CHAPTER 176

Probability distribution of the maximum wave height along a sea wall with finite width

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Abstract

A probability distribution of the simultaneous maximum wave amplitude within a finite width along a virtual vertical plane placed in a 3-dim. irregular sea state is studied theoretically. The probability distribution differs considerably from the Rayleigh distribution even when the width of the structure is an order of 1/10 significant wave length. The difference increases with the interval width. Especially, appearant probability of large amplitude is considerably larger than that expected from the Rayleigh distribution. Coastal structures which were designed with sufficient safety margin have sometimes failes. There are possibilities that the structures were not attacked by "unexpectedly large waves" but large waves which could have been possibly expected if we had taken the width effect into account.

1.Introduction

Probability distribution of zero-crossing irregular wave heights measured by a "fixed wave gauge" agrees well with the Rayleigh probability distribution regardless of a wave spectrum. A zero-crossing wave height is defined as a difference between the maximum and minimum water levels within a zero-crossing wave period in the ordinary definition. When a short crested irregular wave acts on the structure such as a breakwater, wave height along it changes spatially. Although visual observation can roughly recognize a maximum wave height and its location, but single wave gauge can not always catch 3-dim. wave peaks and troughs of the short crested waves but only record a water surface displacement at its fixed position (Fig.1). However, if a big wave hits any part of the structure, it is recognized that the structure is attacked by a big wave. Suppose many wave gauges

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are set up with negligibly small intervals on the structure, even though it seems unpractical. And if the largest one is adopted selectively among measured wave heights with these wave gauges as a simultaneous wave height during an action of a single (short crested) wave, a probability distribution of the wave height defined in this way may change according to the width of the structure. Statistical properties of the selectively defined wave height seems necessary to consider in a design of structure, if the structure is such a type that a local damage may induce a failure of the whole structure. This study deals with a theoretical probability distribution of the spatially maximum wave amplitudes along a structure with a finite width placed in a 3-dim. irregular sea state. A quite different probability distribution of wave amplitudes from the Rayleigh distribution is obtained if the width is not small.

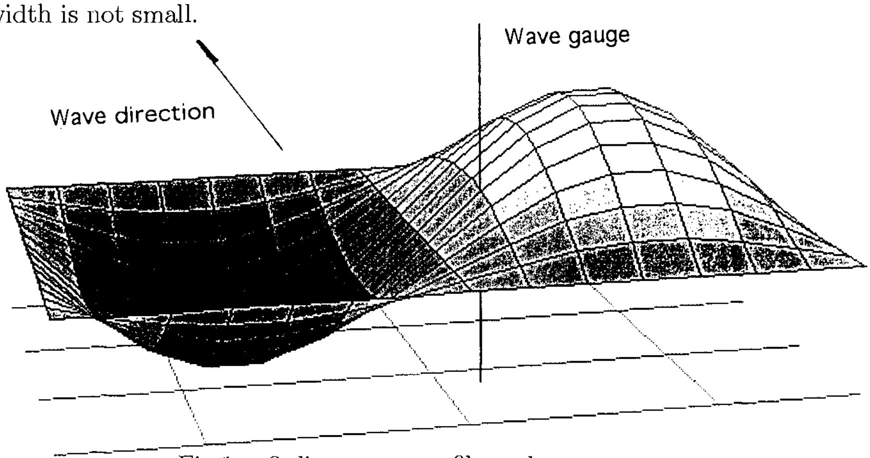


Fig.1 3-dim. wave profile and a wave gauge

2. Wave number spectrum

x and y axes are taken on a still water level so that x axis takes a dominant wave direction and z axis is taken positive upward. Water surface elevation ξ of a 3-dim. short crested irregular wave is expressed as,

$$\xi = \sum_{i=1}^{\infty} c_i \cos\left(\mathbf{k}_i \mathbf{x} + 2\pi f_i t + \varphi_i\right) \tag{1}$$

