented by the Edgeworth expansion. Second, the analytical relationships among the order odd and even moments involved in the energetics-based sediment transport model are derived and verified using laboratory and field data. Third, semi-empirical relationship between the skewness and kurtosis is proposed to facilitate future applications.

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Evolution Equations for Edge Waves and Shear Waves on Longshore Uniform Beaches

James T. Kirby, M.ASCE¹, Uday Putrevu, A.M.ASCE² and H. Tuba Özkan-Haller³

Abstract

A general formalism for computing the nonlinear interactions between triads of coastally-trapped gravity and vorticity waves is developed. An analysis of the linearized problem reveals that gravity (or edge) waves and vorticity (or shear) waves exist as members of the same non-Sturm-Liouville eigenvalue problem, with unstable shear waves representing the complex eigenvalue portion of the resulting spectrum. Interaction equations derived here cover resonant interactions between three edge waves, three shear waves, or a shear wave and two edge waves. Numerical examples are shown for the case of three edge waves on a planar beach in the absence of a longshore current. It is found that edge waves can exchange significant amounts of energy over time scales on the order of ten wave periods, for realistic choices of edge wave parameters.

<u>Introduction</u>

The low frequency wave climate on an open coastal beach contains a complex mix of trapped gravity wave motions (edge waves) as well as vorticity (or shear) waves associated with the instability of the longshore current. To date, there has been a tendancy in the literature to treat both classes of motion as isolated systems in which the principle effect of nonlinearity is through amplitude dispersion. Formulations of this type typically treat the wave field in terms of a wave envelope modulated by cubic nonlinearity, leading to the cubic Schrödinger equation for conservative edge wave systems (Yeh, 1985) or Ginzburg-Landau equation for marginally unstable shear waves (Feddersen, 1998). However, in field conditions, all of these motions occur in a relatively dense spectral environment, and the existence of combinations of waves satisfying three-wave resonance conditions makes it likely that the dominant nonlinear mechanism affecting edge or shear waves would be through resonant interactions at second order.

Direct numerical simulations (Allen et al, 1997; Özkan-Haller and Kirby, 1998)

¹Professor, Center for Applied Coastal Research, University of Delaware, Newark, DE 19716 USA. ²Research Scientist, NorthWest Research Associates, Inc., PO Box 3027, Bellevue, WA 98009-3027 USA.

³Assistant Professor, Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, MI 48109 USA.

suggest that the growth to finite amplitude of the shear wave climate involves strong nonlinear interaction between the various length scales in the motion. It is likely that there are also opportunities for edge waves to undergo strong interactions, although this pathway has not been heavily investigated to date. All of these interactions contribute to the final evolution of the low frequency climate on a beach, which may or may not have some sort of equilibrium configuration.

The goals of present study are to:

- 1. Derive evolution equations describing the nonlinearly-coupled evolution of the discrete modes of the low frequency wave climate.
- 2. Use these equations to investigate the full range of edge wave edge wave, shear wave shear wave, and edge wave shear wave interactions.
- 3. Couple the resulting system to incident wave conditions.
- 4. Investigate the equilibrium statistics of the resulting low-frequency wave climate, and compare to field measurements.

The core of our approach to this problem is the development of a spectral model describing nonlinear interactions between the free waves of the system by means of resonant interactions at second order. To date, the literature has identified the possibility of these resonances for the case of three edge waves (Kenyon, 1970; Bowen, 1976) or three shear waves (Shrira et al, 1997). We wish to add to this list the possibility of a triad involving a single shear wave and two edge waves, either of which can be propagating with or against the shear wave. Figure 1 illustrates such a case with all three waves propagating in the same direction as the longshore current. A general framework for computing these interactions is outlined below, and then specialized to the case of edge waves on a planar beach with no current in order to obtain analytical results.

