CHAPTER 91

THE ROLE OF WAVE-INDUCED SHEAR STRESSES IN THE MOMENTUM BALANCE EQUATIONS

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ABSTRACT

The wave-induced shear stresses, which result from the correlation between horizontal and vertical components of the oscillatory velocity after time-averaging the horizontal momentum balance equations, are shown in this paper to play an important role in vertical circulation analysis, having the same order of magnitude than other wave-induced (normal) stresses. The model of Rivero and Arcilla (1995) to calculate the wave-induced shear stress for a 2DV situation, based on a mathematical identity that relates this stress to the wave-induced normal stresses and the oscillatory vorticity, is now extended to a general 3D flow. The consequences of neglecting the wave-induced shear stresses are shown to be an overprediction of the waves effect on the description of the vertical profiles of the mean (current) velocity. Theoretical examples of such effects are presented and discussed for some simplified situations (the undertow and longshore current vertical profiles).

1. INTRODUCTION

The correlations between horizontal ($\tilde{u}_i$) and vertical ($\tilde{w}$) components of the oscillatory (wave) motion, which appear explicitly in the time-averaged momentum balance equations as wave-induced (effective) shear stresses, have been shown to play an important role in the analysis of the vertical distribution

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of wave-induced currents (see e.g. Deigaard and Fredsøe (1989), De Vriend and Kitou (1990), Stive and De Vriend (1994), and Rivero and Arcilla (1995), to mention some recent references). However, most circulation models have, until very recently, neglected the $<\tilde{u}_i\tilde{w}>$-contribution ($< >$ denotes the time-averaging operator over a wave period) by simply arguing that the two wave velocity components, $\tilde{u}_i$ and $\tilde{w}$, are 90° out of phase. This result, which corresponds to periodic waves of permanent form, is not valid for real waves propagating over a sloping bottom and/or with energy dissipation (be it in the bottom boundary layer or in the free surface area for the roller of breaking waves).

The purpose of this paper is to present new formulations to calculate the wave-induced shear stresses $<\tilde{u}_i\tilde{w}>$, based on mathematical identities that relate these stresses to other wave stresses ($<\tilde{u}_i\tilde{u}_j>$ and $<\tilde{w}^2>$), easier to calculate by any given wave theory (e.g. linear theory), and the vorticity of the oscillatory velocity, and to investigate the relevance of these stresses in vertical circulation analysis.

The outline of the paper is as follows. Section 2 gives an overview of the model of Rivero and Arcilla (1995) to calculate the wave-induced shear stress $<\tilde{u}\tilde{w}>$ in a 2DV situation. This model is extended in section 3 to a general 3D flow, and its implication in the horizontal momentum balance equations are discussed in section 4. Theoretical examples on the role of wave-induced shear stresses in the description of the vertical circulation for some simplified situations are given in sections 5 (undertow profiles) and 6 (longshore current profiles). Finally, section 7 presents the summary and conclusions.

2. WAVE-INDUCED SHEAR STRESS IN A 2DV FLOW

For a 2DV situation (as that encountered in a wave flume), in which waves propagate along the x-direction and the z-axis is directed vertically upwards (Fig. 1), Rivero and Arcilla (1995) derived a mathematical identity which relates the wave-induced shear stress $<\tilde{u}\tilde{w}>$ to other wave-induced stresses ($<\tilde{u}^2>$ and $<\tilde{w}^2>$) and to the (scalar) vorticity of the oscillatory flow $\tilde{\omega} = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x}$:

$$\frac{\partial}{\partial z} <\tilde{u}\tilde{w}> = <\tilde{w}\tilde{\omega}> - \frac{1}{2}\left[\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right)\right] \tag{1}$$

where the oscillatory components of the velocity ($\tilde{u}$ and $\tilde{w}$) are defined such that $<\tilde{u}>=<\tilde{w}>= 0$, and hence, $<\tilde{\omega}>= 0$.

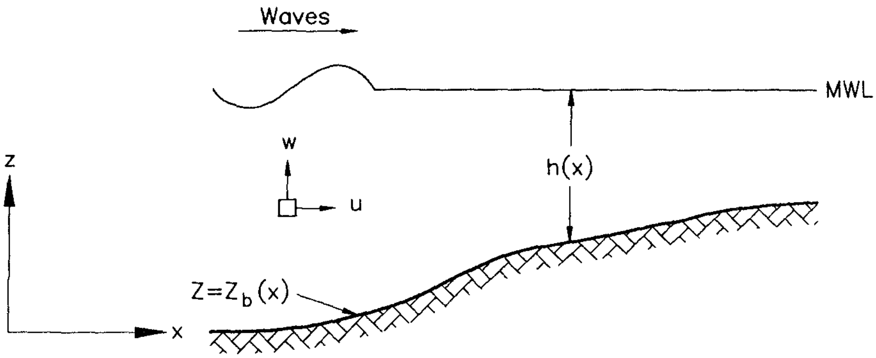

Figure 1. Domain definition sketch -2DV situation-
(after Rivero and Arcilla, 1995).