in which c_i, $\mathbf{k}_i$, f_i and φ_i are amplitude, wave number, frequency and phase angle of the i-th component wave respectively, $\mathbf{x}$=(x, y) and t is time. c_i is determined from

$$\sum_{f_i,\theta_i}^{f_i+df,\ \theta_i+d\theta} c_i^2/2 = E\left(f_i, \theta_i\right) df d\theta = S(f_i)G(\theta_i; f_i) df d\theta \tag{2}$$

where θ and $E(f, \theta)$ are the direction of wave and the directional wave spectrum, $S(f)$ is a power spectrum, $G(\theta; f)$ is a directional function. A wave number spectrum $E(k_{\mathrm{x}}, k_{\mathrm{y}})$ is derived from $E(f,\theta)$ as,

$$E\left(k_{\mathrm{x}}, k_{\mathrm{y}}\right) = C_g/2\pi\mathbf{k}\cdot E\left(f,\theta\right) \tag{3}$$

where k_{x} , k_{y} and C_g are x, y components of a wave number $\mathbf{k}$ and group velocity corresponding to the frequency f respectively. New x, y axes which are anti-clockwise rotation of x and y axes around z axis are taken. The angle between x and x axes is θ. The wave number components on the new axes are expressed as,

$$\begin{aligned} k_x &= k_{\mathrm{x}} \cos\ \theta + k_{\mathrm{y}} \sin\ \theta \\ k_y &= -k_{\mathrm{x}} \sin\ \theta + k_{\mathrm{y}} \cos\ \theta \end{aligned} \tag{4}$$

For simplicity, k_{x}, k_x, k_{y} and k_y are replaced by u, u, v and v respectively. Wave profile in the new coordinate is expressed as,

$$\xi = \sum_{n=1}^{\infty} C_n \cos\left(u_n x + v_n y + 2\pi f_n t + \varepsilon_n\right) \tag{5}$$

and if a vertical virtual plane H is placed on the new x axis, cross section of ξ on H at $t = 0$ is given as,

$$\zeta = \sum_{n=1}^{\infty} C_n \cos\left(u_n x + \varepsilon_n\right) \tag{6}$$

where C_n and ε_n are amplitude and phase angle of the n-th component wave. C_n is determined from,

$$\sum_{u_n, v_n}^{u_n+du, v_n+dv} C_n^2/2 = E\left(u_n, v_n\right) du dv \tag{7}$$

The wave number spectrum $E(\mathrm{u}, \mathrm{v})$ in eq.(3) is transformed into $E(u,v)$ using the relations,

$$\begin{aligned} u &= \mathrm{u} \cos\theta - \mathrm{v} \sin\theta \\ v &= \mathrm{u} \sin\theta + \mathrm{v} \cos\theta \end{aligned} \tag{8}$$

and

$$E(u,v) = \frac{\partial(\mathrm{u}, \mathrm{v})}{\partial(u,v)} E(\mathrm{u}, \mathrm{v}) \tag{9}$$

where $\partial(\mathrm{u}, \mathrm{v})/\partial(u,v) = 1$. One dimensional wave number spectrum $E_\theta(u)$ for ζ is given by an integration of $E(u,v)$ in terms of v,

$$E_\theta(u) du = \sum_{du, v} C^2/2 = du \int_{-\infty}^{\infty} E\left(u,v\right) dv \tag{10}$$

When $\theta = \pi/2$, $E_\theta(u)$ is symmetry about $u = 0$. Small angle shift from $\theta = \pi/2$ brings, however, less unsymmetry on $E_\theta(u)$.