Formulation of the Problem

For simplicity, our attention here is restricted to the case of unforced, undamped nonlinear long wave motions on a longshore uniform beach. The inclusion of forcing would lead to a coupling of the low-frequency motion to the incoming short wave climate (Lippmann et al, 1997). The introduction of longshore variability would extend the present analysis to include both the slow variation of model parameters in the longshore direction as well as the direct scattering of wave modes by wavelength-scale bottom features (Chen and Guza, 1998). These topics will be addressed in extensions of the present work.

The dependent variables in the present analysis are the surface displacement $\eta(x, y, t)$, cross-shore velocity u(x, y, t) and longshore current v(x, y, t) + V(x), where a distinction is made between the mean current profile V(x) and the wave-induced fluctuations v(x, y, t). The governing equations are given by

$$\frac{d\eta}{dt} + (hu)_x + hv_y = -(\eta u)_x - (\eta v)_y \equiv {}^{\eta}N$$
(1)



Figure 1: Diagram illustrating hypothetical resonant triad interaction involving a shear wave and two edge waves. Identifying the shear wave as the first wave in the triad, the origin of the edge wave dispersion curve is translated up the shear wave dispersion curve to the locus of shear wave frequency and wavenumber. Resonances involving two edge waves are then indicated by the intersections of the original and the translated edge wave dispersion curves. The two dashed lines here indicate two edge waves with the same mode number and propagating downstream with the longshore current.

$$\frac{d(hu)}{dt} + gh\eta_x = -huu_x - hvu_y \equiv {}^{u}N$$
(2)

$$\frac{d(hv)}{dt} + V'(hu) + gh\eta_y = -huv_x - hvv_y \equiv {}^{v}N$$
(3)

where a prime denotes differentiation with respect to x, and where

. . .

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + V(x)\frac{\partial}{\partial y} \tag{4}$$

is a time derivative following the local mean current. Eliminating \boldsymbol{u} and \boldsymbol{v} from linear terms gives

$$\frac{d}{dt}\left\{\frac{d^2\eta}{dt^2} - g(h\eta_x)_x - gh\eta_{yy}\right\} + 2ghV'\eta_{xy} = (\epsilon)\text{N.L.T.}$$
(5)

where

N.L.T. =
$$\frac{d}{dt} \left\{ \frac{d \,^{\eta}N}{dt} - (\,^{u}N)_x - (\,^{v}N)_y \right\} + 2V'(\,^{u}N)_y$$
 (6)

and where ϵ denotes a small parameter characterizing the weakness of the wave motions.

The Linearized Problem

We first seek solutions to the linearized problem, obtained by taking the limit $\epsilon = 0$ in (5). Solutions will be of the form

$$\eta = F(x)e^{i(\lambda y - \omega t)} \tag{7}$$

$$u = G(x)e^{i(\lambda y - \omega t)}; \quad G(x) = \frac{-ig}{\sigma}F'(x)$$
(8)

$$v = H(x)e^{i(\lambda y - \omega t)}; \quad H(x) = \frac{g}{\sigma} \left\{ \lambda F - \left(\frac{V'}{\sigma}\right) F' \right\}$$
(9)

where

$$\sigma \equiv \omega - \lambda V(x) \tag{10}$$

is the local intrinsic frequency of the wave with respect to the local longshore current velocity. Substituting (7)-(9) in (5) gives an eigenvalue problem which may be written in self-adjoint form (Howd et al, 1992) as

$$\left(1 - \frac{g\lambda^2 h}{\sigma^2}F\right) + \left(\frac{ghF'}{\sigma^2}\right)' = 0 \qquad 0 \le x \le \infty$$
(11)

$$F$$
 bounded at $x = 0$, $F \downarrow 0$ as $x \to \infty$ (12)

which is not convenient for solution of the eigenvalue problem but which serves as a basis for establishing solvability conditions in the nonlinear problem. The resulting eigenvalue problem is a non-Sturm-Liouville eigenvalue problem for $\{F^r(x), \omega^r\}$ given λ and h(x). There are possible singularities at $\sigma^r = \omega^r - \lambda V_c = 0$, where V_c denotes the critical longshore current velocity. Possible types of solutions include:

- 1. Gravity motions without a critical level in the current profile \rightarrow Distorted "regnlar" edge waves (Howd et al, 1992)
- 2. Gravity motions in the presence of a double set of critical levels, including:
 - (a) Waves trapped against the shore by the faster offshore velocity (Falqués and Iranzo, 1992).
 - (b) Waves trapped between the critical levels, propagating upstream relative to the current (Bryan and Bowen, 1998)
 - (c) Waves trapped between the offshore critical level and deep water (hypothetical).
- 3. Vorticity motions representing the unstable growth of meanders in the longshore current (where ω^r is complex; Bowen and Holman, 1989) or the stable propagation of similar meanders (Falqués and Iranzo, 1992; Bowen and Holman, 1989).

For a given λ , the orthogonality condition for two modes with distinct mode numbers n, m and frequencies ω^n, ω^m is easily established,

$$\int_0^\infty \frac{gh(\sigma^n + \sigma^m)}{(\sigma^n)^2 (\sigma^m)^2} \left\{ F^{n\prime} F^{m\prime} + \lambda^2 F^n F^m \right\} dx = 0$$

but we do not have a theorem for the completeness of the F^n basis. Since the system is of non-Sturm-Liouville form, we expect to obtain a complex spectrum of eigenvalues, of which the components containing positive imaginary parts will correspond to unstable and growing vorticity modes, or shear waves. We wish to emphasize here that the edge waves and shear waves are members of the same basis of eigenfunctions.

The Nonlinear Problem

Returning to the full problem, we follow the usual approach for obtaining evolution equations for variation of modal amplitudes on slow time and longshore space scales. We introduce multiple scales in order to identify slow changes of modal amplitudes in time and in longshore distance.

$$t \to t + \epsilon t = t + T \tag{13}$$

$$y \to y + \epsilon y = y + Y$$
 (14)

We then introduce an expansion for η ,

$$\eta = \eta^{(1)} + \epsilon \eta^{(2)} \tag{15}$$

The solution for $\eta^{(1)}$ corresponds to a superposition of all eigenmodes of the linearized system,

$$\eta^{(1)} = \sum_{n} \sum_{r} \frac{1}{2} A_n^r(Y, T) F_n^r(x) E_n^r + \text{complex conjugate}$$
(16)

where

$$E_n^r = e^{i(\lambda_n y - \omega_n^r t)} \tag{17}$$

is the oscillatory dependence on fast time and longshore distance, and the F_n^r are the eigenmodes of the linear eigenvalue problem. At $O(\epsilon)$, we get a forced problem for each n, r combination. We require the forcing for each component to be orthogonal to the solution of the adjoint of the original eigenvalue problem. Nonlinear terms in the system may be simplified by imposing resonance conditions, given by

$$\pm \lambda_l \pm \lambda_m - \lambda_n = 0 \tag{18}$$

$$Re\left\{\pm\omega_{\ell}^{p}\pm\omega_{m}^{q}-\omega_{n}^{r}\right\} = 0 \tag{19}$$

The final evolution equation for each discrete mode in the system has the form

$$\begin{aligned}
A_{nT}^{r} + C_{gn}^{r} A_{nY}^{r} &= i \sum_{l} \sum_{m} \sum_{p} \sum_{q} \{ + T_{lmn}^{pqr} A_{l}^{p} A_{m}^{q} \delta(l+m-n) \delta(\omega_{l}^{p} + \omega_{m}^{q} - \omega_{n}^{r}) \\
&+ - T_{lmn}^{pqr} A_{l}^{p} A_{m}^{q} * \delta(l-m-n) \delta(\omega_{l}^{p} - \omega_{m}^{q} - \omega_{n}^{r}) \\
&+ - T_{mln}^{qpr} A_{l}^{p*} A_{m}^{q} \delta(m-l-n) \delta(\omega_{m}^{q} - \omega_{l}^{p} - \omega_{n}^{r}) \}
\end{aligned}$$
(20)