Assuming irrotational flow ($\tilde{\omega} = 0$) and invoking linear wave theory to evaluate $<\tilde{u}^2>$, $<\tilde{w}^2>$ and the near-bed value of $<\tilde{u}\tilde{w}>$, which is given by the kinematic boundary condition at the bottom $z = z_b$ (Putrevu and Svendsen, 1993) —see Fig. 2—, the vertical distribution of $<\tilde{u}\tilde{w}>$, after integration of Eq. (1), is found to be linear over depth:

$$<\tilde{u}\tilde{w}> = -G\left(\frac{E}{\rho h}\right)\frac{\partial d}{\partial x} - \left[\frac{\partial}{\partial x}\left(\frac{1}{2}\,G\,\frac{E}{\rho h}\right)\right](z - z_b) \qquad (2)$$

where d is the still-water depth, h is the mean water depth, E is the wave energy density, ρ is the fluid density, and $G = 2kh/\sinh(2kh)$.

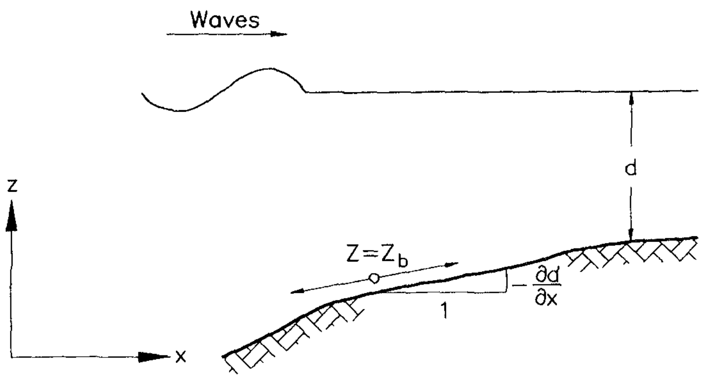

Figure 2. Simplified model for the oscillatory wave motion near the bed.
(after Rivero and Arcilla, 1995)

This general expression (2), using linear wave theory, has been applied to simplified situations and has been shown to coincide or degenerate into other existing formulations with comparable simplifying assumptions, viz. irrotational waves over a sloping bottom in the shallow-water approximation (De Vriend and Kitou, 1990), and surf-zone breaking waves over a horizontal bottom (Deigaard and Fredsøe, 1989).

Expression (2) has also been compared (Rivero and Arcilla, 1997) with experimental results for the vertical distribution of the wave-induced shear stress $<\tilde{u}\tilde{w}>$ in shoaling waves within the framework of a research project (*Dynamics of Beaches*) funded by the EU —see Prinos et al. (1994) for details. Simultaneous time series of horizontal and vertical velocity components, and free surface elevation were measured at several stations along the wave flume and at several points within each station. Various regular and irregular wave conditions were generated with and without a submerged breakwater on a 1:15 sloping beach (Rivero et al, 1996). The experimental results presented here correspond to the test without submerged breakwater under irregular wave action (Test E: Jonswap-type spectrum with $\gamma = 3.3$, peak period T_p=2.50 s and significant wave height H_s=0.25 m in the deeper part of the wave flume, with water depth h=3.06 m). Fig. 3 shows the measured values of $<\tilde{u}\tilde{w}>$ at 3 stations located in the shoaling zone, and the predicted $<\tilde{u}\tilde{w}>$-distribution with expression (2). As can be seen from this figure, both the sign (positive) and the trend (decreasing upwards) of the vertical distribution of $<\tilde{u}\tilde{w}>$ is correctly predicted by the authors' model, as well as quantitavely.