3. Maximum amplitude within a finite interval

Amplitude of zero-crossing wave is approximated with a wave envelope. Solid line in Fig.2 shows schematically a wave envelope $R(x)$ on H for $\zeta|_{t=0}$. $R(x)$ is determined by the Rice(1945) method (eq.20). Taylor series expansion of $R(x)|_{t=0}$ around $x = 0$ is given as,

$$R(x) = R(0) + R'(0)\,x + R''(0)\,x^2/2 + \cdots\cdots\cdots \tag{11}$$

$R(0), R'(0)$ and $R''(0)$ are amplitude of $R(x)$, its first and second derivative in terms of x at $x = 0$ respectively. First three terms in eq.(11) are used to approximate $R(x)$ in this study assuming the width of concern in Fig.2 ($-\Delta L \le x \le \Delta L$: width of the structure) is not large. A difference between the maximum amplitude R_m within the interval and $R(0)$ is given as

$$\begin{aligned} \left|R'(0)/R''(0)\right| &\le \Delta L\,; \\ R_m - R(0) &= -\left\{R'(0)\right\}^2/2R''(0) \end{aligned} \tag{12}$$

$$\begin{aligned} \left|R'(0)/R''(0)\right| &> \Delta L\,; \\ R'(0) &\ge 0 \\ R_m - R(0) &= R'(0)\,\Delta L + R''(0)\,(\Delta L)^2/2 \\ R'(0) &< 0 \\ R_m - R(0) &= -R'(0)\,\Delta L + R''(0)\,(\Delta L)^2/2 \end{aligned} \tag{13}$$

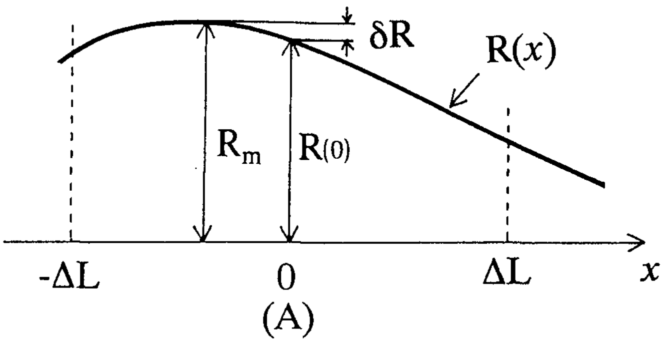


Fig.2 $R(x)$, $R(0)$ and R_m; wave envelope its amplitude at $x = 0$ and its maximum amplitude within the interval $-\Delta L \le x \le \Delta L$

Putting $\delta R = R_m - R(0)$, probability distribution of δR is defined as

$$P(\delta R)_R = \int_S P\left[R'(0), R''(0); R(0)\right] dS \qquad (14)$$

in which $P[R'(0), R''(0); R(0)]$ is a conditional joint probability distribution for $R'(0)$ and $R''(0)$ and S is a region in which $\Delta R < \delta R < \Delta R + dR$ on condition $R(0)$. Figure 3 schematically shows the region S. Since S is symmetry about R'' axis, a half region ($R' > 0$) is shown. A probability distribution for the maximum amplitude R_m within the interval $-\Delta L \leq x \leq \Delta L$ is defined by

$$P_m(R) = \int_S^{\infty} P(\delta R)_R P[R(0)]\, dR(0) \qquad (15)$$

where $P[R(0)]$ is a probability distribution of $R(0)$ (Rayleigh distribution). Connecting eqs.(14) and (15), $P_m(R)$ is calculated from

$$P_m(R) = \int_R^{\infty} \int_S P\left[R(0), R'(0), R''(0)\right] dS dR(0) \qquad (16)$$

in which $P[R(0), R'(0), R''(0)]$ is a joint probability distribution of $R(0), R'(0)$ and $R''(0)$. For simplicity, $R(0), R'(0)$ and $R''(0)$ are expresed as R, R' and R'' respectively.