where $_{+}T$ and $_{-}T$ are complicated interaction coefficients for sum and difference interactions respectively. The group velocity C_{an}^{r} for each mode is given by

$$C_{gn}^{r} = \frac{\int_{0}^{\infty} \left[\frac{2g\lambda_{n}}{(\sigma_{n}^{r})^{2}} h(F_{n}^{r})^{2} + \frac{2V}{\sigma_{n}^{r}} (F_{n}^{r})^{2} - \frac{2g\lambda_{n}VV'hF_{n}^{r}F_{n}^{r'}}{(\sigma_{n}^{r})^{4}} - \frac{2gV'hF_{n}^{r}F_{n}^{r'}}{(\sigma_{n}^{r})^{3}} \right] dx}{\int_{0}^{\infty} \left[\frac{2}{\sigma_{n}^{r}} (F_{n}^{r})^{2} - \frac{2g\lambda_{n}h}{(\sigma_{n}^{r})^{4}} V'F_{n}^{r}F_{n}^{r'} \right] dx}$$
(21)

In the no-current limit, the corresponding group velocity for edge waves on an arbitrary profile reduces to

$$C_{gn}^r = g\left(\frac{\lambda_n}{\omega_n^r}\right) \quad \frac{\int_0^\infty h(F_n^r)^2 dx}{\int_0^\infty (F_n^r)^2 dx} \tag{22}$$

given originally by Pearce & Knobloch (1994).

In order to proceed beyond this point to a numerical determination of a solution, a number of steps need to be carried out. First, a reliable method of determining solutions for the linear eigenvalue problem must be established. Then, given eigenvalue pairs $\{\lambda_n, \omega_n^r\}$, we require an algorithm to reliably search for solutions to resonance conditions. Finally, an accurate means for evaluating integrals in expressions for C_g and the nonlinear coupling coefficients must be developed.

Edge Wave Interactions

In this section, we consider the special case of interaction between triads of edge waves on a planar beach in the absence of currents. In this case, the mode structure and wave dispersion relation is known, and model interaction coefficients may be evaluated analytically.

The possibility of triad interactions between progressive edge waves has been mentioned many times but not often addressed in a direct way. Kenyon (1970) provides a version of the Hasselmann interaction equations for random edge wave interactions, but provided no calculations. Kochergin and Pelinovsky (1989) consider the case of a colinear triad (all waves propagating the same direction) and show results for a single interacting triad. We will establish below that their results are wrong.

For the case of no currents, the interaction coefficients reduce to:

$${}_{\pm}T^{pqr}_{lmn} = \omega^{p}_{l}(\pm\omega^{q}_{m})[8\omega^{r}_{n}\int_{0}^{\infty} (F^{r}_{n})^{2}dx]^{-1} \cdot \\ \int_{0}^{\infty} \left\{ 2(\omega^{p}_{l}\pm\omega^{q}_{m})F^{p'}_{l}F^{q'}_{m}F^{r}_{n} + \omega^{p}_{l}F^{p}_{l}F^{q''}_{m}F^{r}_{n} \pm \omega^{q}_{m}F^{p''}_{l}F^{q}_{m}F^{r}_{n} \right. \\ \left. + \left[2(\omega^{p}_{l}\pm\omega^{q}_{m})\lambda^{p}_{l}(\mp\lambda^{q}_{m}) - \omega^{p}_{l}(\lambda^{q}_{m})^{2} \mp \omega^{q}_{m}(\lambda^{p}_{l})^{2} \right]F^{p}_{l}F^{q}_{m}F^{r}_{n} \right\} dx$$
(23)

For a planar beach, the F_n^r are given in terms of Laguerre polynomials by

$$F_n^r(x) = e^{-|\lambda_n|x} L_r(2|\lambda_n|x)$$
(24)