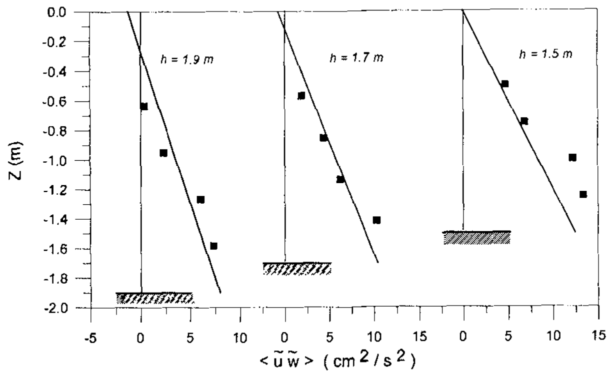

Figure 3. Comparison between experimental data (squares) and this model (Eq. 2) for the vertical distribution of the wave-induced shear stress $<\tilde{u}\tilde{w}>$. Test E: irregular waves, Jonswap-type, H_s=0.25 m, T_p=2.5 s. (after Rivero and Arcilla, 1997)

3. WAVE-INDUCED SHEAR STRESSES IN A 3D FLOW

Explicit expressions for the vertical distribution of the wave-induced shear stresses ($<\tilde{u}\tilde{w}>$, $<\tilde{v}\tilde{w}>$) in a 3D situation may be found from (2) in a horizontally rotated frame of reference, after assuming that the (irrotational) flow is essentially two-dimensional in the direction of wave propagation x' (see Fig. 4), and neglecting the curvature of wave rays:

$$<\tilde{u}\tilde{w}> = -2\mathcal{A}\left(\frac{\partial d}{\partial x}\cos^2\alpha + \frac{\partial d}{\partial y}\sin\alpha\cos\alpha\right) - \left(\frac{\partial \mathcal{A}}{\partial x}\cos^2\alpha + \frac{\partial \mathcal{A}}{\partial y}\sin\alpha\cos\alpha\right)(z - z_b) \tag{3a}$$

$$<\tilde{v}\tilde{w}> = -2\mathcal{A}\left(\frac{\partial d}{\partial x}\sin\alpha\cos\alpha + \frac{\partial d}{\partial y}\sin^2\alpha\right) - \left(\frac{\partial \mathcal{A}}{\partial x}\sin\alpha\cos\alpha + \frac{\partial \mathcal{A}}{\partial y}\sin^2\alpha\right)(z - z_b) \tag{3b}$$

where $\mathcal{A} = \frac{1}{2}G\left(\frac{E}{\rho h}\right)$ is an energy-like magnitude.

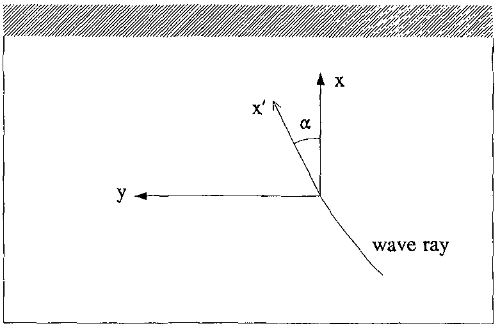

Figure 4. Domain definition sketch -3D situation- (after Rivero and Arcilla, 1997).

A more general (differential) expression for $<\tilde{u}\tilde{w}>$ and $<\tilde{v}\tilde{w}>$, from which their implication in the time-averaged momentum balance equations may be easily assessed, is given by the following straightforward mathematical identities that involve the oscillatory vorticity components,

$$\tilde{\omega}_x = \frac{\partial \tilde{w}}{\partial y} - \frac{\partial \tilde{v}}{\partial z} \qquad \tilde{\omega}_y = \frac{\partial \tilde{u}}{\partial z} - \frac{\partial \tilde{w}}{\partial x}$$

and the continuity equation for the oscillatory velocity components,

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0 \quad :$$

$$\frac{\partial}{\partial z} <\tilde{u}\tilde{w}> = <\tilde{w}\tilde{\omega}_y> -\frac{1}{2}\left[\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right)\right] - <\tilde{u}\frac{\partial \tilde{v}}{\partial y}> \qquad (4a)$$

$$\frac{\partial}{\partial z} <\tilde{v}\tilde{w}> = - <\tilde{w}\tilde{\omega}_x> -\frac{1}{2}\left[\frac{\partial}{\partial y}\left(<\tilde{v}^2> - <\tilde{w}^2>\right)\right] - <\tilde{v}\frac{\partial \tilde{u}}{\partial x}> \qquad (4b)$$

For an irrotational flow ($\tilde{\omega}_x = \tilde{\omega}_y = 0$), these mathematical identities read:

$$\frac{\partial}{\partial z} <\tilde{u}\tilde{w}> = -\frac{1}{2}\left[\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right)\right] - <\tilde{u}\frac{\partial \tilde{v}}{\partial y}> \qquad (5a)$$

$$\frac{\partial}{\partial z} <\tilde{v}\tilde{w}> = -\frac{1}{2}\left[\frac{\partial}{\partial y}\left(<\tilde{v}^2> - <\tilde{w}^2>\right)\right] - <\tilde{v}\frac{\partial \tilde{u}}{\partial x}> \qquad (5b)$$