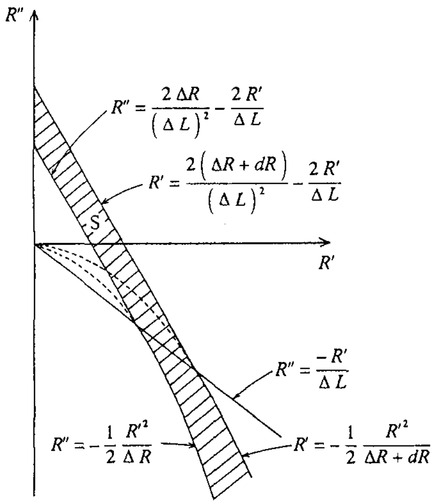


Fig.3 Region of integration S

4. Joint probability distribution for R, R' and R''

Following Rice (1945), joint probability distribution for R, R' and R'' is given as,

$$p\left(R, R', R''\right) = \frac{R^3}{8\pi^3 B}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d\phi' \int_{-\infty}^{\infty} d\phi''$$

$$
\begin{aligned}
\times \quad & \exp\left\{-\frac{1}{2B^2}\left[B_0R^2 + 2B_1R^2\phi' - 2B_2\left(RR'' - R^2{\phi'}^2\right)\right.\right. \\
+ \quad & B_{22}\left({R'}^2 + R^2{\phi'}^2\right) \\
- \quad & 2B_3\left(RR''\phi' - 2{R'}^2\phi' - R'R\phi'' - R^2{\phi'}^3\right) \\
+ \quad & \left.\left. B_4\left({R''}^2 - 2RR''{\phi'}^2 + 4{R'}^2{\phi'}^2 + 4RR'\phi'\phi'' + R^2{\phi'}^4 + R^2{\phi''}^2\right)\right]\right\}
\end{aligned} \tag{17}
$$

in which

$$
\begin{aligned}
B_0 &= \left(b_2b_4 - {b_3}^2\right)B, \quad B_{22} = \left(b_0b_4 - {b_2}^2\right)B, \\
B_1 &= -\left(b_1b_4 - b_2b_3\right)B, \quad B_2 = \left(b_1b_3 - {b_2}^2\right)B, \\
B_3 &= -\left(b_0b_3 - b_1b_2\right)B, \quad B_4 = \left(b_0b_2 - {b_1}^2\right)B, \\
B &= b_0b_2b_4 + 2b_1b_2b_3 - {b_2}^3 - b_0{b_3}^3 - b_4{b_1}^2
\end{aligned} \tag{18}
$$

b_n, $(n = 0, 1, 2, 3, 4)$ is determined as,

$$
\begin{aligned}
b_0 &= \left\langle {I_{c1}}^2\right\rangle = \left\langle {I_{s1}}^2\right\rangle \\
b_2 &= \left\langle {I_{c2}}^2\right\rangle = \left\langle {I_{s2}}^2\right\rangle \\
b_4 &= \left\langle {I_{c3}}^2\right\rangle = \left\langle {I_{s3}}^2\right\rangle \\
b_1 &= \langle I_{c1}I_{s2}\rangle = \langle I_{c2}I_{s1}\rangle \\
b_3 &= \langle I_{s2}I_{c3}\rangle = \langle I_{c2}I_{s3}\rangle
\end{aligned} \tag{19}
$$

in which $\langle\ \rangle$ means an ensemble mean, I_{ci}, I_{si}, $(i = 1, 2, 3)$, R and ϕ are given as,

$$
\begin{aligned}
I_{c1} &= \sum_{n=1}^{\infty} C_n \cos\left(u_nx - u_mx + \varepsilon_n\right) \\
I_{s1} &= \sum_{n=1}^{\infty} C_n \sin\left(u_nx - u_mx + \varepsilon_n\right) \\
R &= \sqrt{I_{c1}^2 + I_{s1}^2} \\
\phi &= \tan^{-1}(I_{s1}/I_{c1})
\end{aligned} \tag{20}
$$

and

$$
I_{c2} = (I_{c1})', \; I_{s2} = (I_{s1})', \; I_{c3} = (I_{c1})'', \; I_{s2} = (I_{s1})'' \tag{21}
$$