Solutions for isolated triads are obtained in terms of Jacobi elliptic functions. In the cases we have investigated, we have found that cases involving counterpropagating

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<u>Wave</u> <u>Mode</u>		Frequency	<u>Wave number</u>		
1	0	ω_1	λ_1		
2	0	$\omega_2 = \frac{1}{2}\omega_1$	$\lambda_2 = -\frac{1}{4}\lambda_1$		
3	1	$\omega_3 = \frac{3}{2}\omega_1$	$\lambda_3 = \frac{3}{4}\lambda_1$		

Table 1: Case 1. Parameters for lowest-order edge wave triad involving counterpropagating zero-mode waves.

waves show strong interactions with energy exchange time scales on the order of 10 wave periods. In contrast, cases involving colinear waves have interaction coefficients of zero, indicating an absence of interaction, contrary to the results of Kochergin and Pelinovsky (1989). Because this result is at odds with the existing literature, we verify it using a direct numerical simulation. The spectral-collocation method of Özkan-Haller and Kirby (1997) is used to obtain direct numerical solutions of the nonlinear shallow water equations with shoreline runup.

Results and Numerical Verification

As a first example, we consider the lowest-order triad involving two counterpropagating zero-mode edge waves, with the relation between frequencies, wavenumbers and mode numbers as indicated in Table 1. The geometry of the triad in wavenumber-frequency space is indicated in Figure 2. The resulting interaction equations are given by

$$A_{1T} = \frac{i\omega_1^3}{8gs^2} A_2^* A_3 \tag{25}$$

$$A_{2T} = \frac{i\omega_1^3}{64gs^2} A_1^* A_3 \tag{26}$$

$$A_{3T} = \frac{9i\omega_1^3}{64as^2} A_1 A_2 \tag{27}$$

$$|A_1|^2 + |A_2|^2 + |A_3|^2 = \text{constant}$$
(28)

In this case, the parameters are chosen such that ω_1 corresponds to a wave with a period of 20s on a beach with a slope of 1 : 10. In the results illustrated in Figure 3, we have initialized the triad by giving waves 1 and 2 amplitudes of 10cm, with wave three having no amplitude to start. The resulting solution for the triad interaction is shown in Figure 3 by the smooth curves. The results indicate a complete exchange of energy between one of the Mode 0 waves and the Mode 1 wave propagating the same direction. The exchange occurs in somewhat less than 20 periods of the Mode 0 wave. The counterpropagating Mode 0 wave is crucial to the interaction but exchanges only a small amount of energy with the other modes. This non-reactivity of the counterpropagating wave has been noted for a wide range of initial conditions.

The analytic results shown in Figure 3 have been verified using direct numerical



Figure 2: Resonant triad edge-wave interaction with counterpropagating components.

simulation with the pseudospectral model of Özkan-Haller and Kirby (1997). Results from that model were obtained by Fourier transforming the longshore dependence of the runup tip. Results are shown in Figure 3 as the curves with smaller-scale jitter in time. (This jitter occurs at wave-period or sub-wave-period scales, and is probably associated with the fact that the linear edge waves input as initial conditions differ from fully nonlinear solutions to the problem.) Agreement between analytical triad results and numerical solutions are close, with the numerical solutions indicating a slightly slower energy exchange time and a tendency for energy to leak out of the three components making up the triad.

The fate of the missing energy can be seen in the plot of the frequency-wavenumber spectrum computed from the numerical solution, shown in Figure 4. The spectrum is dominated by the three waves making up the resonant triad, but there are clear contributions at forced, non-resonant peaks representing sum and difference interactions lying off the edge wave dispersion curves. There has also been an excitation of the Mode 0 edge wave at twice the wavenumber of Wave 1, and at a frequency that is not commensurate with any sum or difference combination in the original triad. The mechanism for exciting this free wave is not clear and may be associated with start-up transients in the initial value problem.