4. WAVE-INDUCED STRESSES IN THE HORIZONTAL MOMENTUM BALANCE EQUATIONS

The "conventional" wave-induced stresses appearing in the time-averaged horizontal momentum balance equations (see e.g. Svendsen and Lorenz, 1989):

X-direction $$\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right) + \frac{\partial}{\partial y} <\tilde{u}\tilde{v}> + \frac{\partial}{\partial z} <\tilde{u}\tilde{w}> \qquad (6a)$$

Y-direction $$\frac{\partial}{\partial y}\left(<\tilde{v}^2> - <\tilde{w}^2>\right) + \frac{\partial}{\partial x} <\tilde{u}\tilde{v}> + \frac{\partial}{\partial z} <\tilde{v}\tilde{w}> \qquad (6b)$$

would read, after invoking identities (4a-b):

X-direction $$<\tilde{w}\tilde{\omega}_y> + \frac{1}{2}\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right) + <\tilde{v}\frac{\partial \tilde{u}}{\partial y}> \qquad (7a)$$

Y-direction $$- <\tilde{w}\tilde{\omega}_x> + \frac{1}{2}\frac{\partial}{\partial y}\left(<\tilde{v}^2> - <\tilde{w}^2>\right) + <\tilde{u}\frac{\partial \tilde{v}}{\partial x}> \qquad (7b)$$

and assuming irrotational flow ($\tilde{\omega}_x = \tilde{\omega}_y = 0$):

$$\text{X-direction} \qquad \frac{1}{2}\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right) + <\tilde{v}\frac{\partial \tilde{u}}{\partial y}> \tag{8a}$$

$$\text{Y-direction} \qquad \frac{1}{2}\frac{\partial}{\partial y}\left(<\tilde{v}^2> - <\tilde{w}^2>\right) + <\tilde{u}\frac{\partial \tilde{v}}{\partial x}> \tag{8b}$$

The implication of taking into account the wave-induced shear stresses, $<\tilde{u}\tilde{w}>$ and $<\tilde{v}\tilde{w}>$, as given by Eqs.(7a-b), or equivalently, Eqs.(8a-b) for an irrotational flow, with respect to the case in which those stresses are neglected from the conventional form of the time-averaged horizontal momentum balance equations (Eqs. 6a-b), are discussed in the following sections for some idealized situations.

5. UNDERTOW VELOCITY PROFILES

For normally incident waves (α=0) on a cilyndrical beach ($\partial d/\partial y$=0), the vertical distribution of mean shear stress $< \tau_{xz} >$ (associated with viscous and/or turbulent effects) can be found from the time-averaged horizontal momentum balance equation in the cross-shore direction after assuming a given mean pressure distribution (usually, $<p>=<p_h> -\rho <\tilde{w}^2>$, where $<p_h>$ is the mean hydrostatic pressure) —see e.g. (Svendsen, 1984) for the assumptions and motivation of this equation—:

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{xz}>}{\rho}\right) = \frac{\partial U^2}{\partial x} + g\frac{\partial <\eta>}{\partial x} + \frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right) + \frac{\partial}{\partial z} <\tilde{u}\tilde{w}> \tag{9}$$

where $U(x,z)$ is the (undertow) mean velocity and $<\eta>(x)$ is the mean water level (set-up/set-down). Eq. (9) shows the vertical variation of the mean shear stress $<\tau_{xz}>$ in the $\{x,z\}$ vertical plane.

As already mentioned before, the $< \tilde{u}\tilde{w} >$ contribution has been, until very recently, neglected throughout the water column except in the bottom boundary layer, in which it led to the streaming solution (Longuet-Higgins, 1953). The relevance of the $<\tilde{u}\tilde{w}>$ contribution in the vertical distribution of $<\tau_{xz}>$ can be easily assessed after substitution of identity (1) into Eq. (9):

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{xz}>}{\rho}\right) = \frac{\partial U^2}{\partial x} + g\frac{\partial <\eta>}{\partial x} + \frac{1}{2}\left[\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right)\right] + <\tilde{w}\tilde{\omega}> \tag{10}$$

Since the $<\tilde{w}\tilde{\omega}>$ term will be, in general, unknown —it depends on the vertical distribution of the current velocity—, it may be set to zero as a first approximation, in which case Eq. (10) would read:

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{xz}>}{\rho}\right) = \frac{\partial U^2}{\partial x} + g\frac{\partial <\eta>}{\partial x} + \frac{1}{2}\left[\frac{\partial}{\partial x}\left(<\tilde{u}^2> - <\tilde{w}^2>\right)\right] \qquad (11)$$