u_m is a mean wave number of $E_\theta(u)$ (eq.10). In these equations $'$ and $''$ mean the first and second derivative in terms of x respectively. For simplicity $E_\theta(u)$ is assumed to be symmetry about $u = 0$. This assumption approximately holds if $|\,\theta - \pi/2|$ is not large. Integrating eq.(17) in terms of ϕ and ϕ'',

$$p\left(R, R', R''\right) = 2\alpha \int_0^\infty \exp\left\{- \beta\phi'^4 - \gamma\phi'^2\right\} d\phi' \tag{22}$$

$$\alpha = \frac{R^2}{(2\pi)^{3/2}\sqrt{B_4}} \exp\left\{-\frac{1}{2B^2}\left(B_0R^2 - 2B_2RR'' + B_{22}R'^2 + B_4R''^2\right)\right\} \tag{23}$$

$$\beta = B_4R^2/2B^2 \tag{24}$$

$$\gamma = \left(B_{22}R^2 - 2B_4RR'' + 2B_2R^2\right) / \left(2B^2\right) \tag{25}$$

Although Rice(1945) gave an analytical solution for eq.(22), calculation is made numerically in this study.

5. Probability distribution of the maximum amplitude

Bretschneider-Mitsuyasu type wave spectrum is applied for $S(f)$ in eq.(2).

$$S(f) = 0.257H_{1/3}^2T_{1/3}(T_{1/3}f)^{-5}\exp\left[-1.03(T_{1/3}f)^{-4}\right] \tag{26}$$

where $H_{1/3}$ and $T_{1/3}$ are significant wave height and period. Mitsuyasu type directional spreading function $G(\theta; f)$ modified by Goda et al. (1975) is used.

$$G(\theta; f) = G_0\cos^{2S}(\theta/2) \tag{27}$$

in which

$$G_0 = 2^{2S-1}\Gamma^2(S+1)/\left[\pi\Gamma(2S+1)\right] \tag{28}$$

and

$$S = \begin{cases} S_{\max} \cdot (f/f_p)^5 & : f \le f_p \\ S_{\max} \cdot (f/f_p)^{-2.5} & : f > f_p \end{cases} \tag{29}$$

f_p is a peak frequency of $S(f)$. Goda et al.(1975) showed that $S_{max} = 10,\ 25$ and 75 are suitable for fully saturated sea state, swell sea states with short and long decay distances respectively. Significant wave height and period of $H_{1/3} = 5.5m, T_{1/3} = 10s$ are used to realize a fully saturated sea state in the present calculation. $L_{1/3}$, $R(0)_{rms}$ and $H_{1/3}/L_{1/3}$ are $156m$, $1.35m$ and 0.035 respectively in deep sea condition.