Figure 5 shows one longshore period of the numerically computed wave field at two instances in time. The top panel shows the situation at 20 wave periods into the simulation, where the wave field is dominated by the higher-frequency Mode 1



Figure 3: Comparison of time series of modal wave amplitudes: analytic and numerical results.



Figure 4: Frequency-wavenumber spectrum for case of counterpropagating waves. Direct numerical simulation.



Figure 5: Snapshot of numerically computed instantaneous wave field showing conditions dominated by Mode 1 wave (top panel) and Mode 0 wave (lower panel).

wave riding on the longer, counterpropagating Mode 0 wave. The lower panel shows the situation at 40 periods (close to the end of the recurrence cycle), where the two counterpropagating Mode 0 waves dominate the wavefield.

As a second example, we consider the case elaborated by Kochergin and Pelinovsky (1989) with all waves travelling the same direction, illustrated in Figure 6. The parameters for the lowest-order case are indicated in Table 2. The present theory indicates that nonlinear interaction coefficients reduce to zero, giving solutions $A_1, A_2, A_3 =$ constant. Figure 7 shows time histories for the first twelve Fourier modes of the longshore runup in a direct numerical simulation, with modes k = 1 and k = 3 corresponding to the initialized low-frequency modes in the triad. The numerical results indicate no interaction between the initialized modes and an absence of growth of the third member of the possible triad. This result is also clear in the resulting frequency-wavenumber spectrum shown in Figure 7, which shows an almost complete lack of energy appearing at the third component, which would appear at scaled wavenumber k = 4 and frequency f = 2.

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Wave	<u>Mode</u>	Frequency	<u>Wave_number</u>		
1	0	ω_1	λ_1		
2	1	$\omega_2 = \omega_1$	$\lambda_2 = \frac{1}{3}\lambda_1$		
3	1	$\omega_3 = 2\omega_1$	$\lambda_3 = rac{4}{3}\lambda_1$		

Table 2: Parameters for lowest order triad with waves propagating in the same direction, as in Kochergin and Pelinovsky (1989).

Conclusions

In this paper, we have described a framework for deriving coupled-mode equations for a sea of edge waves and shear waves. Interaction coefficients have been obtained for the special case of edge waves on a plane beach in the absence of currents. For this system, interactions have been shown to exist and to be fairly rapid for triads involving counterpropagating waves. Triads involving unidirectional propagation have been found to not lead to interaction, in contradiction to the existing literature. We do not yet have a conclusive proof that this result holds for all colinear edge wave triads on a planar beach, but it has been found to hold for all combinations tested so far. Results for both cases have been verified by direct numerical simulation. The close agreement between numerical and analytic results also indicates that a weakly nonlinear formulation is appropriate for examining edge wave interactions. This result is to be expected due to the strongly dispersive nature of the edge wave motions.

The work on edge wave interactions is presently being extended to look at more complicated systems involving multiple coupled triads, leading up to an evaluation of equilibrium distribution of energy in a random sea of edge waves. In order to further this goal, we need to:

- 1. Automate the process of identifying resonances.
- 2. Extend calculations to a large number of components, in order to investigate the assumptions to be made in going over to a stochastic version of the equations.
- 3. Implement the stochastic version and couple it to the incident wave climate.

In addition, the limitation of the present analytical theory to the case of waves on planar beach topographies is restrictive, and needs to be extended to the case of non-planar topographies such as the exponential profile of Ball (1967). It is also possible that the non-interaction of edge wave triads involving waves propagating the same direction, found here for waves on a planar beach, is an anomalous result that will not hold for arbitrary topographies.

For the case with a net longshore current added to the system, we need to elaborate the process for numerically determining the eigenmodes for an arbitrary topography and longshore current distribution, and then repeat the steps outlined above.



Figure 6: Single triad with colinear components. No resulting interaction.

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Figure 7: Time series of modal wave amplitudes for colinear case: Direct numerical simulation.



Figure 8: Wavenumber-frequency spectra for colinear case: Direct numerical simulation.

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