It may be, thus, seen that the $<\tilde{u}\tilde{w}>$-term first effect is to halve the normal wave stress contribution to $<\tau_{xz}>$. This would mean that unrealistic closure submodels for $<\tau_{xz}>$ (i.e. eddy viscosity coefficient ν_t values) must be used to fit measured undertow profiles if the $<\tilde{u}\tilde{w}>$ term is neglected in the momentum balance equation (9). It should be noticed that, although wave stress gradients are usually small compared to $g\frac{\partial <\eta>}{\partial x}$ inside the surf zone (Svendsen and Lorenz, 1989), that is not necessarily the case outside the breaker region or in the transition zone (Putrevu and Svendsen, 1993). This means that the $<\tilde{u}\tilde{w}>$ contribution, and therefore the need to calculate it, is expected to be more significant in areas in which mean water level gradients do not dominate the momentum balance equation.

6. LONGSHORE CURRENT VELOCITY PROFILES

For obliquely incident waves ($\alpha \neq 0$) on a cilyndrical beach ($\partial d/\partial y$=0), the vertical distribution of the mean shear stress $<\tau_{yz}>$, which may be related to the vertical gradients of the longshore current velocity $V(x, z)$, can be found from the time-averaged horizontal momentum balance equation in the alongshore direction —see e.g. (Svendsen and Putrevu, 1994)—:

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{yz}>}{\rho}\right) = \frac{\partial UV}{\partial x} + \frac{\partial}{\partial x} <\tilde{u}\tilde{v}> + \frac{\partial}{\partial z} <\tilde{v}\tilde{w}> \qquad (12)$$

Upon substitution of identity (4b) with the assumption of alongshore uniformity ($\partial/\partial y$=0), Eq. (12) may be written as

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{yz}>}{\rho}\right) = \frac{\partial UV}{\partial x} - <\tilde{w}\tilde{\omega}_x> + <\tilde{u}\frac{\partial \tilde{v}}{\partial x}> \qquad (13)$$

If, as in the preceding section, the term involving the vorticity is disregarded, the governing equation for the vertical distribution of $<\tau_{yz}>$ (13) would read

$$\frac{\partial}{\partial z}\left(\frac{<\tau_{yz}>}{\rho}\right) = \frac{\partial UV}{\partial x} + <\tilde{u}\frac{\partial \tilde{v}}{\partial x}> \qquad (14)$$

In this situation wave stresses are, in the absence of alongshore mean water level gradients, the dominant factors governing the vertical distribution of $<\tau_{yz}>$. Again, it may be noticed that the primary effect of the $<\tilde{v}\tilde{w}>$ term, as given by Eq. (14), is to reduce (approximately, to halve) the contribution of wave stresses ($<\tilde{u}\frac{\partial\tilde{v}}{\partial x}>$ *versus* $\frac{\partial}{\partial x} <\tilde{u}\tilde{v}>$) in the vertical distribution of $<\tau_{yz}>$, with respect to the case in which the $<\tilde{v}\tilde{w}>$ term is neglected in the original equation (12). This fact indicates that, for a given closure submodel of $<\tau_{yz}>$, the predicted curvature of the vertical profiles of longshore currents taking into account the $<\tilde{v}\tilde{w}>$ term (Eq. 14) is smaller than the one predicted neglecting the $<\tilde{v}\tilde{w}>$ term (in Eq. 12), which may help explaining the good fit of logarithmic functions to field measurements of the longshore current vertical profiles (see e.g. Thornton et al, 1995).

7. CONCLUSIONS

The following general conclusions may be drawn from the theoretical derivations presented in this paper:

• New mathematical identities (4a-b) have been derived which express the wave-induced shear stresses ($<\tilde{u}\tilde{w}>$ and $<\tilde{v}\tilde{w}>$) in a 3D flow in terms of other wave-induced stresses ($<\tilde{u}^2>$, $<\tilde{v}^2>$, $<\tilde{w}^2>$ and $<\tilde{u}\tilde{v}>$), easier to calculate by any given wave theory, and the oscillatory vorticity ($\vec{\omega}$).

• In general, the implication of the wave-induced shear stresses ($<\tilde{u}\tilde{w}>$ and $<\tilde{v}\tilde{w}>$) in the horizontal momentum balance equations is to approximately halve the contribution of other wave-induced stresses ($<\tilde{u}^2>$, $<\tilde{v}^2>$, $<\tilde{w}^2>$ and $<\tilde{u}\tilde{v}>$) with respect to the case where horizontal-vertical correlations are neglected, which has been an otherwise common procedure in vertical circulation analysis.

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