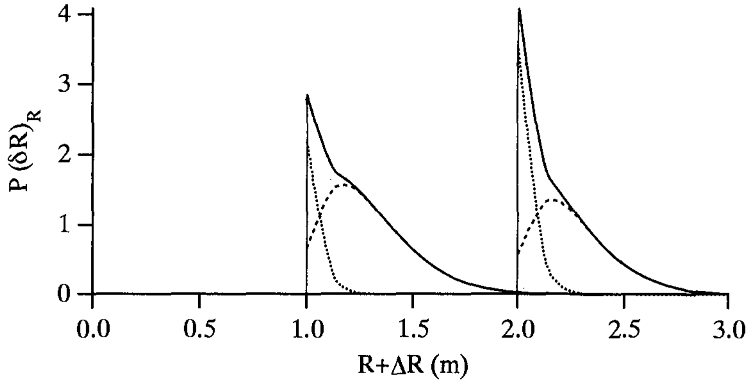
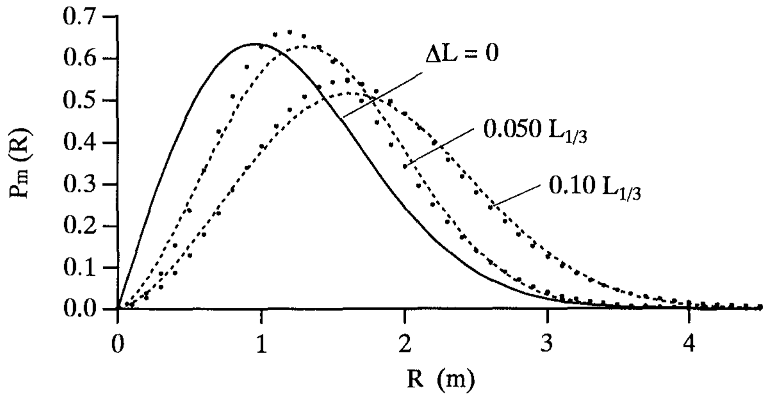
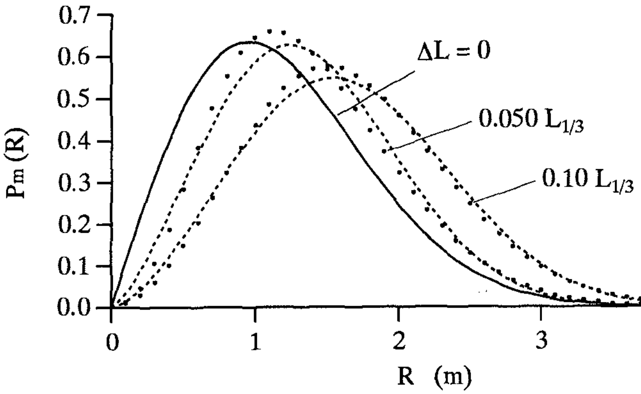


Fig.4 Distributions of δR on conditions $R = 1.0m$ and $R = 2.0m$ $(S_{max} = 10,\ \Delta L = 0.05L_{1/3})$

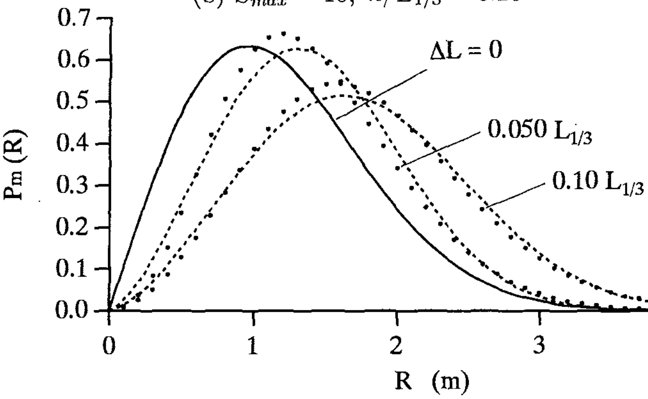
Figure 4 shows $P(\delta R)_R$ when $R = 1.0m$ and $2.0m$ for $\Delta L = 0.05L_{1/3}$, $h/L_{1/3} = 1.0$, and $S_{max} = 10$. Dotted line shows a part of the distribution when local maxima (eq.12) appear within $-\Delta L \leq x \leq \Delta L$ and chain line shows a part of the distribution when local maxima appear outside of the interval. Maximum amplitude within the interval is $R(-\Delta L)$ or $R(\Delta L)$ in the latter distribution (eq.13). Total of the dotted and chain lines gives $P(\delta R)_R$. When $R = 2.0m$, total distribution is narrower than that of $R = 1.0m$. This may correspond that when $R(0)$ is large, R_m presumably exist within a vicinity of $x = 0$. When $R(0)$ is small, on the contrary, the probability that the local maximum exist outside of the interval becomes large.



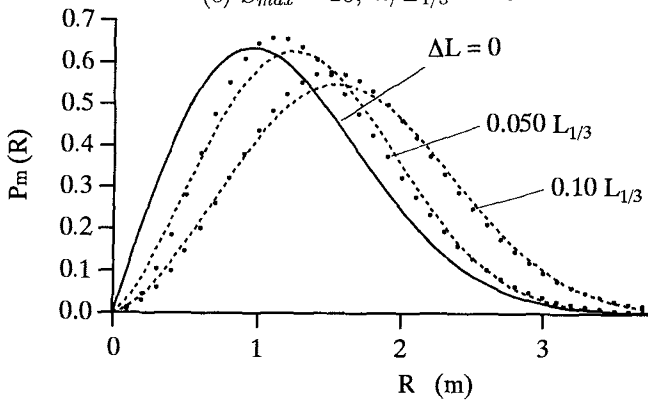
(a) $S_{max} = 10,\ h/L_{1/3} = 1.0$



(b) $S_{max} = 10,\ h/L_{1/3} = 0.25$



(c) $S_{max} = 20,\ h/L_{1/3} = 1.0$



(d) $S_{max} = 20,\ h/L_{1/3} = 0.25$

Fig.5 $P_m(R)$ (•), Rayleigh (solid line) and Weibull (broken line) distributions.

Figure 5 shows a calculated example of $P_m(R)$ ($\bullet$) when (a) $h/L_{1/3} = 1.0$, $S_{max} = 10$, (b) $h/L_{1/3} = 0.25$, $S_{max} = 10$, (c) $h/L_{1/3} = 1.0$, $S_{max} = 20$, and (d) $h/L_{1/3} = 0.25$, $S_{max} = 20$. Two values for ΔL ($0.05L_{1/3}$, $0.1L_{1/3}$) are used in the calculations. Solid line shows the Rayleigh distribution ($\Delta L = 0$). Increasing interval brings a larger departure from the Rayleigh distribution. Broken lines are the Weibull distribution shown for comparisons. Shape parameters for the Weibull distribution are calculated using a Weibull probability paper (Yamauchi, 1972). Agreements of the calculated results and the Weibull distributions are fairly well in all cases.

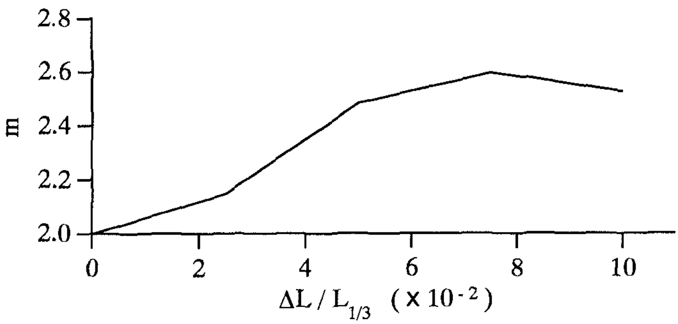


Fig.6 Change of the shape factor with $\Delta L/L_{1/3}$
($h/L_{1/3} = 1.0$, $S_{max} = 10$)

Figure 6 shows a change of the shape factor with $\Delta L/L_{1/3}$ when $h/L_{1/3} = 1.0$, $S_{max} = 10$ (Fig.5(a)). It increases with ΔL while it is small but takes almost constant value $2.5 \sim 2.6$ where $\Delta L/L_{1/3} > 0.06$. Figure 7 shows a change of rms value for R_m calculated from $P_m(R)$ in Fig.5(a). Similar results as shown in Figs.6 and 7 are obtained in other cases. $P_m(R)$ ($\bullet$), Rayleigh (solid line) and Weibull (broken line) distributions are compared for large value of R ($> 3.8m$) in Fig.8 ($h/L_{1/3} = 0.1$, $S_{max} = 25$ and $\Delta L/L_{1/3} = 0.1$). $P_m(R)$ is considerably larger than the Rayleigh and Weibull distribution in this area.

6. Modification of the spectrum

High frequency component of a wave spectrum brings small fluctuations on R (Tayfun et al., 1989). Sometimes a few local maxima appear on R within a given width. Eqs.(12), (13) may give δR between the nearest local maximum and $R(0)$, instead of between the maximum R_m within the interval and $R(0)$ in this case. To eliminate insignificant fluctuations on R, high frequency (large wave number; $\mathbf{k} > 0.2$, $L \leq 0.2L_{1/3}$) part of the spectrum is neglected so that no wave component has shorter wave length than the mentioned widest width ($-\Delta = 0.1L_{1/3}$) in this study.

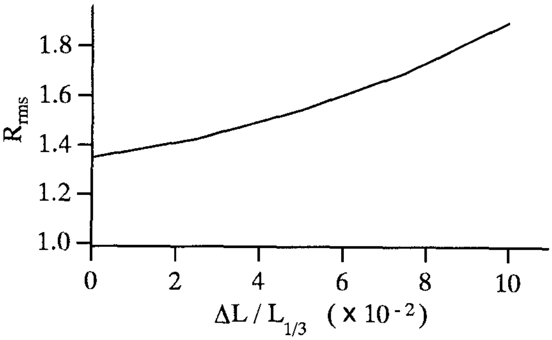


Fig.7 Change of the rms value for R_m with $\Delta L/L_{1/3}$ ($h/L_{1/3} = 1.0$, $S_{max} = 10$)

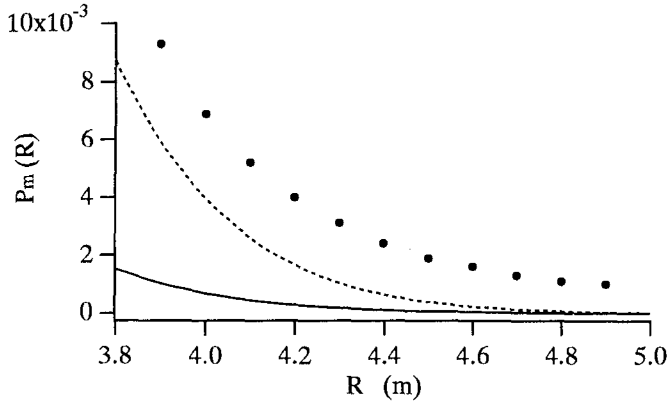


Fig.8 $P_m(R)$ (•), Rayleigh (solid line) and Weibull (broken line) distributions in a large R region. ($h/L_{1/3} = 1.0$, $S_{max} = 10$)

7. Concluding remarks

This study deals with a probability distribution of the maximum wave amplitude within a finite width along a virtual vertical plane placed in a 3-dim. irregular sea state. The probability distribution departs considerably from the Rayleigh distribution. This difference increases with the interval width. Especially, probability of the large amplitude is considerably larger than that of the Rayleigh distribution. It could be considered that unexpectedly large wave, appearance probability of which is very small, attacked when failure of coastal structures took place. However, the width effect for the appearant probability of large waveswe has not been taken into account. Some of the failure may take place since this width effect on the probability distribution of wave heights is neglected. When structures are planed, three-dimensional properties of waves seems necessary to consider.

References

[1] Goda, Y. and Y. Suzuki: Computation of refraction and diffraction of sea waves with Mitsuyasu's directional spectrum, tech. note of the PHRI, No.230, 45p., 1975.

[2] Longuet-Higgins, M. S.: Statistical properties of wave groups in a random sea state, Phil. Trans. Roy. Soc. London, Ser. A, vol. 312, pp.219-250, 1984.

[3] Rice S. O.: Mathematical analysis of random noise, Selected papers on noise and stochastic processes (ed. Nelson Wax), Dover, pp.133-294, 1954.

[4] Tayfun, M. A. and Jen-Men, Lo.: Envelope, phase, and narrow-band models of sea waves, Jour. of Waterway, port, coastal, and ocean eng., vol. 115, No.5, pp.594-613, 1989.

[5] Yamauchi, Z. (ed.): Statistical tables, Japanese standards association, 451p., 1972. (in Japanese)